Problems of Topology and Geometry

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Contents

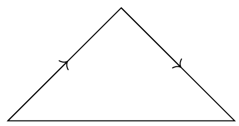
1 Fundamental groups and Van Kampen theorem

Exer 1.1. *Find the fundamental group of the space $X = \mathbb{S}^1 \times \mathbb{S}^1/\{p,q\}$ obtained from the torus by identifying two distinct points p and q to one point.*

X is homotopy equivalent to $\mathbb{S}^1 \times \mathbb{S}^1$ together with a line from p to q.
So $X \simeq \mathbb{S}^1 \times \mathbb{S}^1 \vee \mathbb{S}^1$, *i.e.* $\pi_1(X) = \mathbb{Z}^2 * \mathbb{Z}$ by Van Kampen theorem.

Exer 1.2. *Möbius band is the quotient space $([-1,1] \times [-1,1])/((1,y) \sim (-1,-y))$. Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Determine the fundamental group of X.*

Since Möbius band can given by



So $\pi_1(X) = \langle a, b | a^2 b^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Exer 1.3. *(1) Let A be a single circle in $\mathbb{R}^3$. Compute the fundamental group $\pi_1(\mathbb{R}^3 - A)$.*

(2) Let A and B be disjoint circles in $\mathbb{R}^3$, supported in the upper and lower half-space, respectively. Compute $\pi_1(\mathbb{R}^3 - (A \cup B))$.

(1) $\mathbb{R}^3 - A$ is homotopy equivalent to $\mathbb{S}^2$ together with a line from south pole to north pole.
So $\pi_1(\mathbb{R}^3 - A) = \pi_1(\mathbb{S}^2 \vee \mathbb{S}^1) = \mathbb{Z}$.

(2) Consider the upper half-space and the lower half-space.
By Van Kampen theorem, $\pi_1(\mathbb{R}^3 - (A \cup B)) = \mathbb{Z} * \mathbb{Z}$.

Exer 1.4. *Show that for every group G, there exists topological space X_G with $\pi_1(X_G) \cong G$.*

Every group has a presentation by generators and relations.
Then the generators form the 1-cells of X_G (wedging together at a base point x_0) and the relations form the 2-cells of X_G.

Exer 1.5. *Prove that π_1 of a topological group is abelian*

Consider two loops $\alpha, \beta : [0,1] \to X$ with base point 1.
Then let $H : [0,1]^2 \to X, (s,t) \mapsto \alpha(t)\beta(s)$.
So $H(0,-) = H(1,-) = \alpha, H(-,0) = H(-,1) = \beta$.
Since two path from $(0,0)$ to $(1,1)$ on boundary of square $[0,1]^2$ are homotopy in square.
Hence $[\alpha] * [\beta] = [\beta] * [\alpha]$, *i.e.* $\pi_1(X)$ is abelian.

Exer 1.6. *A topological space X is called an H-space if there exists $e \in X$ and $\mu : X \times X \to X$ such that $\mu(e,e) = e$ and the maps $x \mapsto \mu(e,x)$ and $x \mapsto \mu(x,e)$ are both homotopic to the identity map. Show that the fundamental group of an H-space is Abelian.*

Consider two loops $\alpha, \beta : [0,1] \to X$ with base point e.
Then let $H : [0,1]^2 \to X, (s,t) \mapsto \mu(\alpha(t), \beta(s))$.
So $H(0,-) = H(1,-) = \mu(\alpha, e)$ is homotopic to α, $H(-,0) = H(-,1)$ is homotopic to β.
Since two path from $(0,0)$ to $(1,1)$ on boundary of square $[0,1]^2$ are homotopy in square.
Hence $[\alpha] * [\beta] = [\beta] * [\alpha]$, *i.e.* $\pi_1(X)$ is abelian.

Exer 1.7. *The mapping torus T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = \mathbb{S}^1 \vee \mathbb{S}^1$ with f basepoint-perserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$.*

Let a, b be the 1-cells of X and c be the line from $(e, 0)$ to $(e, 1)$.
Then a, b, c are the 1-cells of T_f and the 2-cells are given by $a^{-1}cf_*(a)c^{-1}, b^{-1}cf_*(b)c^{-1}$.
So $\pi_1(T_f) = \langle a, b, c | a^{-1}cf_*(a)c^{-1} = b^{-1}cf_*(b)c^{-1} = 1\rangle$

Exer 1.8. *What is the fundamental group of* SO(3)*?*

Every elements in SO(3) is given by a unit vector $v \in \mathbb{R}^3$ and an angle $\theta \in [0, \pi]$ with $(v, \pi) \sim (-v, \pi)$.
So $\mathrm{SO}(3) \cong \mathbb{D}^3/\sim \cong \mathbb{RP}^3$.
Hence $\pi_1(\mathrm{SO}(3)) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$.

Exer 1.9. *Show that $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.*

Consider the loop γ_a given by the boundary of square $[0, a]^2$.
Then for any two different $a, b \in \mathbb{R}$, γ_a and γ_b are not homotopic.
This is because there exists $c \in (a, b)$ and γ_a and γ_b are not homotopic in $\mathbb{R}\backslash\{c\}$.
Hence there are uncountable elements in $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$.

Exer 1.10. *For relatively prime positive integers m and n, the torus knot $K = K_{m,n} \subset \mathbb{R}^3$ is the image of the embedding $f : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^3$, given by*

$$f(z) = (z^m, z^n),$$

where the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is embedded in $\mathbb{R}^3$ in the standard way. Compute the fundamental group $\pi_1(\mathbb{R}^3\backslash K)$.

$\mathbb{R}^3\backslash K \simeq \mathbb{S}^3\backslash K$.
Notice that $\mathbb{S}^1 \times \mathbb{S}^1$ divides $\mathbb{S}^3$ into two solid torus A, B.
Let $\pi_1(A) = \langle a\rangle, \pi_1(B) = \langle b\rangle$ and $\pi_1(T\backslash K) = \pi_1(K) = \langle c\rangle$.
Then a, b corresponds to two generators of $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ and $c = ma + nb \in \pi_1(T)$.
Consider $U = A \cup (\mathbb{S}^1 \times \mathbb{S}^1\backslash K), V = B \cup (\mathbb{S}^1 \times \mathbb{S}^1\backslash K)$.
By Van Kampen theorem, $\pi_1(\mathbb{R}^3\backslash K) = \langle a, b | a^m = b^n\rangle$.

2 Covering spaces

Exer 2.1. *Suppose K is a finite connected simplicial complex. True or false:*

(1) If $\pi_1(K)$ is finite, then the universal cover of K is compact.

(2) If the universal cover of K is compact, then $\pi_1(K)$ is finite.

(1) Since $\pi_1(K)$ is finite.
So $\tilde{K}$ is also a finite simplicial complex, *i.e.* K is compact.

(2) Consider a point $p \in K$ and the covering map $\pi : \tilde{K} \to K$.
Then the number of $\pi^{-1}(p)$ is $\pi_1(K)$.
And since $\tilde{K}$ is compact.
So $\pi^{-1}(p)$ has no limit point, *i.e.* $\pi_1(K)$ is finite.

Exer 2.2. *Let X be path-connected and locally path-connected, and let Y be a finite Cartesian product of circles. Show that if $\pi_1(X)$ is finite, then every continuous map from X to Y is null-homotopic.*

Consider $f_* : \pi_1(X) \to \pi_1(Y)$.
Since $\pi_1(Y) = \mathbb{Z}^n$ and $\pi_1(X)$ is finite.
So every element of $\pi_1(X)$ has finite order, *i.e.* $f_*([\gamma]) \equiv 0$.
Therefore f can lift to a map $\tilde{f} : X \to \mathbb{R}^n$, which is null-homotopic since $\mathbb{R}^n$ is contractible.
Hence f is null-homotopic.

Exer 2.3. *Let X be a path connected, locally path connected, semilocally path connected space. Recall that a path connected covering space $\tilde{X} \to X$ is abelian if $\pi_1(\tilde{X})$ is normal in $\pi_1(X)$ and the quotient is abelian. Show that there is a universal abelian cover: this is an abelian cover $\tilde{X} \to X$ such that for any other abelian cover $\tilde{Y} \to X$, there is a covering map $\tilde{X} \to \tilde{Y}$ factoring the map $\tilde{X} \to X$.*

Consider $N = [\pi_1(X), \pi_1(X)]$.
Then N is the minimal normal subgroup of $\pi_1(X)$ such that $\pi_1(X)/N$ is abelian.
So the corresponding covering space X_N is the universal abelian cover.

Exer 2.4. *Let $X = \mathbb{S}^2 \vee \mathbb{RP}^2$ be the wedge of the 2-sphere and the real projective plane. Describe the universal covering space of X.*

Since the universal covering space of $\mathbb{RP}^2$ is $\mathbb{S}^2$ and each point of $\mathbb{RP}^2$ has two preimages that are antipodal.
So the universal covering space of X is $\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2$.

Exer 2.5. *Show that every subgroup of a free group is free.*

The covering spaces of $\bigvee^n \mathbb{S}^1$ must only have 1-cells.
So their fundamental groups is free.

Exer 2.6. *Let Σ_2 be the closed orientable surface of genus 2.*

(1) What is $G = \pi_1(\Sigma_2)$?

(2) Why is G non-abelian?

(3) Why does G contain a subgroup of index 7?

(4) Show that Σ_2 is not a non-trivial cover of any orientable surface.

(5) Show that Σ_2 is a non-trivial cover of a space.

(1) $G = \langle a, b, c, d | aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$

(2) $ab \neq ba$ otherwise $aba^{-1}b^{-1} = 1$.

(3) Consider the normal subgroup H generated by a^7, b, c, d.
Then $G/H = \langle a | a^7 = 1 \rangle$, *i.e.* H is of index 7.

(4) Since $\chi(\Sigma_2) = 2 - 4 = -2$.
So if Σ_2 is a non-trivial cover of some surface S, then $\chi(S) = -1$.
Hence S is not orientable.

(5) Take $S = \mathbb{RP}^2 \# \mathbb{T}$.
Then the 2-sheeted covering space of S is $\mathbb{T} \# \mathbb{S}^2 \# \mathbb{T} \cong \Sigma_2$.

Remark 2.1. For (3), we can also construct a 7-sheeted covering space directly. It is of genus 8, given similarly as example 1.41 in Hatcher.

Exer 2.7. *$X = \mathbb{S}^1 \vee \mathbb{S}^1$ is the wedge sum of two circles, let a, b denote the two standard generators of $\pi_1(X) \cong F_2$.*

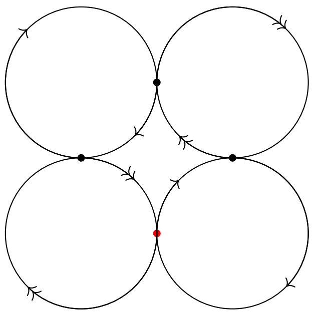
(1) Draw a picture of the directed graph of the covering space of X that corresponds to the following subgroup

$$H = \langle a^2, b^2, ab^2a, ba^2b, (ab)^2 \rangle.$$

On each edge of the directed graph, you should draw an arrow for its direction and label "a" or "b" for its image under the covering map.

(2) Is the covering above a normal covering? Prove your claim.

(1) the red point is base point and edges with one arrow are a, with two arrows are b.



(2) Notice that the graph is symmetric to four points.

So for any $g \in F_2$, no matter which point the base point was sent to by g, h corresponds to a loop at that point for $h \in H$.

Then by g^{-1} the point is sent back to base point, *i.e.* ghg^{-1} corresponds to a loop.

Hence $ghg^{-1} \in H$, *i.e.* H is normal.

