Complex Geometry
Exercises Solution

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Contents

Chapter 1

Local Theory

1.1 Holomorphic Functions of Several Variables

Exer 1.1.1. *Show that every holomorphic map* $f : \mathbb{C} \to \mathbb{H} := \{z | \operatorname{Im}(z) > 0\}$ *is constant.*

Consider $g \colon \mathbb{H} \to \mathbb{D}, z \mapsto \frac{z-1}{z+1}$.
Then g is biholomorphic, *i.e.* $g \circ f$ is holomorphic.
And since $|(g \circ f)(z)| < 1$.
So by maximum principle, $g \circ f$ is constant, *i.e.* f is constant.

Exer 1.1.2. *Show that real and imaginary part u respectively v of a holomorphic function $f = u + iv$ are harmonic,* i.e. $\sum\limits_i \frac{\partial^2 u}{\partial x_i^2} + \sum\limits_i \frac{\partial^2 u}{\partial y_i^2} = 0$ *and similarly for v.*

$$\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} = \frac{\partial}{\partial x_i}\frac{\partial v}{\partial y_i} - \frac{\partial}{\partial y_i}\frac{\partial v}{\partial x_i} = 0.$$

Exer 1.1.3. *Deduce the maximum principle and the identity theorem for holomorphic functions of several variables from the corresponding one-dimensional results.*

Identity theorem:
Let $h = f - g$.
Then $= 0$ for all $z \in V$.
For any $w \in U$, restrict h to the "line" connecting z and w, denote it by h_0.
Then h_0 is an one variable holomorphic map and $h_0 = 0$ in a open neighborhood of z.
So $h_0 \equiv 0, i.e. h(w) = 0$.
Hence $f \equiv g$.
Maximum principle:
Suppose $|f|$ has local maximum at z and $|f(z)|$ is maximal in polydisc V.
Then for any $w \in V$, restrict f to the "line" connecting z and w, denote it by g.
Then g is an one variable holomorphic map and $|g|$ has local maximum at z.
So g is constant, *i.e.* $f(w) = f(z)$.
Hence f is constant in V, contradiction!

Exer 1.1.4. *Prove the chain rule* $\frac{\partial(f \circ g)}{\partial z} = \frac{\partial f}{\partial w}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{g}}{\partial z}$ *and its analogue for* $\frac{\partial}{\partial \bar{z}}$*. Use this to show that the composition of two holomorphic functions is holomorphic.*

Let $g = g_x + ig_y, w = u + iv$

$$\begin{aligned}
\frac{\partial(f\circ g)}{\partial z} &= \frac{1}{2}\left(\frac{\partial(f\circ g)}{\partial x} - i\frac{\partial(f\circ g)}{\partial y}\right)\\
&= \frac{1}{2}\left(\frac{\partial f}{\partial u}\frac{\partial g_x}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial g_y}{\partial x} - i\frac{\partial f}{\partial u}\frac{\partial g_x}{\partial y} - i\frac{\partial f}{\partial v}\frac{\partial g_y}{\partial y}\right)\\
&= \frac{1}{4}\left(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}\right)\left(\frac{\partial g_x}{\partial x} + i\frac{\partial g_y}{\partial x} - i\frac{\partial g_x}{\partial y} + \frac{\partial g_y}{\partial y}\right)\\
&+ \frac{1}{4}\left(\frac{\partial f}{\partial u} + i\frac{\partial f}{\partial v}\right)\left(\frac{\partial g_x}{\partial x} - i\frac{\partial g_y}{\partial x} - i\frac{\partial g_x}{\partial y} - \frac{\partial g_y}{\partial y}\right)
\end{aligned}$$

When f, g are holomorphic, $\frac{\partial f\circ g}{\partial \bar z}\frac{\partial f}{\partial w}\frac{\partial g}{\partial \bar z} + \frac{\partial f}{\partial \bar w}\frac{\partial \bar g}{\partial \bar z} = 0$.
So $f\circ g$ is holomorphic.

Exer 1.1.5. *Deduce the implicit function theorem for holomorphic functions $f : U \to \mathbb{C}$ from the Weierstrass preparation theorem.*

WLOG, assume $z_0 = 0$.
Then $\frac{\partial f}{\partial z_1} \neq 0$, $i.e. f_0(z_1) \not\equiv 0$ and $f_0(0)$ is of order 1.
So by WPT, there exists $g(z_1, w) = z_1 + \alpha_1(w)$ and holomorphic h such that $f = g\cdot h$ around 0 and $h(0) \neq 0$.
Therefore around 0, $f = 0$ iff $g = 0$, *i.e.* $z_1 = -\alpha_1(w)$.

Exer 1.1.6. *Consider the function $f : \mathbb{C}^2 \to \mathbb{C}, (z_1, z_2) \mapsto z_1^3 z_2 + z_1 z_2 + z_1^2 z_2^2 + z_2^2 + z_1 z_2^3$ and find an explicit decomposition $f = h\cdot g_w$ as claimed by the WPT.*

Let

$$g = z_2\left(z_2 + \frac{z_1^2 + 1 - \sqrt{1 - 2z_1^2 - 3z_1^4}}{2z_1}\right), h = z_1 z_2 + \frac{z_1^2 + 1 + \sqrt{1 - 2z_1^2 - 3z_1^4}}{2}.$$

Then $f = z_2(z_1 z_2^2 + (z_1^2 + 1)z_2 + z_1^3 + z_1) = g\cdot h, h(0) = 0$ and

$$\frac{z_1^2 + 1 - \sqrt{1 - 2z_1^2 - 3z_1^4}}{2z_1} = \frac{z_1}{2} + \frac{3z_1^3 + 2z_1}{2\left(1 + \sqrt{1 - 2z_1^2 - 3z_1^4}\right)} \to 0$$

as $z_1 \to 0$.
So g is the Weierstrass polynomial.

Exer 1.1.7. *State and prove the product formula for $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$. Show that the product $f\cdot g$ of two holomorphic functions f and g is holomorphic and that $\frac{1}{f}$ is holomorphic on the complement of the zero set $Z(f)$.*

$$\begin{aligned}
\frac{\partial fg}{\partial z} &= \frac{1}{2}\left(\frac{\partial fg}{\partial x} - i\frac{\partial fg}{\partial y}\right)\\
&= \frac{1}{2}\left(f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g - if\frac{\partial g}{\partial y} - i\frac{\partial f}{\partial y}g\right)\\
&= f\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}g.
\end{aligned}$$

Similarly, we have $\frac{\partial fg}{\partial \bar z} = f\frac{\partial g}{\partial \bar z} + \frac{\partial f}{\partial \bar z}g$.
When f, g are holomorphic, $\frac{\partial fg}{\partial \bar z} = f\frac{\partial g}{\partial \bar z} + \frac{\partial f}{\partial \bar z}g = 0$.
So fg is holomorphic.
And on the complement of $Z(f)$, $\frac{\partial}{\partial \bar z}\frac{1}{f} = -\frac{1}{f^2}\cdot\frac{\partial f}{\partial \bar z} = 0$.
Hence $\frac{1}{f}$ is holomorphic on the complement of $Z(f)$.

Exer 1.1.8. *Let $U \subset \mathbb{C}^n$ be open and connected. Show that for any non-trivial holomorphic function $f : U \to \mathbb{C}$ the complement $U \backslash Z(f)$ of the zero set of f is connected and dense in U.*

Since $Z(f)$ is closed.
So by identity theorem, $Z(f)$ has no interior point, *i.e.* $U\backslash Z(f)$ is dense.
For $z, w \in U\backslash Z(f)$, restrict f to the "line" connecting z and w, denote it by g.
Then g is an one variable holomorphic map, *i.e.* $Z(g)$ is discrete.
So there exists a path in $\mathbb{C}\backslash Z(g)$ that connects z, w.
Hence $U\backslash Z(f)$ is connected.

Exer 1.1.9. *Let $U \subset \mathbb{C}^n$ be open and connected. Show that the set $K(U)$ of meromorphic functions on U is a field. What is the relation between $K(U)$ and the quotient field of $\mathcal{O}_{\mathbb{C}^n,z}$ for $z \in U$?*

For $\left(\frac{g_i}{h_i}, U_i\right), \left(\frac{p_j}{q_j}, V_j\right) \in K(U)$, let $W_{ij} = U_i \cap V_j$.
Then $h_i q_j \cdot \left(\frac{g_i}{h_i} + \frac{p_j}{q_j}\right) = g_i q_j + p_j h_i$, $h_i q_j \cdot \left(\frac{g_i}{h_i}\frac{p_j}{q_j}\right) = g_i p_j$ in W_{ij}.
And in U_i, $h_i \cdot \left(-\frac{g_i}{h_i}\right) = -g_i$, $g_i \cdot \frac{h_i}{g_i} = h_i$ on the complement of $Z(g_i)$.
So these are all meromorphic functions, *i.e.* $K(U)$ is a field.
And since $f \in K(U)$ is given by $\frac{g}{h}$ with $g, h \in \mathcal{O}_{\mathbb{C}^n,z}$ in a neighborhood of z.
Hence the quotient field of $\mathcal{O}_{\mathbb{C}^n,z}$ is the stalk of presheaf $K(U)$.

Exer 1.1.10. *Let $U := B_\varepsilon(0) \subset \mathbb{C}^n$ and consider the ring $\mathcal{O}(U)$ of holomorphic functions on U. Show that $\mathcal{O}(U)$ is naturally contained in $\mathcal{O}_{\mathbb{C}^n,0}$. What is the relation between the localization of $\mathcal{O}(U)$ at the prime ideal of all functions vanishing at the origin and $\mathcal{O}_{\mathbb{C}^n,0}$? Is this prime ideal maximal?*

The map $\mathcal{O}(U) \to \mathcal{O}_{\mathbb{C}^n,0}, f \mapsto (f, U)$ is an inclusion.
Moreover, $\mathcal{O}(U)_\mathfrak{p} \to \mathcal{O}_{\mathbb{C}^n,0}$, $\frac{f}{g} \mapsto \left(\frac{f}{g}, U\backslash Z(g)\right)$ is injective.
But it may not be surjective, when $n = 1$, $\exp\left(\frac{1}{z-\frac{\varepsilon}{2}}\right) \in \mathcal{O}_{\mathbb{C}^n,0}$ has an essential singular point at $z = \frac{\varepsilon}{2}$, while $\frac{f}{g} \in \mathcal{O}(U)_\mathfrak{p}$ can only have poles.
And since $\mathcal{O}(U)/\mathfrak{p} \cong \mathbb{C}$.
So $\mathfrak{p}$ is maximal in $\mathcal{O}(U)$.