Exer 2.8. *Let X be a topological space. We say that two covering spaces $f : Y \to X$ and $g : Z \to X$ are isomorphic if there exists a homeomorphism $h : Y \to Z$ such that $g \circ h = f$. If X is a compact oriented surface of genus g (that is, a g-holed torus), how many connected 2-sheeted covering spaces does X have, up to isomorphism?*

Let $\tilde{X}$ be the 2-sheeted covering spaces of X.
Then $\pi_1(\tilde{X})$ is a subgroup of $\pi_1(X)$ of index 2, *i.e.* it is normal.
So $\pi_1(\tilde{X})$ can be given by the kernel of $\varphi : \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$.
Since φ must passing through the abelization $\mathbb{Z}^{2g}$ of $\pi_1(X)$.
So there are $2^{2g} - 1$ different non-trivial φ by appointing each generator of $\mathbb{Z}^{2g}$ to 0 or 1.
Hence there are $2^{2g} - 1$ different $\tilde{X}$ up to isomorphism.

Exer 2.9. *Find all the connected 2-sheeted and 3-sheeted covering spaces of $\mathbb{S}^1 \vee \mathbb{S}^1$ up to isomorphism of covering spaces without basepoints. Indicate which covering spaces are normal.*

Similar to the above exercise, every 2-sheeted covering space is normal and the number of 2-sheeted covering spaces are 2.

For 3-sheeted covering spaces, there are 3 that are normal.

And for not normal case, there are 2:

$$\langle a^2, b^2, aba, bab \rangle, \langle a, b^3, bab, ba^{-1}b \rangle.$$

Exer 2.10. *Let M be a connected non-orientable manifold whose fundamental group G is simple (that is, has no non-trivial normal subgroup). Prove that G must be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Consider the oriented 2-sheet covering space $\tilde{M}$ of M.
Then $\pi_1(\tilde{M})$ is an index 2 subgroup of $\pi_1(M)$, *i.e.* it is normal.
So $\pi_1(M) = \mathbb{Z}/2\mathbb{Z}$.

Exer 2.11. *A covering space is abelian if it is normal and its group of deck transformations is abelian. Determine all connected abelian covering spaces of $\mathbb{S}^1 \vee \mathbb{S}^1$.*

Let $X = \mathbb{S}^1 \vee \mathbb{S}^1$ and $\tilde{X}$ be an abelian covering space of X.
Then $\pi_1(\tilde{X})$ can be seen as kernel of homomorphism $\varphi : \pi_1(X) \to G$ where $G = \pi_1(X)/\pi_1(\tilde{X})$ is abelian.
Since φ must passing through the abelization $\mathbb{Z}^2$ of $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$.
So G must be the subgroup of $\mathbb{Z}^2$, *i.e.* $G = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for some $m, n \in \mathbb{N}$.
By appointing each generator of $\mathbb{Z}^2$ to G, we can easily determine all connected abelian covering spaces of $\mathbb{S}^1 \vee \mathbb{S}^1$.

Exer 2.12. *Consider a map $f : \mathbb{T}^2 \to \mathbb{RP}^2$ from the torus to the projective plane. Suppose the induced map $f_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{RP}^2)$ is trivial. Is f always null-homotopic? Prove your conclusion.*

No. Consider a map $g : \mathbb{T}^2 \to \mathbb{S}^2$ with degree 1 and the quotient map $q : \mathbb{S}^2 \to \mathbb{RP}^2$.
Then $f = q \circ g$ induced a trivial map but is not null-homotopic.

Exer 2.13. *Show that $\mathbb{CP}^{2n}$ does not cover any manifold except itself.*

Suppose $\mathbb{CP}^{2n}$ is the covering space of X.
Then there exists a non-trivial deck transformation $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$.
Consider $f^* : H^*(\mathbb{CP}^{2n}) \to H^*(\mathbb{CP}^{2n})$ and let $f^*(a) = ka$ where a is the generator of $H^2(\mathbb{CP}^{2n})$.
So the Lefschetz number of f is $\tau(f) = 1 + k + k^2 + \cdots + k^{2n} \neq 0$.
By Lefschetz fixed point theorem, f has a fixed point, contradiction!

Exer 2.14. *Let F_n be the free group of rank n.*

(1) Give an example of a finite connected graph such that its fundamental group is F_2.

(2) Does F_2 contain a proper subgroup isomorphic to F_2?

(3) Does F_2 contain a proper finite index subgroup isomorphic to F_2?

(1) Take a graph with one point and two self-loops on it.

Then the graph is homeomorphic to $\mathbb{S}^1 \vee \mathbb{S}^1$, *i.e.* its fundamental group is F_2.

(2) Yes, such as $\langle a^2, b \rangle$.

(3) Suppose there exists a finite index subgroup G of F_2 that is isomorphic to F_2.

Then it corresponds to a n-sheeted covering space X of $\mathbb{S}^1 \vee \mathbb{S}^1$ where $n = [F_2 : G]$.

So X is a graph with n points and $2n$ edges.

Hence starting from a base point, there are at least $n + 1$ independent loops, *i.e.* $G \cong F_m$ for some $m \geqslant n + 1$, contradiction!

Exer 2.15. *Describe all the connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$. Here $\vee$ is the wedge sum.*

$\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) = \langle a, b | a^2 = b^2 = 1 \rangle$.
The elements in it are $(ab)^n, (ab)^n a, (ba)^n$ and $b(ab)^n$.
Let H be a proper subgroup of $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2)$.
Take the shortest element h of H.
If $h = (ab)^n$, then $(ab)^m a, b(ab)^m \notin H$.
Otherwise let $m = nq + r$, we have $(ab)^r a \in H$ and it is shorter than h.
Moreover, H is generated by h, *i.e.* the correspondent covering map is $\bigvee^{2n} \mathbb{S}^2$.
If $h = (ab)^n a$ and $b(ab)^m \in H$ for some m, let m be the smallest one.
So $(ab)^{n+m+1} \in H$.
Suppose $(ab)^k a \in H$ for some k such that $(n + m + 1) \nmid (k - n)$.
Then $h(ab)^k a = a(ba)^n (ab)^k a = (ba)^{k-n} \in H$.
Let $k - n = (n + m + 1)q + r$.
So $(ab)^r \in H$ and $(ab)^{n-r} a \in H$.
When $r \leqslant n$, $(ab)^{n-r} a \in H$ is shorter than h, contradiction!
When $r > n$, $(ab)^{n-r} a = (ba)^{r-n-1} b$ is shorter than $b(ab)^m$, contradiction!
Thus H is generated by h and $b(ab)^m$, *i.e.* it corresponds to $\mathbb{RP}^2 \vee \bigvee^{n+m} \mathbb{S}^2 \vee \mathbb{RP}^2$.
If $h = (ab)^n a$ but no $b(ab)^m \in H$, then H is generated by h.
So it corresponds to $\mathbb{RP}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2 \vee \cdots$.

Exer 2.16. *Let X be a topological space and $\pi : \mathbb{R}^2 \to X$ a covering map. Let K be a compact subset of X and B the closed unit ball centered at the origin in $\mathbb{R}^2$.*

(1) Suppose $\pi : \mathbb{R}^2 \backslash B \to X \backslash K$ is a homeomorphism. Show that $\pi : \mathbb{R}^2 \to X$ is a homeomorphism.

(2) Suppose $\mathbb{R}^2 \backslash B$ is homeomorphic to $X \backslash K$, where the homeomorphism may mot be π. Is X necessarily homeomorphic to $\mathbb{R}^2$? Prove your assertion.

(1) Since $\pi^{-1}(K) \subset B$ is closed and bounded.

So $\pi^{-1}(K)$ is compact, *i.e.* π is of finite-sheeted.

And since $\mathbb{R}^2 \backslash \pi^{-1}(K)$ is the covering space of $X \backslash K$.

Therefore $\pi_1(\mathbb{R}^2 \backslash \pi^{-1}(K))$ is the finite index subgroup of $\pi_1(X \backslash K) = \mathbb{Z}$, *i.e.* it must be $\mathbb{Z}$.

Suppose $|\pi_1(X)| > 1$.

Then $\pi^{-1}(K)$ has at least two components.

So $\pi_1(X \backslash \pi^{-1}(K))$ has at least two generators given by the boundaries of components of $\pi^{-1}(K)$.

Contradiction!

(2) No. Let $X = \mathbb{T}^2$, K is the complement of an open annulus.

Then $X \backslash K$ is an open annulus, which is homeomorphic to $\mathbb{R}^2 \backslash B$.

Exer 2.17. *Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ (*i.e. *, X and Y are homotopy equivalent), then $\tilde{X} \simeq \tilde{Y}$.*

Let $f : X \to Y$ be the homotopy equivalent with inverse $g : Y \to X$ and $g \circ f \overset{H}{\simeq} \mathrm{Id}_X$
Take base point $x_0 \in X$, $y_0 = f(x_0) \in Y$.
Then $\tilde{X}$ is consist of the path γ from x_0 to $p \in X$.
Consider $\tilde{f} : \tilde{X} \to \tilde{Y}, [\gamma] \mapsto [f \circ \gamma]$.
So $\tilde{g} \circ \tilde{f}([\gamma]) = [g \circ f \circ \gamma]$.
Therefore $\tilde{g} \circ \tilde{f} \overset{\tilde{H}}{\simeq} \mathrm{Id}_{\tilde{X}}$ where $\tilde{H}([\gamma], t) = [H(\gamma, t)]$.
Hence $\tilde{X} \simeq \tilde{Y}$.

Exer 2.18. *Show that $\mathbb{S}^2 \times \mathbb{S}^1$ is a double cover of the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$.*

$\mathbb{RP}^3 \cong \mathbb{D}^3/\sim$ with $x \sim -x$ for $x \in \mathbb{S}^2$.
Let $D_1 = \{x \in \mathbb{R}^3 | |x| \leqslant 3\}, D_2 = \{x \in \mathbb{R}^3 | |x| \leqslant 1\}$.
Then $\mathbb{RP}^3 \# \mathbb{RP}^3 \cong (D_1 \backslash D_2)/\sim$ with $x \sim -x$ for $x \in \partial D_1 \cup \partial D_2$.
So consider map $\pi : \mathbb{S}^2 \times \mathbb{S}^1 \to (D_1 \backslash D_2)/\sim, (x, \theta) \mapsto \left(3 - 2\left|\frac{\theta}{\pi} - 1\right|\right) x$ where $\theta \in [0, 2\pi)$.
It is easy to see that π is a covering map.

Exer 2.19. *Let G and H be connected topological groups and $\phi : G \to H$ a continuous homomorphism. If ϕ is a covering space map, show that the kernel of ϕ is contained in the center of G.*

Take $x \in \ker \phi$, Consider continuous map $f_x : G \to \ker \phi, g \mapsto gxg^{-1}$.
f_x is well-defined since $\phi(gxg^{-1}) = \phi(g)\phi(x)\phi(g)^{-1} = e$.
But since ϕ is covering space map.
So $\ker \phi$ is discrete, *i.e.* f_x must be constant for every $x \in \ker \phi$.
Hence $gxg^{-1} = x$ for all $g \in G$ and $x \in \ker \phi$, *i.e.* $\ker \phi \subset Z(G)$.

3 Classification of closed surface

Exer 3.1. *Is the surface of genus 3 a covering space of the surface of genus 2? Is the surface of genus 2 a covering space of the surface of genus 3? If so, then exhibit an example of such a covering.*

$\chi(\Sigma_3) = -4, \chi(\Sigma_2) = -2$.
So the surface of genus 2 is not a covering space of the surface of genus 3.
The construction of $\pi : \Sigma_3 \to \Sigma_2$ is similar as the example 1.41 in Hatcher.

Exer 3.2. *For $n \geqslant 2$, let X_n be the space obtained from a regular $(2n)$-gon by identifying the opposite sides with parallel orientations. This description produces a cell decomposition of X_n.*

(1) Write down the associated cellular chain complex.

(2) Show that X_n is a surface, and find its genus.

(1) If $2|n$, then there are one 0-cell e^0, n 1-cells $e_1^1, \dots, e_n^1$ and one 2-cells e^2.
And the gluing data is $e_1^1 e_2^1 \cdots e_n^1 \left(e_1^1\right)^{-1} \cdots \left(e_n^1\right)^{-1}$.
If $2 \nmid n$, then there are two 0-cells e_1^0, e_2^0, n 1-cells $e_1^1, \dots, e_n^1$ and one 2-cells e^2.
And e_k^1 is from e_1^0 to e_2^0 if $2 \nmid k$, from e_2^0 to e_1^0 if $2|k$.
The gluing data is $e_1^1 e_2^1 \cdots e_n^1 \left(e_1^1\right)^{-1} \cdots \left(e_n^1\right)^{-1}$.

(2) X_n is a manifold of two dimensional, *i.e.* it is a surface.
And it is orientable, its Euler number is $\chi(X_n) = \left(n - 2\left[\frac{n}{2}\right] + 1\right) - n + 1 = 2 - 2\left[\frac{n}{2}\right]$.
So its genus is $\left[\frac{n}{2}\right]$.

Exer 3.3. *Consider the following group with $2n$ generators and 1 relation:*

$$G_n = \langle a_1, b_1, a_2, b_2, \dots, a_n, b_n | a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-2} \rangle.$$

For which pairs (m, n) does G_n contain a finite index subgroup isomorphic to G_m?

G_n is the fundamental group of Σ_n.
So G_n contains a finite index subgroup isomorphic to G_m iff Σ_m is a covering space of Σ_n.
And since $\chi(\Sigma_m) = 2 - 2m, \chi(\Sigma_n) = 2 - 2n$.
Hence $(n-1)|(m-1)$.

Exer 3.4. *Which topological surfaces can arise from gluing the edges (via linear maps on the edges) of a 2024-gon?*

$\chi(X) = V - 1012 + 1 \geqslant -1010$.
On the other hand, every surface X with $\chi(X) \geqslant -1010$ can be arise from a polygon with $4 - 2\chi(X)$ edges.
And by divide a pair of antipodal edge into m small pieces, where $m = 1010 + \chi(X)$, we can easily construct a 2024-gon that can be arised to X.
Hence all surface X with $\chi(X) \geqslant -1010$ are eligible.

Exer 3.5. *Show that a closed orientable surface M_g of genus g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ (the product of n copies of $\mathbb{Z}$) if and only if $n \leqslant 2g$.*

Let X be the normal covering space of M_g such that $\pi_1(M_g)/\pi_1(X) \cong \mathbb{Z}^n$.
Since the abelization of $\pi_1(X)$ is $\mathbb{Z}^{2g}$.
So $\mathbb{Z}^n \subset \mathbb{Z}^{2g}$, *i.e.* $n \leqslant 2g$.
Let $a_1, a_2, \ldots, a_{2g}$ be the generator of M_g.
Take $\pi_1(X)$ generated by $[G, G]$ and $a_1, \ldots, a_{2g-n}$ and their conjugation.
Then $\pi_1(M_g)/\pi_1(X) \cong \mathbb{Z}^n$.

Exer 3.6. *Let $M = \mathbb{T}^2 \backslash \mathbb{D}^2$ be the complement of a disk inside the two-torus. Determine all connected surfaces that can be described as 3-fold covers of M.*

Let S be a 3-fold covers of M with genus g and k-punctures.
Then $\chi(S) = 2 - 2g - k = 3\chi(M) = 3 \cdot (2 - 2 - 1) = -3$.
So S must be a torus with 3-punctures or a genus 2 surface with 1-puncture.

Exer 3.7. *Let $n \geqslant 2$. Classify (up to diffeomorphism) all connected surfaces that can be represented as n-fold covers of the Klein bottle.*

Let S be the n-fold cover of K.
Then $\chi(S) = n\chi(K) = 0$.
So S can only be the Klein bottle or torus.
On the other hand, torus is its 2-fold cover.

Exer 3.8. *Let $X = (\mathbb{S}^1 \times \mathbb{S}^1) \backslash \{p\}$ be a once-punctured torus.*

(1) How many connected, 3-sheeted covering spaces $f : Y \to X$ are there?

(2) Show that for any of these covering spaces, Y is either a 3-times punctured torus or a once-punctured surface of genus 2.

(1) 5, given by exercise 2.9.

(2) By exercise 3.6 there must be one of these two cases.

On the other hand, both cases can be realized. 3-times punctured torus corresponds to normal covering spaces, and once-punctured surface of genus 2 corresponds to covering spaces that are not normal.

4 Singular and Cellular homology, long exact sequence

Exer 4.1. *Let X be the quotient space of $\mathbb{S}^2$ under the identifications $x \sim -x$ for x in the equator $\mathbb{S}^1$. Compute the homology groups $H_n(X)$. Do the same for $\mathbb{S}^3$ with antipodal points of the equator $\mathbb{S}^2 \subset \mathbb{S}^3$ identified.*

By cellular homology,

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

$$(a,b) \longmapsto 2a - 2b$$

So $H_0(X) = \mathbb{Z}, H_1(X) = \mathbb{Z}/2\mathbb{Z}, H_2(X) = \mathbb{Z}$.
For $\mathbb{S}^3$, by cellular homology,

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

In particular, $\mathbb{Z}^2 \to \mathbb{Z}$ is trivial since $\mathbb{S}^2 \to \mathbb{RP}^2$ has degree 0.
So $H_0(X) = \mathbb{Z}, H_1(X) = \mathbb{Z}/2\mathbb{Z}, H_2(X) = 0, H_3(X) = \mathbb{Z}^2$.

Exer 4.2. *Let $\mathbb{S}^2 \xleftarrow{q_1} \mathbb{S}^2 \vee \mathbb{S}^2 \xrightarrow{q_2} \mathbb{S}^2$ be the maps that crush out one of the two summands. Let $f : \mathbb{S}^2 \to \mathbb{S}^2 \vee \mathbb{S}^2$ be a map such that $q_i \circ f : \mathbb{S}^2 \to \mathbb{S}^2$ is a map of degree d_i. Compute the integral homology groups of $(\mathbb{S}^2 \vee \mathbb{S}^2) \underset{f}{\cup} \mathbb{D}^3$. Here $\mathbb{D}^3$ is the unit solid ball with boundary $\mathbb{S}^2$.*

By cellular homology,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(d_1,d_2)} \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

So $H_0(X) = \mathbb{Z}, H_1(X) = 0, H_2(X) = \mathbb{Z} \oplus \mathbb{Z}/\gcd(d_1,d_2)\mathbb{Z}, H_3(X) = 0$ if $d_1 d_2 \neq 0$.
If $d_1 = d_2 = 0$, then $H_3(X) = \mathbb{Z}$.

Exer 4.3. *Let A be a CW complex, and n a positive integer. Assuming that every homology class in degree $d \geqslant n$ is in the image of $H_d(\mathbb{S}^d;\mathbb{Z})$ for some map $\mathbb{S}^d \to A$, show that there is a CW complex X, and a cellular map $A \to X$ such that the inclusion map*

$$H_i(A;\mathbb{Z}) \to H_i(X;\mathbb{Z})$$

is an isomorphism whenever $i < n$, and vanishes otherwise.

For each generator c of $H_d(A;\mathbb{Z})$, we attach a $(d+1)$-cell on it, this is well-defined since we have $\mathbb{S}^d \to A$ that induce c.

This construct a CW complex X and $A \to X$ induce isomorphism homology group for dimension $i < n$, and vanishes otherwise.

Exer 4.4. *Let X be $(\mathbb{S}^2 \times \mathbb{S}^2) \underset{\mathbb{S}^2}{\cup} \mathbb{D}^3$, where we attach the 3-disk via the map $\mathbb{S}^2 \to \mathbb{S}^2 \vee \mathbb{S}^2$ which crushes a great circle connecting the north and south poles. Compute the homology groups of X.*

By cellular homology,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

So $H_0(X) = \mathbb{Z}, H_1(X) = 0, H_2(X) = \mathbb{Z}, H_3(X) = 0, 4(X) = \mathbb{Z}$.

Exer 4.5. *Let X be a topological space and $p \in X$. The reduced suspension ΣX of X is the space obtained from $X \times [0,1]$ by contracting $(X \times \{0,1\}) \cup (\{p\} \times [0,1])$ to a point. Describe the relation between the homology groups of X and ΣX.*

Let $CX = (X \times [0,1])/(X \times \{1\} \cup \{p\} \times [0,1])$.
Then $\Sigma X = CX/(X \times \{0\})$.
So we have long exact sequence

$$\cdots \longrightarrow \tilde{H}_{i+1}(CX) \longrightarrow \tilde{H}_{i+1}(\Sigma X) \longrightarrow \tilde{H}_i(X) \longrightarrow \tilde{H}_i(CX) \longrightarrow \cdots$$

Notice that CX is contractible.
So $\tilde{H}_i(X) \cong \tilde{H}_{i+1}(\Sigma X)$.

Exer 4.6. *Construct a space X with $H_0(X) = \mathbb{Z}, H_1(X) = \mathbb{Z}_2 \times \mathbb{Z}_3, H_2(X) = \mathbb{Z}$, and all other homology groups of X vanishing.*

Let X be a CW complex with one 0-cell e^0, one 1-cell e^1 and two 2-cells e_1^2, e_2^2.
And the attach map is $6e^1$ and 0 resp.

Exer 4.7. *Show that $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.*

By cellular homology, both of them are

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So they have the same homology groups in all dimensions.
But their universal covering spaces are $\mathbb{R}^2$ and the complete quadtree wedging a $\mathbb{S}^2$ at each vertices, so they do not have the same homology group in dimension 2.

Exer 4.8. *Compute all homology groups of the m-skeleton of an n-simplex, $0 \leqslant m \leqslant n$.*

Consider the simplicial homology of Δ^n

$$0 \longrightarrow \Delta^n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

By definition, C_m is the m-skeleton of Δ^n.
So the simplicial homology of C_m is given by

$$0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Hence $H_k(C_m) = 0$ for $0 < k < m$, $H_0(C_m) = \mathbb{Z}$ and

$$\begin{aligned} \dim H_m(C_m) &= \dim C_{m+1} - \dim \ker \partial_{m+1} \\ &= \binom{n+1}{m+2} - \binom{n}{m+2} \\ &= \binom{n}{m+1} \end{aligned}$$

that is, $H_m(C_m) = \mathbb{Z}^{\binom{n}{m+1}}$.

Exer 4.9. *Suppose that X is contractible and that some point a of X has a neighborhood homeomorphic to $\mathbb{R}^k$. Prove that $H_n(X \backslash \{a\}) \cong H_n(\mathbb{S}^{k-1})$ for all n.*

Let U be the neighborhood that homeomorphic to $\mathbb{R}^k$.
So by excision theorem, $H_n(X \backslash \{a\}, U \backslash \{a\}) = \tilde{H}_n(X, U) = 0$.
Therefore we have long exact sequence

$$\cdots \longrightarrow H_{n+1}(X \backslash \{a\}, U \backslash \{a\}) \longrightarrow \tilde{H}_n(U \backslash \{a\}) \longrightarrow \tilde{H}_n(X \backslash \{a\}) \longrightarrow H_n(X \backslash \{a\}, U \backslash \{a\}) \longrightarrow \cdots$$

Hence $\tilde{H}_n(X \backslash \{a\}) = \tilde{H}_n(U \backslash \{a\}) = \tilde{H}(\mathbb{S}^{k-1})$.

Exer 4.10. *Let $X \to Y$ be a covering map, and d a positive integer not equal to 0 or 1. Given examples where the homology $H_d(X, \mathbb{Q})$ is non-trivial and the map $H_d(X, \mathbb{Q}) \to H_d(Y, \mathbb{Q})$*

(1) is an isomorphism,

(2) is surjective but not an isomorphism,

(3) is injective but not an isomorphism.

(1) Let X, Y be the torus and $f : X \to Y$ be a two-sheeted covering space map.
Then $H_2(X, \mathbb{Q}) = H_2(Y, \mathbb{Q}) \cong \mathbb{Q}$ and $f_*(a) = 2a$ for $d = 2$, *i.e.* it is an isomorphism.