Exer 1.1.11. *The notion of irreduciblility for analytic germs generalizes in a straightforward way to the corresponding notion for analytic sets $X \subset \mathbb{C}^n$. Given an example of an irreducible analytic set that does not define irreducible analytic germs at every point and of an analytic set whose induced germs are all irreducible, but the set is not.*

Consider $f(z_1, z_2) = z_1^2 - z_2^2(z_2 + 1), X = Z(f)$.
Then X is reducible near 0 since $\sqrt{z_2 + 1}$ is well-defined around 0.
And X is irreducible analytic set(hard to prove here).
On the other hand, $\mathbb{C} \times \{\pm 1\} \subset \mathbb{C}^2$ induces irreducible germs at every point, but is not irreducible set.

Exer 1.1.12. *Let $U \subset \mathbb{C}^n$ be an open subset and let $f : U \to \mathbb{C}$ be holomorphic. Show that for $n \geqslant 2$ the zero set $Z(f)$ cannot consist of a single point. Analogously, show that for a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}, n \geqslant 2$ and $w \in \mathrm{Im}(f)$ there exists $z \in f^{-1}(w)$ such that $\|z\| >> 0$.*

WLOG, assume $Z(f) = \{0\}$.
Then by WPT, $f = g_w \cdot h$ with $g_w(z_1) = z_1^d + a_1(w)z_1^{d-1} + \cdots + a_d(w)$.
For any w and sufficiently small ε, $|a_i(\varepsilon w)|$ can be arbitrary small.

So $g_{\varepsilon w}(z_1) = 0$ for some z_1 near 0, *i.e.* $(z_1, \varepsilon w) \in Z(f)$, contradiction!
Suppose $Z(f)$ is bounded.
Then there exists a polydisc $B_r(0)$ containing $Z(f)$ and $\partial B_r(0) \cap Z(f) \neq \varnothing$.
For $(z_1, w) \in \partial B_r(0) \cap Z(f)$, there exists some $(z_1', w + \varepsilon w) \in Z(f)$, contradiction!

Exer 1.1.13. *Show that the product of two analytic germs is in a natural way an analytic germ*

Consider germs $Z(f_1, \ldots, f_k), Z(g_1, \ldots, g_l)$ where $f_i \in \mathcal{O}_{\mathbb{C}^m,0}, g_i \in \mathcal{O}_{\mathbb{C}^n,0}$.
Let $p_i(z_1, \ldots, z_{m+n}) = f_i(z_1, \ldots, z_m), q_j(z_1, \ldots, z_{m+n}) = g_j(z_{m+1}, \ldots, z_{m+n})$.
Then $Z(p_1, \ldots, p_k, q_1, \ldots, q_l) = Z(f_1, \ldots, f_k) \times Z(g_1, \ldots, g_l)$.

Exer 1.1.14. *Let $X \subset \mathbb{C}^n$ be an irreducible analytic set of dimension d. A point $x \in X$ is called singular if X cannot be defined by $n - d$ holomorphic functions locally around x for which x is regular. Then the set of singular points $X_{\text{sing}} \subset X$ is empty or an analytic subset of dimension $< d$. Although the basic idea behind this result is very simple, its complete proof is rather technical. Try to prove the fact in easy cases, e.g. when X is defined by a single holomorphic function.*

If x is a regular point, i.e. $x \in X_{\text{reg}} := X \backslash X_{\text{sing}}$*, the $n - d$ holomorphic functions defining X near x can be completed to a local coordinate system.*

Let $X = Z(f)$.
Then for $z \in X_{\text{sing}}$, $\frac{\partial f}{\partial z_i} = 0$ for any $i = 1, \ldots, n$.
So $X_{\text{sing}} = Z\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ is analytic subset with dimension $< d$.

Exer 1.1.15. *Consider the holomorphic map $f : \mathbb{C} \to \mathbb{C}^2, z \mapsto (z^2 - 1, z^3 - z)$. Is the image an analytic set?*

Let $g(z_1, z_2) = z_2^2 - z_1^2(z_1 + 1)$.
Then $g \circ f(z) = (z^3 - z)^2 - (z^2 - 1)^2 z^2 = 0$ and for $z \in Z(g)$, $f\left(\frac{z_2}{z_1}\right) = z$.
So $\operatorname{Im} f = Z(g)$ is an analytic set.

Exer 1.1.16. *The aim of this exercise is to establish the theorem of Poincaré stating that the polydisc $B_{(1,1)}(0) \subset \mathbb{C}^2$ and the unit disc $D := \{z \in \mathbb{C}^2 | \|z\| < 1\}$ are not biholomorphic. (Thus the Riemann mapping theorem does not generalize to higher dimensions.)*

(a) Recall the description of the group of automorphisms of the unit disc in the complex plane. Show that the group of unitary matrices of rank two is a subgroup of the group of biholomorphic maps of D which leave the origin fixed.