(2) Let $X = \mathbb{S}^2, Y = \mathbb{RP}^2$ and $f : X \to Y$ be a two-sheeted covering space map.
Then $H_2(X, \mathbb{Q}) \cong \mathbb{Q}, H_2(Y, \mathbb{Q}) = 0$ and $f_* = 0$ for $d = 2$, *i.e.* it is surjective.

(3) Let $X = \mathbb{R}^2, Y = \mathbb{T}^2$ and $f : X \to Y$ be the universal covering space map.
Then $H_2(X, \mathbb{Q}) = 0, H_2(Y, \mathbb{Q}) \cong \mathbb{Q}$ and $f_* = 0$ for $d = 2$, *i.e.* it is injective.

Exer 4.11. *Show that if $\mathbb{R}^m$ and $\mathbb{R}^n$ have homeomorphic open subset then $m = n$.*

Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be two homeomorphic open subset.
Then by excision theorem, $H_k(U, U\backslash\{p\}) = H_k(\mathbb{R}^m, \mathbb{R}^m\backslash\{p\}) = H_k(\mathbb{S}^m)$.
So $H_m(U, U\backslash\{p\}) = H_m(V, V\backslash\{q\}) = \mathbb{Z}$ only if $m = n$.

Exer 4.12. *Show that $\mathbb{Z}/2\mathbb{Z}$ is the only group that can act freely on $\mathbb{S}^{2n}$.*

Let G be a group that can act freely on $\mathbb{S}^{2n}$ and $f \in G$.
Then f is an automorphism without fixed point, its Lefschetz number is $\tau(f) = 1 + \deg(f)$.
So by Lefschetz fixed point theorem, $\deg(f) = -1$.
And since $\deg(f \circ f) = 1$.
Therefore $f \circ f$ has a fixed point, *i.e.* it must be Id.
Hence $G = \mathbb{Z}/2\mathbb{Z}$.

Exer 4.13. *For a topological space X, denote the suspension of X by SX.*

(1) Show that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ for all n. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

(2) Making the preceding problem more concrete, construct explicit chain maps $s : C_n(X) \to C_{n+1}(SX)$ inducing isomorphisms $\tilde{H}_n(X) \to \tilde{H}_{n+1}(SX)$.

(1) Since $SX = (X \times [0, 1])/(X \times \{0, 1\}) = CX/(X \times \{0\})$.
So we have long exact sequence

$$\cdots \longrightarrow \tilde{H}_{i+1}(CX) \longrightarrow \tilde{H}_{i+1}(SX) \longrightarrow \tilde{H}_i(X) \longrightarrow \tilde{H}_i(CX) \longrightarrow \cdots$$

And notice that CX is contractible.
Therefore $\tilde{H}_{n+1}(SX) = \tilde{H}_n(X)$.
For more general C_nX, by MV sequence,

$$\cdots \longrightarrow \tilde{H}_i(CX) \longrightarrow \tilde{H}_i(C_{n-1}X) \oplus \tilde{H}_i(SX) \longrightarrow \tilde{H}_i(C_nX) \longrightarrow \tilde{H}_{i-1}(CX) \longrightarrow \cdots$$

Hence $\tilde{H}_i(C_nX) \cong \tilde{H}_i(C_{n-1}X) \oplus \tilde{H}_i(SX) = \left(\tilde{H}_{i-1}(X)\right)^{n-1}$.

(2) Let $SX = C_1 \cup C_2$ where $C_1 = \left(X \times \left[\frac{1}{2}, 1\right]\right)/(X \times \{1\}), C_2 = \left(X \times \left[0, \frac{1}{2}\right]\right)/(X \times \{0\})$.

Then for any $f : \Delta^n \to X$, we have $C_1 f : \Delta^{n+1} \to C_1, C_2 f : \Delta^{n+1} \to C_2$.

So $\partial(C_1 f - C_2 f) = C_1 \partial f - C_2 \partial f = (C_1 - C_2)\partial f$, *i.e.* $C_1 - C_2$ induce a chain map.

Consider the boundary map in long exact sequence $D : \tilde{H}_{i+1}(SX) \to \tilde{H}_i(X)$.

Then $D(C_1 - C_2)_*([f]) = D(C_1)_*([f]) - D(C_2)_*[f] = [f] - [f] = 0$.

So $C_1 - C_2$ induces an isomorphism on $\tilde{H}_n(X) \to \tilde{H}_{n+1}(SX)$.

Exer 4.14. *If $f : M \to N$ is a degree one map between compact connected oriented manifolds without boundary, show that the induced map on π_1 is surjective.*

Suppose $f_* : \pi_1(M) \to \pi_1(N)$ is not surjective.

Then we can induce f to $\tilde{f} : M \to \tilde{N}$ where $\tilde{N}$ is an n-sheeted covering space of N with $\pi : \tilde{N} \to N$ and $f_*(\pi_1(M)) \subset \pi_1(\tilde{N})$.

So $\deg f = \deg \tilde{f} \cdot \deg \pi = \deg \tilde{f} \cdot n > 1$, contradiction!

Exer 4.15. *If a finite-dimensional CW complex X is a $K(G,1)$ space, then the group $G = \pi_1(X)$ must be torsion-free.*

Suppose G is not torsion-free and $a \in G$ is of order m.

Then the covering space of $K(G,1)$ corresponding to $\mathbb{Z}/m\mathbb{Z}$ is homeomorphic to $K(\mathbb{Z}/m\mathbb{Z}, 1)$.

But $H_k(K(\mathbb{Z}/m\mathbb{Z}, 1))$ has infinite many nonzero terms, contradiction!

Remark 4.1. The space $K(\mathbb{Z}/m\mathbb{Z}, 1) \cong \mathbb{S}^\infty/\mathbb{Z}_m$ is the infinite-dimensional lens space and we will not prove the detail because it is too complicated. You can see the example 2.43 on Hatcher.

Exer 4.16. *Let $M'_h \subset M_g$ be a compact subsurface of genus h with one boundary circle, so M'_h is homeomorphic to M_h with an open disk removed. Show that there is no retraction $M_g \to M'_h$ if $h > \frac{g}{2}$.*

Suppose retraction $M_g \to M'_h$ exists, restrict it on $M_g \backslash M'_h \cong M'_{g-h}$ and denote by f.

Then we have $f : M'_{g-h} \to M'_h$ and $f\big|_{\partial M'_h}$ is homeomorphism.

So f induces a map $F : M_{g-h} \to M_h$ and $\deg F = 1$.

Let $a_1, b_1, \ldots, a_{g-h}, b_{g-h}$ be generators of $H^1(M_{g-h})$ and $c_1, d_1, \ldots, c_h, d_h$ be those of $H^1(M_h)$.

Since $h > \frac{g}{2}$, *i.e.* $h > g - h$, so let

$$\sum_{i=1}^{h} C_i f^*(c_i) + \sum_{i=1}^{h} D_i f^*(d_i) = 0.$$

And WLOG, we assume $C_1 \neq 0$.

But we have

$$\begin{aligned}\left(\sum C_i f^*(c_i) + D_i f^*(d_i)\right) \smile f^*(d_i) &= f^*\left(\left(\sum C_i c_i + D_i d_i\right) \smile d_i\right) \\ &= f^*(C_1[M]) = C_1[N] \neq 0\end{aligned}$$

Contradiction!

Hence there is no retraction $M_g \to M'_h$.

Remark 4.2. More generally, we can use Poincaré duality to prove that degree 1 maps induce surjective map on $H^k(-)$.

5 Mayer-Vietoris sequence

Exer 5.1. *Let $f : X \to Y$ be a continuous, pointed map. Let $\Sigma^n(f) : \Sigma^n X \to \Sigma^n Y$ be the n-th (pointed) suspension of f. Show that if for some n, $\Sigma^n(f)$ induces the trivial map on reduced homology, then it does for all n.*

By exercise 4.5, $\tilde{H}_{i+n}(\Sigma^n X) = \tilde{H}_i(X), \tilde{H}_{i+n}(\Sigma^n Y) = \tilde{H}_i(Y)$.
And moreover, $\Sigma^n(f)_* : \tilde{H}_{i+n}(\Sigma^n X) \to \tilde{H}_{i+n}(\Sigma^n Y)$ is the same as $f_* : \tilde{H}_i(X) \to \tilde{H}_i(Y)$.
So if one of them is trivial, then all of them are trivial.

Exer 5.2. *If $f : X \to X$ is a self-map, then the "mapping torus of f" is the quotient*

$$T_f := (X \times [0,1])/(x,0) \sim (f(x),1), \forall x \in X.$$

For $n \in \mathbb{Z}$, let f_n be a degree n map on $\mathbb{S}^3$. Compute the integral homology groups of T_{f_n}.

Let $U = \left(X \times [0, \frac{1}{2}]\right) / \sim, V = \left(X \times [\frac{1}{2}, 1]\right) / \sim$.
Then by MV sequence,

$$\cdots \longrightarrow \tilde{H}_{k+1}(T_{f_n}) \xrightarrow{\partial} \tilde{H}_k\left(X \times \{0, \tfrac{1}{2}\}\right) \xrightarrow{i_k} \tilde{H}_k(U) \oplus \tilde{H}_k(V) \longrightarrow \tilde{H}_k(T_{f_n}) \xrightarrow{\partial} \cdots$$
$$(a,b) \longmapsto (a+b, f_*(a)+b)$$

where $\tilde{H}_k(U) = \tilde{H}_k(V) = \tilde{H}_k(X)$.
So for $k \neq 1,3,4$, $\tilde{H}_k(T_{f_n}) = 0$ and $\tilde{H}_1(T_{f_n}) = \tilde{H}_0\left(X \times \{0, \frac{1}{2}\}\right) \cong \mathbb{Z}$.
Moreover, $\tilde{H}_3(T_{f_n}) = \tilde{H}_3(X)^2 / \operatorname{Im}(i_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$.
And $\tilde{H}_4(T_{f_n}) = \ker(i_n) = 0$ for $n \neq 1$ and $\tilde{H}_4(T_{f_n}) \cong \mathbb{Z}$ if $n = 1$.

Exer 5.3. *The join of X and Y is defined to be*

$$X * Y := (X \times [0,1] \times Y)/\sim$$

where $\sim$ is generated by

$$(x,0,y_1) \sim (x,0,y_2)\ \forall x \in X, \forall y_1, y_2 \in Y$$
$$(x_1,1,y) \sim (x_2,1,y)\ \forall x_1, x_2 \in X, \forall y \in Y.$$

Prove that there is an isomorphism for all $i > 0$:

$$H_i(X \times Y) \cong H_i(X) \oplus H_i(Y) \oplus H_{i+1}(X * Y).$$

Let $U = \left(X \times [0, \frac{1}{2}] \times Y\right) / \sim, V = \left(X \times [\frac{1}{2}, 1] \times Y\right) / \sim$.
Then by MV sequence,

$$\cdots \longrightarrow H_{k+1}(X * Y) \xrightarrow{\partial} H_k(X \times Y) \xrightarrow{i_k} H_k(U) \oplus H_k(V) \longrightarrow H_k(X * Y) \xrightarrow{\partial} \cdots$$
$$a \longmapsto (\pi_{1*}(a), \pi_{2*}(a))$$

where $H_k(U) = H_k(X), H_k(V) = H_k(Y)$.
So i_k is a surjective and it has a left inverse.
Hence we have a short split exact sequence

$$0 \longrightarrow H_{k+1}(X * Y) \xrightarrow{\partial} H_k(X \times Y) \xrightarrow{i_k} H_k(U) \oplus H_k(V) \longrightarrow 0$$

And $H_i(X \times Y) \cong H_i(X) \oplus H_i(Y) \oplus H_{i+1}(X * Y)$.