(b) Show that for any $z \in B_{(1,1)}(0)$ there exists a biholomorphic map $f : B_{(1,1)}(0) \to B_{(1,1)}(0)$ with $f(z) = 0$.

(c) Show that group of biholomorphic maps of $B_{(1,1)}(0)$ which leave invariant the origin is abelian.

(d) Show that D and $B_{(1,1)}(0)$ are not biholomorphic.

(a) Since $\|Uz\| = z^* U^* U z = z^* z = \|z\|$.

So $D \mapsto D, z \mapsto Uz$ is a biholomorphic map, which leave the origin fixed.

(b) Let $z = (z_1, z_2)$ and $f \colon B_{(1,1)}(0) \to B_{(1,1)}(0), (w_1, w_2) \mapsto \left(\frac{z_1 - w_1}{1 - \bar{z}_1 w_1}, \frac{z_2 - w_2}{1 - \bar{z}_2 w_2}\right)$.

Then $f(z) = 0$ and f is biholomorphic.

(c) This statement is wrong.

Consider $f(z_1, z_2) = (z_2, z_1), g(z_1, z_2) = (-z_1, z_2)$.

Then $f \circ g(z_1, z_2) = (z_2, -z_1), g \circ f(z_1, z_2) = (-z_2, z_1)$ are not the same.

(d) I have no idea how to prove this since (c) is wrong.

Exer 1.1.17. *Let $X \subset \mathbb{C}^n$ be an analytic subset. Show that locally around any point $x \in X$ the regular part X_{sing} has zero volume. This will be needed later when we integrate differential forms over singular subvarieties.*

Remark 1.1.1. The origin statement of the exercise in the book is rediculous, according to page 142 where the exercise is needed, I think this statement is what author want us to show.

By exercise 1.1.14, X_{sing} is an analytic subset of dimension $< \dim X$.
So X_{sing} has zero volume.

Exer 1.1.18. *Let $f : U \to V$ be holomorphic and let $X \subset V$ be an analytic set. Show that $f^{-1}(X) \subset V$ is analytic. What is the relation between the irreducibility of X and $f^{-1}(X)$?*

For $z \in f^{-1}(X)$, let $w = f(z)$ and W be an open neighborhood of w such that $W \cap X = \{w | f_1(w) = \cdots = f_k(w) = 0\}$.
Then $f^{-1}(W)$ is an open neighborhood of z and $f^{-1}(W) \cap f^{-1}(X) = \{z | f_1 \circ f(z) = \cdots = f_k \circ f(z) = 0\}$.
So $f^{-1}(X)$ is analytic.
Consider $X = \mathbb{C} \times \{0, 1\}, f : \mathbb{C} \times \{0\} \hookrightarrow \mathbb{C}^2$.
Then X is reducible but $f^{-1}(X)$ is irreducible.
Consider $X = Z(z_2^2 - z_1^2(z_1 + 1)), f : \mathbb{C} \times \{0\} \hookrightarrow \mathbb{C}^2$.
Then X is irreducible but $f^{-1}(X)$ is reducible.

Exer 1.1.19. *Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be the ideal generated by $z_1^2 - z_2^3 + z_1$ and $z_1^4 - 2z_1z_2^3 + z_1^2$. Describe $\sqrt{I}$*

$(z_1^4 - 2z_1z_2^3 + z_1^2) - 2z_1(z_1^2 - z_2^3 + z_1) = z_1^2(z_1^2 - 2z_1 - 1) \in I$.
And notice that $\frac{1}{z_1^2 - 2z_1 - 1} \in \mathcal{O}_{\mathbb{C}^2,0}$.
So $z_1 \in \sqrt{I}$.
Therefore $z_2^3 \in \sqrt{I}, i.e. z_2 \in \sqrt{I}$.
Hence $\sqrt{I} = (z_1, z_2)$.

Exer 1.1.20. *Let $U \subset \mathbb{C}^n$ be an open subset and $f : U \backslash \mathbb{C}^{n-2} \to \mathbb{C}$ a holomorphic map. Show that there exists a unique holomorphic extension $\tilde{f} : U \to \mathbb{C}$ of f.*

WLOG, assume the $\mathbb{C}^{n-2}$ is $\{(0, 0)\} \times \mathbb{C}^{n-2} \subset \mathbb{C}^n$.
Then using the proof of Hartogs' theorem, we can extend f_w to a holomorphic map on $U \cap \{(z, w) | z \in \mathbb{C}^2\}$.
Hence we extend f to a holomorphic map on U, which is unique.