Exer 5.4. *Consider the CW-complexes $A = \mathbb{S}^n \vee \mathbb{S}^n, X = \mathbb{S}^n \times \mathbb{S}^n, B = \mathbb{S}^n \times [0,1]/(* \times [0,1])$, where $*$ is the basepoint of $\mathbb{S}^n$. There are inclusions $A \to X$ given by the pairs of points where at least one is the basepoint, and $A \to B$ which takes one $\mathbb{S}^n$ to $\mathbb{S}^n \times 0$ and the other to $\mathbb{S}^n \times 1$. Compute the homology of*

$$Y = X \underset{A}{\cup} B.$$

Let $f : A \to X, g : A \to B$ be the inclusions.

By MV sequence,

$$\cdots \longrightarrow \tilde{H}_{k+1}(Y) \xrightarrow{\partial} \tilde{H}_k(A) \xrightarrow{i_k} \tilde{H}_k(X) \oplus \tilde{H}_k(B) \longrightarrow \tilde{H}_k(Y) \xrightarrow{\partial} \cdots$$

$$(a,b) \longmapsto (f_*(a,b), a+b)$$

So for $k \neq n, n+1, 2n$, $\tilde{H}_k(Y) = 0$.

Moreover, $\tilde{H}_n(Y) = (\tilde{H}_n(X) \oplus \tilde{H}_n(B))/\operatorname{im} i_n$ and $f_*(a,b) = (a,b)$ for $(a,b) \in \tilde{H}_n(A) \cong \mathbb{Z}^2$.

So $\tilde{H}_n(Y) \cong \mathbb{Z}, \tilde{H}_{n+1}(Y) \cong \ker i_n = 0, \tilde{H}_{2n}(Y) = (\tilde{H}_{2n}(X) \oplus \tilde{H}_{2n}(B))/\operatorname{im} i_{2n} = \mathbb{Z}$.

Exer 5.5. *Suppose the space X is the union of open sets $A_1, \ldots, A_n$ such that each intersection $A_{i_1} \cap \cdots \cap A_{i_k}$ is either empty or has trivial reduced homology groups. Show that $\tilde{H}_i(X) = 0$ for $i \geqslant n-1$, and give an example showing this inequality is best possible, for each n.*

We prove it by induction on n.

For $n = 1$ it is trivial.

Suppose the statement is true for $n = t-1$, now consider the case that $n = t$.

Let $U = A_1 \cup \cdots \cup A_{n-1}, V = A_2 \cup \cdots \cup A_n$.

Then By MV sequence,

$$\cdots \longrightarrow \tilde{H}_{k+1}(X) \longrightarrow \tilde{H}_k(U \cap V) \longrightarrow \tilde{H}_k(U) \oplus \tilde{H}_k(V) \longrightarrow \tilde{H}_k(X) \longrightarrow \cdots$$

Since $U \cap V = A_2 \cup \cdots \cup A_{n-1} \cup (A_1 \cap A_n)$.

So $\tilde{H}_k(U \cap V) = \tilde{H}_k(U) = \tilde{H}_k(V) = 0$ for $k \geqslant n-2$.

Hence $\tilde{H}_k(X) = 0$ for $k \geqslant n-1$.

And we give an example by induction on n.

Suppose we can cover $X = \mathbb{S}^{n-2}$ with n open sets such that each intersection has trivial reduced homology groups.

Then for $X = \mathbb{S}^{n-1}$, take A_1 covering the lower hemisphere and $A_2, \ldots, A_n$ be the cone of open set covering $\mathbb{S}^{n-2}$.

Exer 5.6. *A compact surface (without boundary) of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact 3-dimensional region called a handlebody H (the region "inside" the surface). Let*

$$X = (H \times \{0,1,2\})/\sim,$$

where $(x,i) \sim (x,j)$ for all $x \in \partial H$ and $i,j \in \{0,1,2\}$. Compute the homology of X.

Let $Y = (H \times \{0,1\})/\sim, U = H \times \{0\}, V = H \times \{1\}$ and $i : \partial H \to H$.

Then by MV sequence,

$$\cdots \longrightarrow H_{k+1}(Y) \xrightarrow{\partial} H_k(\partial H) \xrightarrow{i_k} H_k(U) \oplus H_k(V) \longrightarrow H_k(Y) \xrightarrow{\partial} \cdots$$

$$a \longmapsto (i_*(a), i_*(a))$$

And since

$$H_k(\partial H) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^{2g} & k=1 \\ \mathbb{Z} & k=2 \end{cases}, H_k(H) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^g & k=1 \end{cases}$$

So $H_0(Y)=\mathbb{Z}, H_1(Y)=H_1(H)^2/\operatorname{im} i_1 \cong \mathbb{Z}^g, H_2(Y)=\ker i_1 \cong \mathbb{Z}^g, H_3(Y)=H_2(\partial H)=\mathbb{Z}$.

Let $Z=H\times\{2\}$ and $j:\partial H\to Y$.

By MV sequence,

$$\cdots \longrightarrow H_{k+1}(X) \xrightarrow{\partial} H_k(\partial H) \xrightarrow{j_k} H_k(Y)\oplus H_k(Z) \longrightarrow H_k(X) \xrightarrow{\partial} \cdots$$

$$a \longmapsto (j_*(a), i_*(a))$$

So $H_0(X)=\mathbb{Z}, H_1(X)=H_1(Y)\oplus H_1(Z)/\operatorname{im} j_1 \cong \mathbb{Z}^g, H_2(X)=\ker j_1 \oplus H_2(Y)/\operatorname{im} j_2 \cong \mathbb{Z}^{2g}$, $H_3(X)=\ker j_2\oplus H_3(Y)=\mathbb{Z}^2$.

Exer 5.7. *Let M be a compact odd-dimensional manifold with boundary ∂M. Show that the Euler characteristics of M and ∂M are related by:*

$$\chi(M)=\frac{1}{2}\chi(\partial M)$$

Let $N=(M\times\{0,1\})/\sim$ where $(x,0)\sim(x,1)$ for all $x\in\partial M$.

Then N is a compact odd-dimensional manifold without boundary, *i.e.* $\chi(N)=0$.

And by MV sequence,

$$\cdots \longrightarrow H_{k+1}(N) \longrightarrow H_k(\partial M) \longrightarrow H_k(M)\oplus H_k(M) \longrightarrow H_k(N) \longrightarrow \cdots$$

So $2\chi(M)=\chi(\partial M)+\chi(N)=\chi(\partial M)$.

Exer 5.8. *Consider the quotient space*

$$X=([0,1]\times\mathbb{S}^1\times\mathbb{S}^1)/\sim.$$

where the equivalence relation $\sim$ is generated by

$$(0,x,y)\sim(0,z,w) \text{ if } xy=zw,$$

and

$$(1,x,y)\sim(1,z,w) \text{ if } x^2y^6=z^2w^6.$$

Here we treat $\mathbb{S}^1$ as the space of unit complex numbers. Compute $H_n(X;\mathbb{Z})$ for all n.

Let $U=\left(\left[0,\frac{1}{2}\right]\times\mathbb{S}^1\times\mathbb{S}^1\right)/\sim, V=\left(\left[\frac{1}{2},1\right]\times\mathbb{S}^1\times\mathbb{S}^1\right)/\sim$ and $\pi_1:X\to X/\sim_0, \pi_2:X\to X/\sim_1$.

By MV sequence,

$$\cdots \longrightarrow H_{k+1}(X) \xrightarrow{\partial} H_k(\mathbb{S}^1\times\mathbb{S}^1) \xrightarrow{i_k} H_k(U)\oplus H_k(V) \longrightarrow H_k(X) \xrightarrow{\partial} \cdots$$

$$a \longmapsto (\pi_{1*}(a), \pi_{2*}(a))$$

where $H_k(U)=H_k(V)=H_k(\mathbb{S}^1)$.

And $\pi_{1*}:H_1(\mathbb{S}^1\times\mathbb{S}^1)\to H_1(\mathbb{S}^1), (a,b)\mapsto a+b, \pi_{2*}:H_1(\mathbb{S}^1\times\mathbb{S}^1)\to H_1(\mathbb{S}^1), (a,b)\mapsto 2a+6b$.

So $H_1(X)=H_1(\mathbb{S}^1)^2/\operatorname{im} i_1\cong\mathbb{Z}/4\mathbb{Z}, H_2(X)=\ker i_1=0, H_3(X)=H_2(\mathbb{S}^1\times\mathbb{S}^1)=\mathbb{Z}$.

Exer 5.9. *Let $X=\mathbb{R}^4/\sim$, where*

$$(x_1,x_2,x_3,x_4)\sim(x_1,x_2+1,x_3,x_4)$$

$$(x_1,x_2,x_3,x_4)\sim(x_1,x_2,x_3,x_4+1)$$

$$(x_1,x_2,x_3,x_4)\sim(x_1+1,x_2,x_3,x_4)$$

$$(x_1,x_2,x_3,x_4)\sim(x_1,x_2+x_4,x_3+1,x_4)$$

Compute $H_1(X,\mathbb{Z})$.

Let $Y = \mathbb{R}^3/\sim$ with coordinate (x_2, x_3, x_4).
Then $X = Y \times \mathbb{S}^1$.
And since the first two equivalent relation gives a torus.
So $Y \cong T_f$ where $f : \mathbb{T}^2 \to \mathbb{T}^2, (x_2, x_4) \mapsto (x_2 + x_4, x_4)$.
Let $U = \mathbb{T}^2 \times \left[0, \frac{1}{2}\right], V = \mathbb{T}^2 \times \left[\frac{1}{2}, 1\right]$.
Then by MV sequence,

$$\cdots \longrightarrow H_1(\mathbb{T}^2) \oplus H_1(\mathbb{T}^2) \xrightarrow{i_1} H_1(U) \oplus H_1(V) \longrightarrow H_1(Y) \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$(a, b) \longmapsto (a + b, f_*(a) + b)$$

where $H_1(U) = H_1(V) = H_1(\mathbb{T}^2)$
So $H_1(Y) = \mathbb{Z} \oplus H_1(U) \oplus H_1(V)/\operatorname{im} i_1 \cong \mathbb{Z}^2$.
Hence $H_1(X) = \mathbb{Z}^3$.

Exer 5.10. *(1) Show that $\mathbb{RP}^{2n}$ cannot be the boundary of a compact manifold.*

(2) Show that $\mathbb{RP}^3$ is the boundary of some compact manifold.

(1) $\chi(\mathbb{RP}^{2n}) = 1$ is odd.

So by exercise 5.7, it cannot be the boundary of a compact manifold.

(2) $\mathbb{RP}^3 \cong \mathrm{SO}(3)$ is the unit tangent bundle of $\mathbb{S}^2$.

So $\mathbb{RP}^3$ is the boundary of $\left\{(x, v) \in T\mathbb{S}^2 \big| |v| \leqslant 1\right\}$.

Exer 5.11. *Use the Mayer-Vietoris sequence to show that a nonorientable closed surface X, cannot be embedded as a subspace of $\mathbb{R}^3$ in such a way as to have a neighborhood homeomorphic to the mapping cylinder of some map from a closed orientable surface to X.*

Suppose there exists an open set $U \subset \mathbb{R}^3$ with $U \cong C_h$ where $h : Y \to X$.
Then U can deformation retract to X, *i.e.* X can be embedded in to $\mathbb{R}^3$ with image in U.
So let $V = \mathbb{R}^3/X$.
Since $U \cap V \cong C_h \backslash X \cong Y \times [0, 1] \simeq Y$.
By MV sequence,

$$\cdots \longrightarrow H_{k+1}(\mathbb{R}^3) \longrightarrow H_k(Y) \longrightarrow H_k(X) \oplus H_k(V) \longrightarrow H_k(\mathbb{R}^3) \longrightarrow \cdots$$

Therefore $H_1(X) \oplus H_1(V) \cong H_1(Y) \cong \mathbb{Z}^{2g}$.
But $H_1(X)$ has torsion, contradiction!

Exer 5.12. *Let M be the quotient space*

$$\left([0, 1] \times \mathbb{CP}^2\right) / (0, [z_0, z_1, z_2]) \sim (1, [\bar{z}_0, \bar{z}_1, \bar{z}_2]) .$$

Compute the homology group $H_k(M; \mathbb{Z})$ for all $k \geqslant 0$.

Let $U = \left(\left[0, \frac{1}{2}\right] \times \mathbb{CP}^2\right) / \sim, V = \left(\left[\frac{1}{2}, 1\right] \times \mathbb{CP}^2\right) / \sim$.
By MV sequence,

$$\cdots \longrightarrow H_{k+1}(M) \xrightarrow{\partial} H_k(\left\{0, \tfrac{1}{2}\right\} \times \mathbb{CP}^2) \xrightarrow{i_k} H_k(U) \oplus H_k(V) \longrightarrow H_k(M) \xrightarrow{\partial} \cdots$$

$$(a, b) \longmapsto (a + b, f_*(a) + b)$$

where $H_k(U) = H_k(V) = H_k(\mathbb{CP}^2)$.
And since the generator of $H_2(\mathbb{CP}^2)$ is $\mathbb{CP}^1 \cong \mathbb{S}^2$.
So $f_* : H_2(\mathbb{CP}^2) \to H_2(\mathbb{CP}^2), a \mapsto -a$ and $f_* : H_4(\mathbb{CP}^2) \to H_4(\mathbb{CP}^2), a \mapsto a$.
Hence $H_0(M) = \mathbb{Z}, H_1(M) = \ker i_0 \cong \mathbb{Z}, H_2(M) = H_2(\mathbb{CP}^2)^2/\operatorname{im} i_2 \cong \mathbb{Z}/2\mathbb{Z}, H_3(M) = \ker i_2 = 0, H_4(M) = H_4(\mathbb{CP}^2)^2/\operatorname{im} i_4 \cong \mathbb{Z}, H_5(M) = \ker i_4 = \mathbb{Z}$.

6 Universal coefficient theorem and Kunneth formula

Exer 6.1. *Let X be the Klein bottle.*

(1) Compute the homology groups $H_n(X;\mathbb{Z})$.

(2) Compute the homology groups $H_n(X;\mathbb{Z}/2\mathbb{Z})$.

(3) Compute the homology groups $H_n(X\times X;\mathbb{Z}/2\mathbb{Z})$.

(1) By cellular homology,

$$0\longrightarrow \mathbb{Z}\overset{(0,2)}{\longrightarrow}\mathbb{Z}\times\mathbb{Z}\overset{0}{\longrightarrow}\mathbb{Z}\longrightarrow 0$$

So $H_0(X;\mathbb{Z})=\mathbb{Z}, H_1(X;\mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}, H_2(X;\mathbb{Z})=0$.

(2) By Universal coefficient theorem,

$$H_0(X;\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z},$$

$$H_1(X;\mathbb{Z}/2\mathbb{Z})=(\mathbb{Z}/2\mathbb{Z})^2\oplus \mathrm{Tor}_1(\mathbb{Z},\mathbb{Z}/2\mathbb{Z})=(\mathbb{Z}/2\mathbb{Z}^2),$$
$$H_2(X;\mathbb{Z}/2\mathbb{Z})=\mathrm{Tor}_1(\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}.$$

(3) By Kunneth formula,

$$H_0(X\times X;\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z},$$
$$H_1(X\times X;\mathbb{Z}/2\mathbb{Z})=\left((\mathbb{Z}/2\mathbb{Z})^2\otimes\mathbb{Z}/2\mathbb{Z}\right)^2=(\mathbb{Z}/4\mathbb{Z})^4,$$
$$H_2(X\times X;\mathbb{Z}/2\mathbb{Z})=\left((\mathbb{Z}/2\mathbb{Z})^2\otimes(\mathbb{Z}/2\mathbb{Z})^2\right)\oplus(\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z})^2=(\mathbb{Z}/2\mathbb{Z})^6,$$
$$H_3(X\times X;\mathbb{Z}/2\mathbb{Z})=\left((\mathbb{Z}/2\mathbb{Z})^2\otimes\mathbb{Z}/2\mathbb{Z}\right)^2=(\mathbb{Z}/4\mathbb{Z})^4,$$
$$H^4(X\times X;\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z}=\mathbb{Z}/2\mathbb{Z}.$$

Exer 6.2. *What are the homology groups of the 5-manifold $\mathbb{RP}^2\times\mathbb{RP}^3$:*

(1) with coefficients in $\mathbb{Z}$?

(2) with coefficient in $\mathbb{Z}/2\mathbb{Z}$?

(3) with coefficient in $\mathbb{Z}/3\mathbb{Z}$?

(1) By Kunneth formula,

$$H_0(M;\mathbb{Z})=\mathbb{Z}, H_1(M;\mathbb{Z})=(\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z})^2=(\mathbb{Z}/2\mathbb{Z})^2,$$

$$H_2(M;\mathbb{Z})=(\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z})\oplus\left(\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z})\right)^2=\mathbb{Z}/2\mathbb{Z},$$

$$H_3(M;\mathbb{Z})=(\mathbb{Z}\otimes\mathbb{Z})\oplus\left(\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z})\right)^2=\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z},$$

$$H_4(M;\mathbb{Z})=(\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z})\oplus\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})=\mathbb{Z}/2\mathbb{Z},$$
$$H_5(M;\mathbb{Z})=\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})=0.$$

(2) By universal coefficient theorem,

$$H_0(M;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z},$$
$$H_1(M;\mathbb{Z}/2\mathbb{Z}) = \big((\mathbb{Z}/2\mathbb{Z})^2 \otimes \mathbb{Z}/2\mathbb{Z}\big) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^2,$$
$$H_2(M;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}) \oplus \operatorname{Tor}_1^{\mathbb{Z}}\big((\mathbb{Z}/2\mathbb{Z})^2,\mathbb{Z}/2\mathbb{Z}\big) = (\mathbb{Z}/2\mathbb{Z})^3,$$
$$H_3(M;\mathbb{Z}/2\mathbb{Z}) = ((\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^3,$$
$$H_4(M;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^2$$
$$H_5(M;\mathbb{Z}/2\mathbb{Z}) = \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}.$$

(3) By universal coefficient theorem,

$$H_0(M;\mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z},$$
$$H_1(M;\mathbb{Z}/3\mathbb{Z}) = \big((\mathbb{Z}/2\mathbb{Z})^2 \otimes \mathbb{Z}/3\mathbb{Z}\big) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/3\mathbb{Z}) = 0,$$
$$H_2(M;\mathbb{Z}/3\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z}) \oplus \operatorname{Tor}_1^{\mathbb{Z}}\big((\mathbb{Z}/2\mathbb{Z})^2,\mathbb{Z}/3\mathbb{Z}\big) = 0,$$
$$H_3(M;\mathbb{Z}/3\mathbb{Z}) = ((\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{Z}/3\mathbb{Z}) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z},$$
$$H_4(M;\mathbb{Z}/3\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z}) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},\mathbb{Z}/3\mathbb{Z}) = 0$$
$$H_5(M;\mathbb{Z}/3\mathbb{Z}) = \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/3\mathbb{Z}) = 0.$$

Exer 6.3. *Find a degree one map $\mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{RP}^3$, or prove that no such map exists.*

Suppose $f : \mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{RP}^3$ with $\deg f = 1$.
By Kunneth theorem, $H^*(\mathbb{S}^1 \times \mathbb{S}^2;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[a,b]/(a^2,b^2)$ with $|a| = 1, |b| = 2$.
And since $H^*(\mathbb{RP}^3;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[c]/(c^4)$ with $|c| = 1$.
Let $f^*(c) = ka + lb$.
Then $f^*(c^2) = k^2a^2 + 2klab + l^2b^2 = 2klab$.
But $\left\langle f^*(c^2), \left[\mathbb{S}^1 \times \mathbb{S}^2\right]\right\rangle = \deg f \cdot \left\langle c^2, \left[\mathbb{RP}^3\right]\right\rangle = 1 \neq 2kl \left\langle ab, \left[\mathbb{S}^1 \times \mathbb{S}^2\right]\right\rangle = 2kl$, contradiction!

Exer 6.4. *Consider $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $Y = \mathbb{CP}^2 \vee \mathbb{CP}^1$.*

(1) For any abelian coefficient group G, compute the homology groups $H_*(X;G)$ and $H_*(Y;G)$.

(2) Show that X and Y are not homotopy equivalent.

(1) By Kunneth formula, $H_0(X;\mathbb{Z}) = H_4(X;\mathbb{Z}) = \mathbb{Z}, H_1(X;\mathbb{Z}) = H_3(X;\mathbb{Z}) = 0, H_2(X;\mathbb{Z}) = \mathbb{Z}^2$

And by MV sequence, $H_*(Y;\mathbb{Z})$ are the same, *i.e.* $H_*(X;G) = H^*(Y;G)$.

By UCT, $H_0(X;G) = H_4(X;G) = \mathbb{Z} \otimes G = G, H_1(X;G) = H_3(X;G) = 0$ and $H_2(X;G) = \mathbb{Z}^2 \otimes G = G^2$ since $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z},-) = 0$.

(2) By Kunneth formula, $H^*(X;\mathbb{Z}) = \mathbb{Z}[a,b]/(a^2,b^2)$ with $|a| = |b| = 2$.

And $H^*(Y;\mathbb{Z}) = \mathbb{Z}[c,d]/(c^3,d^2,cd)$ with $|c| = |d| = 2$.

Suppose there exists isomorphism $f : H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ and let $f(a) = pc + qd, f(b) = rc + sd$.

Then $f(a^2) = p^2c^2 + 2pqcd + q^2d^2 = p^2c^2$.

So $p = 0$ and similarly $r = 0$, contradiction!

Hence X and Y are not homotopy equivalent.

Exer 6.5. *Show that a continuous map $f : X \to \mathbb{RP}^n$ factors through $\mathbb{S}^n \to \mathbb{RP}^n$ if and only if the induced map $f^* : H^1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \to H^1(X; \mathbb{Z}/2\mathbb{Z})$ is zero.*

If f factors through $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ then $f^* = 0$ since $H^1(\mathbb{S}^n; \mathbb{Z}/2\mathbb{Z}) = 0$.
Conversely, by universal coefficient theorem, $f_* : H_1(X; \mathbb{Z}/2\mathbb{Z}) \to H_1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$ is zero.
And since $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$ is abelian.
So $f_* : \pi_1(X) \to \pi_1(\mathbb{RP}^n)$ factors through $\mathbf{Ab}(\pi_1(X)) = H_1(X)$.
Hence $f_* = 0$, *i.e.* $f : X \to \mathbb{RP}^n$ factors through π.

Exer 6.6. *Is it true that if a continuous map $f : X \to Y$ induces isomorphisms of homology groups with $\mathbb{Z}$-coefficients, then it induces an isomorphism of the cohomology rings with $\mathbb{Z}$-coefficients? If true, give a proof. If false, give a counterexample and prove that your example is indeed a counterexample.*

By universal coefficient theorem and five lemma, $f^* : H^k(X; \mathbb{Z}) \to H^k(Y; \mathbb{Z})$ is isomorphism for all k.
And since $f^* : H^*(X; \mathbb{Z}) \to H^*(Y; \mathbb{Z})$ is ring homomorphic.
So $f^* : H^*(X; \mathbb{Z}) \to H^*(Y; \mathbb{Z})$ is ring isomorphic.