1.2 Complex and Hermitian Structures

Exer 1.2.1. *Let $(V, \langle\, , \rangle)$ be a four-dimensional euclidian vector space. Show that the set of all compatible almost complex structures consist of two copies of $\mathbb{S}^2$.*

WLOG, assume $\langle\,,\rangle$ is the standard inner product.
Since $I^2 = -id$ and $I^T I = id$.
So $I^T I^2 = -I^T = I$, *i.e.* I is skew-symmetric, take

$$I = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

Then $a^2 + b^2 + c^2 = 1, bd + ce = 0, ad = cf, ae + bf = 0, a^2 + d^2 + e^2 = 1, ab + ef = 0, ac = df, b^2 + d^2 + f^2 = 1, bc + de = 0, c^2 + e^2 + f^2 = 1$.
So $a^2 = f^2, b^2 = e^2, c^2 = d^2, ae = -bf, bd = -ce, ad = cf$.
Let $a = \cos\varphi, b = \sin\varphi\cos\theta, c = \sin\varphi\sin\theta$.
Then $d = \pm\sin\varphi\sin\theta, e = \mp\sin\varphi\cos\theta, a = \pm\cos\varphi$.
Hence I must be one of the following two forms:

$$\begin{bmatrix} 0 & \cos\varphi & \sin\varphi\cos\theta & \sin\varphi\sin\theta \\ -\cos\varphi & 0 & \sin\varphi\sin\theta & -\sin\varphi\cos\theta \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & 0 & \cos\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & -\cos\varphi & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & \cos\varphi & \sin\varphi\cos\theta & \sin\varphi\sin\theta \\ -\cos\varphi & 0 & -\sin\varphi\sin\theta & \sin\varphi\cos\theta \\ -\sin\varphi\cos\theta & \sin\varphi\sin\theta & 0 & -\cos\varphi \\ -\sin\varphi\sin\theta & -\sin\varphi\cos\theta & \cos\varphi & 0 \end{bmatrix}$$

Each form of I forms a $\mathbb{S}^2$.

Exer 1.2.2. *Show that the two decompositions*

$$\bigwedge^k V^* = \bigoplus_{i\geqslant 0} L^i P^{k-2i}, L^i P^{k-2i} = \bigoplus_{p+q=k-2i} L^i P^{p,q}$$

are orthogonal with respect to the Hodge-Riemann pairing.

Consider $\alpha \in P^{k-2i}, \beta \in P^{k-2j}$ with $i < j$.
Then $Q(L^i\alpha, L^j\beta) = \omega^i \wedge \alpha \wedge \omega^j \wedge \beta \wedge \omega^{n-k} = \omega^{n-k+2i+1} \wedge \alpha \wedge \omega^{j-i-1} \wedge \beta = 0$
Consider $\alpha \in P^{p,q}, \beta \in P^{p',q'}$.
Then $Q(L^i\alpha, L^i\beta) = \omega^i \wedge \alpha \wedge \omega^i \wedge \beta \wedge \omega^{n-k} = Q(\alpha, \beta) = 0$.

Exer 1.2.3. *Prove the following identities:* $*\Pi^{p,q} = \Pi^{n-q,n-p}*$ *and* $[L, \mathbf{I}] = [\Lambda, \mathbf{I}] = 0$.

By definition, for $\alpha \in \bigwedge^{p,q} V^*$, $*\alpha \in \bigwedge^{n-q,n-p} V^*$.
So $*\Pi^{p,q} = \Pi^{n-q,n-p}*$.
And $[L, \mathbf{I}](\alpha) = i^{(p+1)-(q+1)}L(\alpha) - L\left(i^{p-q}\alpha\right) = 0, [\Lambda, \mathbf{I}] = i^{(p-1)-(q-1)}\Lambda(\alpha) - \Lambda\left(i^{p-q}\alpha\right) = 0$.

Exer 1.2.4. *Is the product of two primitive forms again primitive?*

No, take $\alpha \in P^i, \beta \in P^j$ with $i, j < n, i + j > n$.
Then $\alpha \wedge \beta \in \bigwedge^{i+j} V^*$, while $P^{i+j} = 0$.

Exer 1.2.5. *Let* $(V, \langle\,,\rangle)$ *be an euclidian vector space and let* $I, J,$ *and* K *be compatible almost complex structures where* $K = I \circ J = -J \circ I$. *Show that* V *becomes in a natural way a vector space over the quaternions. The associated fundamental forms are denoted by* $\omega_I, \omega_J,$ *and* ω_K. *Show that* $\omega_J + i\omega_K$ *with respect to* I *is a form of type* $(2, 0)$. *How many natural almost complex structures do you see in this context?*

$(a+bi+cj+dk)\cdot v = a\cdot v + b\cdot I(v) + c\cdot J(v) + d\cdot K(v)$.
Since $I^2 = J^2 = K^2 = -\mathrm{Id}$ and $K = IJ = -JI$.
So the quaternions structure is well-defined.
And $(\omega_J + i\omega_K)(v,w) = \langle J(v),w\rangle + i\langle J\circ I(v),w\rangle = \langle J(v),w\rangle - i\langle I\circ J(v),w\rangle$.
Therefore $(\omega_J + i\omega_K)(I(v),w) = \langle I\circ J(v),w\rangle + i\langle J(v),w\rangle = i(\omega_J+i\omega_K)(v,w)$,
$(\omega_J + i\omega_K)(v,I(w)) = -\langle J\circ I(v),w\rangle + i\langle J(v),w\rangle = i(\omega_J+i\omega_K)(v,w)$.
Hence $\omega_J + i\omega_K$ is of type $(2,0)$.
Moreover, every $aI+bJ+cK$ with $a^2+b^2+c^2=1$ is an almost complex structure.