Exer 6.7. *Prove that $\mathbb{RP}^n \times \mathbb{S}^2$ does not have an open cover consisting of k contractible open subsets if $k \leqslant n + 1$.*

Let $X = \mathbb{RP}^n \times \mathbb{S}^2$
By Kunneth formula, $H^*(X; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[a, b]/(a^{n+1}, b^2)$ with $|a| = 1, |b| = 2$.
So $a^n b \neq 0 \in H^*(X; \mathbb{Z}/2\mathbb{Z})$.
Suppose X can be covered by $k \leqslant n + 1$ contractible open subsets $U_1, \cdots, U_k$.
Consider cup product $H^*(X, U_1; \mathbb{Z}/2\mathbb{Z}) \times \cdots \times H^*(X, U_k; \mathbb{Z}/2\mathbb{Z}) \to H^*(X, X; \mathbb{Z}/2\mathbb{Z})$.
Since $H^*(X, U_i; \mathbb{Z}/2\mathbb{Z}) = H^*(X)$ and $H^*(X, X; \mathbb{Z}/2\mathbb{Z}) = 0$.
So any k-cup product $H^*(X; \mathbb{Z}/2\mathbb{Z})$ is zero, contraction!

Exer 6.8. *Recall a space X is called an H-space if there exists a point $e \in X$ and a continuous map $\mu : X \times X \to X$ such that the map*

$$X \to X \text{ defined by } x \mapsto \mu(e, x)$$

and the map

$$X \to X \text{ defined by } x \mapsto \mu(x, e)$$

are both homotopic to the identity map. Show that $\mathbb{CP}^n$ is not an H-space for any $1 \leqslant n < \infty$.

Let $p_1 : X \to X \times X, x \mapsto (x, e)$ and $p_2 : X \to X \times X, x \mapsto (e, x)$.
Then $p_1^* \circ \mu^* = p_2^* \circ \mu^* = \mathrm{Id}_{H^*(X)}$.
By Kunneth formula, $H^*(X \times X) = \mathbb{Z}[a, b]/\left(a^{n+1}, b^{n+1}\right)$ with $|a| = |b| = 2$.
So $p_1^*(a) = a, p_1^*(b) = 0, p_2^*(a) = 0, p_2^*(b) = b$.
Let $\mu^* : H^*(X) \to H^*(X \times X), c \mapsto ka + lb$ where c is the generator of $H^2(X)$.
Then $p_1^* \circ \mu^*(c) = ka, p_2^* \circ \mu^*(c) = lb$.
So $k, l = \pm 1$.
But $\mu^*(c^{n+1}) = (ka + lb)^{n+1} = (\pm a \pm b)^{n+1} \neq 0$, contradiction!

Exer 6.9. *X and Y be locally contractible, connected spaces with fixed basepoints. Let $X \vee Y$ be the wedge sum at the basepoints. Note the canonical inclusion $f : X \vee Y \to X \times Y$. Assume that X and Y have abelian fundamental groups. Show that the map f_* on fundamental groups exhibits $\pi_1(X \times Y)$ as the abelianization of $\pi_1(X \vee Y)$.*

Let $\gamma = (\gamma_1, \gamma_2)$ be a loop in $X \times Y$.
Consider $H : [0, 1]^2 \to X \times Y, (s, t) = (\gamma_1(s), \gamma_2(t))$.
Then $H(t, t) = \gamma(t)$, *i.e.* $[\gamma] = [H(-, 0)] + [H(0, -)] = f_*[\gamma_1] + f_*[\gamma_2] = f_*[\gamma_1 * \gamma_2]$.
So $f_* : \pi_1(X) * \pi_1(Y) \to \pi_1(X) \times \pi_1(Y)$ is abelianization.

7 Cohomology ring, cup product and cap product

Exer 7.1. *Show that any self homeomorphism of* $\mathbb{CP}^2$ *is orientation preserving.*

Since $H^*(\mathbb{CP}^2) = \mathbb{Z}[a]/(a^3)$ with $|a| = 2$.
Let $f^*(a) = ka$.
Then $f^*(a^2) = k^2a^2$, *i.e.* $\deg f = k^2 > 0$.
So f is orientation preserving.

Exer 7.2. *Let* $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ *be the 2-torus with the standard orientation, and let* $F : \mathbb{T}^2 \to \mathbb{T}^2$ *be a smooth map of degree* 1 *such that* $F \circ F = \mathrm{Id}$ *and* F *has no fixed points. Prove that the induced map* $F_* : H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^2)$ *is the identity.*

Since $H^*(\mathbb{T}^2) = \Lambda_{\mathbb{Z}}[a,b]/(a^2,b^2)$ with $|a| = |b| = 1$.
Let $F^*(a) = pa + qb, F^*(b) = ra + sb$.
Then $(F \circ F)^*(a) = (p^2 + qr)a + (pq + qs)b = a, (F \circ F)^*(b) = (rp + sr)a + (rq + s^2)b = b$ and $F^*(ab) = pra^2 + (ps - qr)ab + qsb^2 = (ps - qr)ab = ab$.
So $p^2 + qr = 1, pq + qs = 0, rp + sr = 0, rq + s^2 = 1, ps - qr = 1$.
If $q = 0$, then $p^2 = s^2 = ps = 1, r(p + s) = 0$.
So $p = s = \pm 1, r = 0$.
If $q \neq 0$, $p + s = 0, qr = 1 - p^2 = ps - 1$.
So $1 = -1$, contradiction!
And by Lefschetz fixed point theorem, $\tau(F) = 1 - (p + s) + 1 = 0$.
Therefore $p = s = 1, q = r = 0$.
By universal coefficient theorem, $F_* : H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^2)$ is identity.

Exer 7.3. *Show that the fundamental group of an* H*-space is abelian and that* $\mathbb{S}^{even}$ *is not an* H*-space.*

By exercise 1.6, the fundamental group is abelian.
Let $X = \mathbb{S}^{2n}$, $p_1 : X \to X \times X, x \mapsto (x, e)$ and $p_2 : X \to X \times X, x \mapsto (e, x)$.
Then $p_1^* \circ \mu^* = p_2^* \circ \mu^* = \mathrm{Id}_{H^*(X)}$.
By Kunneth formula, $H^*(X \times X) = \mathbb{Z}[a,b]/\left(a^2, b^2\right)$ with $|a| = |b| = 2n$.
So $p_1^*(a) = a, p_1^*(b) = 0, p_2^*(a) = 0, p_2^*(b) = b$.
Let $\mu^* : H^*(X) \to H^*(X \times X), c \mapsto ka + lb$ where c is the generator of $H^{2n}(X)$.
Then $p_1^* \circ \mu^*(c) = ka, p_2^* \circ \mu^*(c) = lb$.
So $k, l = \pm 1$.
But $\mu^*(c^2) = k^2a^2 + 2klab + l^2b^2 = 2klab \neq 0$, contradiction!

Exer 7.4. *(1) Show that* $\mathbb{S}^2 \times \mathbb{S}^2$ *and* $\mathbb{CP}^2 \vee \mathbb{S}^2$ *are not homotopy equivalent.*

(2) Let $\eta : \mathbb{S}^3 \to \mathbb{S}^2 \cong \mathbb{CP}^1$ *be the attaching map of the* 4*-cell in* $\mathbb{CP}^2$*. Show that* η *is not null-homotopic.*

(1) See exercise 6.4 ($\mathbb{CP}^1 \cong \mathbb{S}^2$).

(2) Suppose η is null-homotopic.

Then $\mathbb{CP}^4 \simeq \mathbb{S}^4 \vee \mathbb{S}^2$ by the homotopy from η to constant map.

But $a^2 \neq 0$ for $a \in H^2(\mathbb{CP}^4)$, contradiction!

Exer 7.5. *Let* $\overline{\mathbb{CP}^2}$ *denote the complex projective plane with the opposite orientation.*

(1) Show that $\mathbb{S}^2 \times \mathbb{S}^2$ *and* $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ *have the same cohomology groups but different cohomology rings.*

(2) Show that $\left(\mathbb{S}^2\times\mathbb{S}^2\right)\#\overline{\mathbb{CP}^2}$ *and* $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ *have the same cohomology rings.*

(1) $H^*(\mathbb{S}^2\times\mathbb{S}^2)=\mathbb{Z}[a,b]/(a^2,b^2)$ and $H^*\left(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2\right)=\mathbb{Z}[c,d]/(c^3,d^3,cd,c^2+d^2)$ with $|a|=|b|=|c|=|d|=2$.

Suppose there exists a ring isomorphism $f:H^*\left(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2\right)\to H^*(\mathbb{S}^2\times\mathbb{S}^2)$.

Let $f(c)=pa+qb, f(d)=ra+sb$.

Then $f(c^2)=2pqab\neq\pm ab$, contradiction!

(2) $H^*\left(\left(\mathbb{S}^2\times\mathbb{S}^2\right)\#\overline{\mathbb{CP}^2}\right)=\mathbb{Z}[a,b,c]/(a^2,b^2,c^3,ab+c^2,ac,bc)$ with $|a|=|b|=|c|=2$.

$H^*\left(\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}\right)=\mathbb{Z}[x,y,z]/(x^3,y^3,z^3,xy,yz,zx,x^2+y^2,x^2+z^2)$ with $|x|=|y|=|z|=2$.

Take $f:H^*\left(\left(\mathbb{S}^2\times\mathbb{S}^2\right)\#\overline{\mathbb{CP}^2}\right)\to H^*\left(\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}\right)$ with $f(a)=x+y, f(b)=x+z, f(c)=x+y+z$.

Then $f(a^2)=x^2+y^2=0, f(b^2)=x^2+z^2=0, f(c^2)=x^2+y^2+z^2=-x^2, f(ab)=x^2, f(ac)=x^2+y^2=0, f(bc)=x^2+z^2=0$.

Hence f is a ring isomorphism.

Exer 7.6. *Let* $\mathbb{RP}^m$ *be the projective space of dimension* m *over the real number* $\mathbb{R}$*. Describe the cohomology ring* $H^*(\mathbb{RP}^m\times\mathbb{RP}^n;\mathbb{Z})$*.*

WLOG, we assume $m\leqslant n$.
If $i=0$, then $H_i(\mathbb{RP}^m\times\mathbb{RP}^n;\mathbb{Z})=\mathbb{Z}$.
If $0<i<m$, then

$$\begin{aligned}&H_i(\mathbb{RP}^m\times\mathbb{RP}^n;\mathbb{Z})\\&=\left(\bigoplus_{j=0}^{i}H_j(\mathbb{RP}^m;\mathbb{Z})\otimes H_{i-j}(\mathbb{RP}^n;\mathbb{Z})\right)\oplus\left(\bigoplus_{j=0}^{i-1}\operatorname{Tor}\left(H_j(\mathbb{RP}^m;\mathbb{Z}),H_{i-1-j}(\mathbb{RP}^n;\mathbb{Z})\right)\right)\\&=\begin{cases}(\mathbb{Z}/2\mathbb{Z})^{\frac{i}{2}} & 2|i\\(\mathbb{Z}/2\mathbb{Z})^{\frac{i+3}{2}} & 2\nmid i\end{cases}\end{aligned}$$

If $i=m<n$, then

$$H_i(\mathbb{RP}^m\times\mathbb{RP}^n;\mathbb{Z})=\begin{cases}(\mathbb{Z}/2\mathbb{Z})^{\frac{m}{2}} & 2|m\\(\mathbb{Z}/2\mathbb{Z})^{\frac{m+1}{2}}\oplus\mathbb{Z} & 2\nmid m\end{cases}$$

If $m<i<n$, then

$$\begin{aligned}&H_i(\mathbb{RP}^m\times\mathbb{RP}^n;\mathbb{Z})\\&=\left(\bigoplus_{j=0}^{m}H_j(\mathbb{RP}^m;\mathbb{Z})\otimes H_{i-j}(\mathbb{RP}^n;\mathbb{Z})\right)\oplus\left(\bigoplus_{j=0}^{m}\operatorname{Tor}\left(H_j(\mathbb{RP}^m;\mathbb{Z}),H_{i-1-j}(\mathbb{RP}^n;\mathbb{Z})\right)\right)\\&=\begin{cases}(\mathbb{Z}/2\mathbb{Z})^{\lceil\frac{m}{2}\rceil} & 2|i\\(\mathbb{Z}/2\mathbb{Z})^{\lfloor\frac{m}{2}\rfloor+1} & 2\nmid i\end{cases}\end{aligned}$$