Exer 1.2.6. *Let $\omega \in \bigwedge^2 V^*$ be non-degenerate,* i.e. *the induced homomorphism $\tilde{\omega} : V \to V^*$ is bijective. Study the relation between the two isomorphisms $L^{n-k} : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ and $\bigwedge^k V^* \cong \bigwedge^{2n-k} V \cong \bigwedge^{2n-k} V^*$, where the latter is given by $\bigwedge^{2n-k}\tilde{\omega}$. Here, $2n = \dim_{\mathbb{R}}(V)$.*

Let $\{x_i, y_i = I(x_i)\}$ be a basis of V such that $\omega = \sum\limits_{i=1}^{n} x^i \wedge y^i$.

Then $\bigwedge^k V^* \cong \bigwedge^{2n-k} V \cong \bigwedge^{2n-k} V^*$ is given by

$$x^I \wedge y^J \mapsto (-1)^{\mathrm{sgn}(I,J,I^c,J^c)} x_{I^c} \wedge y_{J^c} \mapsto (-1)^{\mathrm{sgn}(I,J,I^c,J^c)+|J^c|} y^{I^c} \wedge x^{J^c}$$

Remark 1.2.1. Since this isomorphism is definitely different from L^{n-k}, so I don't know what author what us to study.

Exer 1.2.7. *Let V be a vector space endowed with a scalar product and a compatible almost complex structure. What is the signature of the pairing $(\alpha,\beta) \mapsto \frac{\alpha\wedge\beta\wedge\omega^{n-2}}{\mathrm{vol}}$ on $\bigwedge^2 V^*$?*

Let $\{x_i, y_i = I(x_i)\}$ be a basis of V such that $\omega = \sum\limits_{i=1}^{n} x^i \wedge y^i$.

Consider the basis $\{x^1\wedge y^1, \dots, x^n \wedge y^n\} \cup \{x^i\wedge x^j, y^i \wedge y^j\} \cup \{x^i \wedge y^j\}$ of $\bigwedge^2 V^*$.

Then the pairing is a blocked diagonal matrix with n^2-n+1 parts, corresponding to n^2-n+1 parts of basis resp.

First part is given by $\{1-\delta_{ij}\}$, *i.e.* signature is $(1, n-1)$.
For other n^2-n parts, each of them has signature $(1,1)$.
Hence signature of the pairing is (n^2-n+1, n^2-1).

Exer 1.2.8. *Let $\alpha \in P^k$ and $s \leqslant r$. Prove the following formula $\Lambda^s L^r \alpha = r(r-1)\cdots(r-s+1)(n-k-r+1)\cdots(n-k-r+s)L^{r-s}\alpha$.*

$$*L^r\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{r!}{(n-k-r)!} L^{n-k-r}\mathbf{I}(\alpha)$$

$$*L^{r-s}\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{(r-s)!}{(n-k-r+s)!} L^{n-k-r+s}\mathbf{I}(\alpha)$$

So we have

$$\Lambda^s L^r \alpha = *^{-1} L^s * L^r \alpha = \frac{(n-k-r+s)!}{(r-s)!} \cdot \frac{r!}{(n-k-r)!} L^{r-s}\alpha$$

Exer 1.2.9 (Wirtinger inequality). *Let $(V, \langle\,,\rangle)$ be an euclidian vector space endowed with a compatible almost complex structure I and the associated fundamental form ω. Let $W \subset V$ be an oriented subspace of dimension $2m$. The induced scalar product on W together with the chosen orientation define a natural volume form* $\mathrm{vol}_W \in \bigwedge^{2m} W^*$. *Shot that*

$$\omega^m\big|_W \leqslant m! \cdot \mathrm{vol}_W$$

and that equality holds if and only if $W \subset V$ is a complex subspace, i.e. *$I(W) = W$ and the orientation is the one induced by the almost complex structure. (The inequality is* *meant with respect to the isomorphism $\bigwedge^{2m} W^* \cong \mathbb{R}, \mathrm{vol}_W \mapsto 1$. Hint: Use that there exists an oriented orthonormal base $e_1, \ldots, e_{2m}$ such that $\omega\big|_W = \sum_{i=1}^{m} \lambda_i e^{2i} \wedge e^{2i-1}$ and the Cauchy-Schwarz inequality.)*

Since $\omega\big|_W$ is anti-symmetric.
So there exists orthonormal base $e_1, \cdots, e_{2m}$ such that $\omega\big|_W = \sum_{i=1}^{m} \lambda_i e^{2i} \wedge e^{2i-1}$.
Notice that $|\lambda_i| = \left|\omega(e^{2i}, e^{2i-1})\right| \leqslant \|e_{2i}\| \cdot \|e_{2i-1}\| = 1$.
Hence $\frac{\omega^m|_W}{\mathrm{vol}_W} \leqslant \sum_{i_1 \leqslant \cdots \leqslant i_m} \prod_{j=1}^{m} \left|\lambda_{i_j}\right| \leqslant m!$

Exer 1.2.10. *Choose an orthonormal basis $x_1, y_1 = I(x_1), \ldots, x_n, y_n = I(x_n)$ of an euclidian vector space V endowed with a compatible almost complex structure I. Show that the dual Lefschetz operator applied to a two-form α is explicitly given by $\Lambda\alpha = \sum \alpha(x_i, y_i)$.*

Since $x_1, y_1, \ldots, x_n, y_n$ is orthonormal basis.
So $\omega = \sum_{i=1}^{n} x^i \wedge y^i$.
For $\alpha = x^i \wedge y^j$, $\Lambda\alpha = (-1)^{n+i+j} *^{-1} L\left(x^{\{i\}^c} \wedge y^{\{j\}^c}\right) = \delta_{ij} *^{-1} (\mathrm{vol}) = \delta_{ij}$.
Similarly, for $\alpha = x^i \wedge x^j, y^i \wedge y^j$, $\Lambda\alpha = 0$.
Hence $\Lambda\alpha = \sum \alpha(x_i, y_i)$.

1.3 Differential Forms

Exer 1.3.1. *Let $f : U \to V$ be a holomorphic map. Show that the natural pull-back $f^* : \mathcal{A}^k(V) \to \mathcal{A}^k(U)$ induces maps $\mathcal{A}^{p,q}(V) \to \mathcal{A}^{p,q}(U)$.*

Since f is holomorphic.
So $f^*\mathcal{A}^{1,0}(V) \subset \mathcal{A}^{1,0}(U), f^*\mathcal{A}^{0,1}(V) \subset \mathcal{A}^{0,1}(V)$.
Hence $f^*\left(\mathcal{A}^{p,q}(V)\right) \subset \mathcal{A}^{p,q}(U)$.

Exer 1.3.2. *Show that $\overline{\partial\alpha} = \bar{\partial}\bar{\alpha}$. In particular, this implies that a real (p,p)-form $\alpha \in \mathcal{A}^{p,p}(U) \cap \mathcal{A}^{2p}(U)$ is ∂-closed(exact) if and only if α is $\bar{\partial}$-closed(exact). Formulate the ∂-version of the three Poincaré lemmas.*

For $\alpha \in \mathcal{A}^{p,q}(U)$, $\overline{\partial\alpha} = \overline{\Pi^{p+1,q}(\mathrm{d}\alpha)} = \Pi^{q,p+1}\left(\overline{\mathrm{d}\alpha}\right) = \bar{\partial}\bar{\alpha}$.
Now let α be an real (p,p)-form.
α is ∂-closed$\Leftrightarrow \partial\alpha = 0 \Leftrightarrow \bar{\partial}\alpha = \bar{\partial}\bar{\alpha} = 0 \Leftrightarrow \alpha$ is $\bar{\partial}$-closed.
α is ∂-exact$\Leftrightarrow \alpha = \partial\beta \Leftrightarrow \alpha = \bar{\alpha} = \bar{\partial}\bar{\beta} \Leftrightarrow \alpha$ is $\bar{\partial}$-exact.
∂-Poincaré lemma in one variable:
Consider an open neighborhood of the closure of a bounded one-dimensional disc $B_\varepsilon \subset \bar{B}_\varepsilon \subset U \subset \mathbb{C}$. For $\alpha = f\mathrm{d}z \in \mathcal{A}^{1,0}(U)$ the function

$$g(z) = \frac{1}{2\pi i} \frac{\bar{f}(w)}{w - z} \mathrm{d}w \wedge \mathrm{d}\bar{w}$$

on B_ε satisfies $\alpha = \partial\bar{g}$.
∂-Poincaré lemma in several variables:
Let U be an open neighborhood of the closure of a bounded polydisc $B_\varepsilon \subset \bar{B}_\varepsilon \subset U \subset \mathbb{C}^n$. If $\alpha \in \mathcal{A}^{p,q}(U)$ is ∂-closed and $q > 0$, then there exists a form $\beta \in \mathcal{A}^{p-1,q}(B_\varepsilon)$ with $\alpha = \partial\beta$ on B_ε.
∂-Poincaré lemma on the open disc:
If $\alpha \in \mathcal{A}^{p,q}(B)$ is ∂-closed and $q > 0$, then there exists $\beta \in \mathcal{A}^{p-1,q}(B)$ with $\alpha = \partial\beta$.

Exer 1.3.3. *Let $B \subset \mathbb{C}^n$ be a polydisc and let $\alpha \in \mathcal{A}^{p,q}(B)$ be a* d*-closed form with $p, q \geqslant 1$. Show that there exists a form $\gamma \in \mathcal{A}^{p-1,q-1}(B)$ such that $\partial\bar{\partial}\gamma = \alpha$. (This is a local version of the $\partial\bar{\partial}$-lemma for compact Kähler manifolds Corollary 3.2.10)*

By Poincaré lemma, there exists $\beta \in \mathcal{A}^{p+q-1}(B)$ such that $\mathrm{d}\beta = \alpha$.
Let $\beta = \beta_1 + \beta_2$ with $\beta_1 \in \mathcal{A}^{p-1,q}(B), \beta_2 \in \mathcal{A}^{p,q-1}(B)$.
Then $\bar{\partial}\beta_1 = 0, \partial\beta_2 = 0$.
So By $\partial, \bar{\partial}$-Poincaré lemma, $\beta_1 = \bar{\partial}\gamma_1, \beta_2 = \partial\gamma_2$.
Hence $\alpha = \partial\beta_1 + \bar{\partial}\beta_2 = \partial\bar{\partial}(\gamma_1 - \gamma_2)$.

Exer 1.3.4. *Show that for a polydisc $B \subset \mathbb{C}^n$ the sequence*

$$\mathcal{A}^{p-1,q-1}(B) \xrightarrow{\partial\bar{\partial}} \mathcal{A}^{p,q}(B) \xrightarrow{\mathrm{d}} \mathcal{A}_{\mathbb{C}}^{p+q+1}(B)$$

is exact. (For more general open subsets or complex manifolds the sequence is no longer exact. This gives rise to the so called Bott-Chern cohomology which shall be introduced in Exercise 2.6.7.)

For $\alpha \in \ker \mathrm{d}$, $\mathrm{d}\alpha = 0$.
So by exercise 1.3.3, $\partial\bar{\partial}\gamma = \alpha$ for some $\gamma \in \mathcal{A}^{p-1,q-1}(B)$.
Hence $\ker \mathrm{d} \subset \mathrm{im}(\partial\bar{\partial})$, *i.e.* the sequence exact.

Exer 1.3.5. *Show that $\omega = \frac{i}{2\pi}\partial\bar{\partial}\log\left(|z|^2 + 1\right) \in \mathcal{A}^{1,1}(\mathbb{C})$ is the fundamental form of a compatible metric g that osculates to order two in any point. (This is the local shape of the Fubini-Study Kähler form on $\mathbb{P}^1$, cf. Section 3.1.)*

$\omega = \frac{i}{2\pi}\partial\left(\frac{z\mathrm{d}\bar{z}}{1+|z|^2}\right) = \frac{i}{2\pi}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{\left(1+|z|^2\right)^2}$.
So ω is the fundamental form of metric $g = \frac{1}{\pi\left(1+|z|^2\right)^2}\mathrm{d}z \otimes \mathrm{d}\bar{z}$.
Moreover, $\mathrm{d}\omega = 0$, *i.e.* g osculates to order two in any point.

Exer 1.3.6. *Analogously to Exercise 1.3.5, study the form $\omega = \frac{1}{2\pi i}\partial\bar{\partial}\log\left(1 - |z|^2\right)$ on $B_1 \subset \mathbb{C}$. (This is the local example of a negatively curved Kähler structure. See Section 3.1.)*

Remark 1.3.1. The original statement is wrong (including the statement in section 3.1), it seems that author forget a sign while differentiating.

$\omega = \frac{1}{2\pi i}\partial\left(\frac{-z\mathrm{d}\bar{z}}{1-|z|^2}\right) = \frac{i}{2\pi}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{\left(1-|z|^2\right)^2}$.
So ω is the fundamental form of metric $g = \frac{1}{\pi\left(1-|z|^2\right)^2}\mathrm{d}z \otimes \mathrm{d}\bar{z}$, which is also osculates to order two in any point.

Exer 1.3.7. *Let $\omega = \frac{i}{2\pi}\sum \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$ be the standard fundamental form on $\mathbb{C}^n$. Show that one can write $\omega = \frac{i}{2\pi}\partial\bar{\partial}\varphi$ for some positive function φ and determine φ. The function φ is called the Kähler potential.*

Let $\varphi = \sum |z_i|^2$.
Then $\omega = \frac{i}{2\pi}\partial\bar{\partial}\varphi$.

Exer 1.3.8. *Let $\omega \in \mathcal{A}^{1,1}(B)$ be the fundamental form associated to a compatible metric on a polydisc $B \subset \mathbb{C}^n$ which osculates in every point $z \in B$ to order two. Show that $\omega = \frac{i}{2\pi}\partial\bar{\partial}\varphi$ for some real function $\varphi \in \mathcal{A}^0(B)$.*

Since $\mathrm{d}\omega = 0$.
So by exercise 1.3.3, there exists $\gamma \in \mathcal{A}^{0,0}(\mathbb{C}^n)$ such that $\frac{i}{2\pi}\partial\bar{\partial}\gamma = \omega$.
And notice that ω is real.
Therefore $\frac{i}{2\pi}\partial\bar{\partial}\gamma = \frac{i}{2\pi}\partial\bar{\partial}\bar{\gamma}$.
Let $\varphi = \frac{1}{2}\left(\gamma + \bar{\gamma}\right)$.
Then φ is real and $\omega = \frac{i}{2\pi}\partial\bar{\partial}\varphi$.

Exer 1.3.9. *Let g be a compatible metric on $U \subset \mathbb{C}^n$ that osculates to order two in any point $z \in U$. For which real function f has the conformally equivalent metric $\mathrm{e}^f \cdot g$ the same property?*

Let ω_0 be the fundamental form of metric$e^f \cdot g$.
Then $\omega_0 = e^f \cdot g(I(), ()) = e^f \cdot \omega$.
So $\mathrm{d}\omega_0 = \mathrm{d}e^f \cdot \omega = \mathrm{d}f \cdot \omega_0$.
Hence f must be constant.