If $i=n>m$, then

$$H_i(\mathbb{RP}^m\times\mathbb{RP}^n;\mathbb{Z})=\begin{cases}(\mathbb{Z}/2\mathbb{Z})^{\lceil\frac{m}{2}\rceil} & 2|n\\(\mathbb{Z}/2\mathbb{Z})^{\lfloor\frac{m}{2}\rfloor}\oplus\mathbb{Z} & 2\nmid n\end{cases}$$

If $i = m = n$, then

$$H_i(\mathbb{RP}^m \times \mathbb{RP}^n; \mathbb{Z}) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\frac{m}{2}} & 2|m \\ (\mathbb{Z}/2\mathbb{Z})^{\frac{m-1}{2}} \oplus \mathbb{Z}^2 & 2\nmid m \end{cases}$$

If $n < i < m + n$, then

$$\begin{aligned} & H_i(\mathbb{RP}^m \times \mathbb{RP}^n; \mathbb{Z}) \\ = & \left(\bigoplus_{j=i-n}^{m} H_j(\mathbb{RP}^m; \mathbb{Z}) \otimes H_{i-j}(\mathbb{RP}^n; \mathbb{Z}) \right) \oplus \left(\bigoplus_{j=i-1-n}^{m} \operatorname{Tor}\left(H_j(\mathbb{RP}^m; \mathbb{Z}), H_{i-1-j}(\mathbb{RP}^n; \mathbb{Z})\right) \right) \\ = & \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\lceil \frac{m}{2} \rceil - \lfloor \frac{i-n}{2} \rfloor} & 2|i \\ (\mathbb{Z}/2\mathbb{Z})^{\lfloor \frac{m}{2} \rfloor - \lfloor \frac{i-n}{2} \rfloor + 1} & 2\nmid i \end{cases} \end{aligned}$$

If $i = m + n$, then

$$H_i(\mathbb{RP}^m \times \mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & 2\nmid m, 2\nmid n \\ 0 & 2|m \text{ or } 2|n \end{cases}$$

Remark 7.1. This problem is too complicated for me and I'm not sure whether my answer is true or not.

Exer 7.7. *Let $V \subset W$ be complex vector spaces of dimension k and n respectively, and $\mathbb{P}V \subset \mathbb{P}W$ the corresponding projective space of one-dimensional subspaces of V and W. Find the cohomology ring $H^*(X; \mathbb{Z})$ of the complement $X = \mathbb{P}W \backslash \mathbb{P}V$.*

Let $(z_0, \ldots, z_n)$ be the coordinate of W and $V = \{(z_0, \ldots, z_k, 0, \ldots, 0)\}$.
Then $X = \{[z_0 : \cdots : z_n] | z_{k+1} z_{k+2} \cdots z_n \neq 0\}$.
Consider $H : X \times [0, 1] \to X, ([z_0 : \cdots : z_n], t) = [tz_0 : \cdots : tz_k : z_{k+1} : \cdots : z_n]$.
Then $X \overset{H}{\simeq} \mathbb{CP}^{n-k-1}$.
So $H^*(X; \mathbb{Z}) \cong H^*(\mathbb{CP}^{n-k-1}; \mathbb{Z}) \cong \mathbb{Z}[a]/(a^{n-k})$ with $|a| = 2$.

Exer 7.8. *Show that for $m > n$ there does not exist an antipodal map $f : \mathbb{S}^m \to \mathbb{S}^n$, that is, a continuous map carrying antipodal points to antipodal points.*

Since $f : \mathbb{S}^m \to \mathbb{S}^n$ maps antipodal points to antipodal points.
So it induce a map $g : \mathbb{RP}^m \to \mathbb{RP}^n$ and $g_* : \pi_1(\mathbb{RP}^m) \to \pi_1(\mathbb{RP}^n)$ is nontrivial.
Therefore $g_* : H_1(\mathbb{RP}^m; \mathbb{Z}/2\mathbb{Z}) \to H_1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$ is nontrivial.
By universal coefficient theorem, $g^* : H^1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^m; \mathbb{Z}/2\mathbb{Z})$ is nontrivial.
But since $H^*(\mathbb{RP}^m; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a]/(a^{m+1}), H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[b]/(b^{n+1})$.
So $g^*(b^{n+1}) = (g^*(b))^{n+1} \neq 0$, contradiction!

Exer 7.9. *For which natural number n is it the case that every continuous map from $\mathbb{CP}^n$ to itself has a fixed point? The same question for $\mathbb{RP}^n$.*

By exercise 2.13, n must be odd.
And for odd n, let $f : \mathbb{CP}^n \to \mathbb{CP}^n, [z_0 : \cdots : z_n] \mapsto [\bar{z}_1 : -\bar{z}_0 : \bar{z}_3 : -\bar{z}_2 : \cdots : \bar{z}_n : -\bar{z}_{n-1}]$.
Since $\bar{z}_{2i+1} z_{2i+1} = |z_{2i+1}|^2 \geqslant 0$ while $-\bar{z}_{2i} z_{2i} = -|z_{2i}|^2 \leqslant 0$.
So $f([z_0 : \cdots : z_n]) = [z_0 : \cdots : z_n]$ only if $z_0 = \cdots = z_n = 0$, which is impossible.
For $\mathbb{RP}^n$, using $\mathbb{Z}/2\mathbb{Z}$-coefficient cohomology, we can also deduce that n must be odd.
And for odd n, let $f : \mathbb{RP}^n \to \mathbb{RP}^n, [x_0 : \cdots : x_n] \mapsto [x_1 : -x_0 : \cdots : x_n : -x_{n-1}]$.
Then f has no fixed point similarly to the $\mathbb{CP}^n$ case.

Exer 7.10. *Show that the inclusion $i : \mathbb{RP}^{n-1} \to \mathbb{RP}^n$ of a projective hyperplane admits no retraction (*i.e.* there is no map $r : \mathbb{RP}^n \to \mathbb{RP}^{n-1}$ with $r \circ i = \mathrm{Id}$). Is the same true for $i : \mathbb{RP}^n \to \mathbb{CP}^n$ the inclusion as the subset of real points.*

Since $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a]/(a^{n+1})$ with $|a| = 1$.
So $i^*(a) = a$.
Suppose such r exists.
Then $r^*(a) = i^*(r^*(a)) = a$.
But $r^*(a^n) = a^n \neq 0$, contradiction!
For $i : \mathbb{RP}^n \to \mathbb{RP}^n$, $i^*(b) = a^2$ where $b \in H^2(\mathbb{CP}^n;\mathbb{Z}/2\mathbb{Z})$ is the generator.
Suppose such r exists.
Then $r^*(a) = 0$ since $H^1(\mathbb{CP}^n;\mathbb{Z}/2\mathbb{Z}) = 0$.
But $i^*(r^*(a^2)) = i^*(0) = 0 \neq a^2$, contradiction!

Exer 7.11. *For each of the properties listed below either show the existence of a CW complex X with those properties or else show that there doesn't exists such a CW complex.*

(1) The fundamental group of X is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$.

(2) The cohomology ring $H^*(X;\mathbb{Z})$ is isomorphic to the graded ring freely generated by one element in degree 2.

*(3) The CW complex is finite (*i.e. *is built out of a finite number of cells) and the cohomology ring of its universal covering space is not finitely generated.*

(4) The cohomology ring $H^*(X;\mathbb{Z})$ is generated by its elements of degree 1 *and has nontrivial elements of degree* 100.

(1) By exercise 1.4, this statement is true.

(2) Let $X = \mathbb{CP}^\infty$.
Then $H^*(X;\mathbb{Z}) = \mathbb{Z}[a]$ with $|a| = 2$.

(3) Let $X = \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$.
Then $\tilde{X}$ is a complete quadtree wedging a sphere on each vertex.
So $H^2(\tilde{X})$ is not finitely generated.

(4) Consider $\mathbb{T}^{100} = (\mathbb{S}^1)^{100}$.
Then $H^*(\mathbb{T}^{100}) = \Lambda_{\mathbb{Z}}[x_1, \ldots, x_{100}]/(x_1^2, \ldots, x_{100}^2)$ with $|x_1| = \cdots = |x_{100}| = 1$.
So $x_1 \cdot s x_{100}$ is non trivial and of degree 100.

8 Orientation on manifolds, fundamental class, mapping degree

Exer 8.1. *Prove that the real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd.*

$$H_n(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & 2 \nmid n \\ 0 & 2 | n \end{cases}$$

So $\mathbb{RP}^n$ is orientable iff n is odd.

Exer 8.2. *Let M be a closed smooth n-manifold.*

*(1) Does there always exists a smooth map $f : M \to \mathbb{S}^n$ from M into the n-sphere, such that f is essential (*i.e. *f is not homotopic to a constant map)? Justify your answer.*

(2) Same question, replacing $\mathbb{S}^n$ by the n-torus $\mathbb{T}^n$.

(1) Let p be a point in M and U is a chart around p such that $U \cong \mathbb{D}^n$.

Take f as the quotient map $M \to M/(M\backslash U)$.

Then $M/(M\backslash U) \cong \bar{U}/\partial U \cong \overline{\mathbb{D}^n}/\mathbb{S}^{n-1} = \mathbb{S}^n$.

And f is essential since $f_* : H^n(M; \mathbb{Z}/2\mathbb{Z}) \to H^n(\mathbb{S}^n; \mathbb{Z}/2\mathbb{Z})$ is nontrivial.

(2) Take $M = \mathbb{S}^n$, Suppose such f exists.

Then $f_* : \pi_1(\mathbb{S}^n) \to \pi_1(\mathbb{T}^n)$ is trivial since $\pi_1(\mathbb{S}^n) = 0$.

So f can lift to $\tilde{f} : \mathbb{S}^n \to \mathbb{R}^n$.

And since $\tilde{f}$ is null-homotopic.

Therefore f is null-homotopic, contradiction!

Exer 8.3. *Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and $f : \mathbb{S}^n \to \mathbb{S}^n$ a continuous map. Assume that the degree of f is odd. Show that there exists $x_0 \in \mathbb{S}^n$ such that $f(-x_0) = -f(x_0)$.*

Suppose $f(-x_0) \neq -f(x_0)$ for all $x_0 \in \mathbb{S}^n$, consider

$$H : \mathbb{S}^n \times [0,1] \to \mathbb{S}^n, (x,t) \mapsto \frac{tf(x) + (1-t)f(-x)}{|tf(x) + (1-t)f(-x)|}.$$

Then $f \overset{H}{\simeq} g$ where $g = H\left(-, \frac{1}{2}\right)$.
And since $g(x) = g(-x)$.
So g factor through $\pi : \mathbb{S}^n \to \mathbb{RP}^n$.
Therefore $\deg g = \deg \pi \cdot k = 2k$ or 0 for some integer k.
But $\deg f$ is odd, contradiction!

Exer 8.4. *Show that there is no degree one map from $\mathbb{S}^2 \times \mathbb{S}^2$ to $\mathbb{CP}^2$.*

Suppose such map $f : M \to N$ exists.
Since $H^*(\mathbb{S}^2 \times \mathbb{S}^2) = \mathbb{Z}[a,b]/(a^2, b^2), H^*(\mathbb{CP}^2) = \mathbb{Z}[c]/(c^3)$ with $|a| = |b| = |c| = 2$.
Let $f^*(c) = ka + lb$.
Then $c^2 = k^2a^2 + 2klab + l^2b^2 = 2klab$.
So $\deg f = 2kl \neq 1$, contradiction!

Exer 8.5. *Let $\Sigma \hookrightarrow \mathbb{R}^3$ be an embedded, closed surface of genus 3. Define the Gauss map, $\phi : \Sigma \to \mathbb{S}^2$ by associating to each point $x \in \Sigma$, the out ward pointing, unit normal vector.*
Let ω denote the volume form (total volume$= 4\pi$) on $\mathbb{S}^2$. Compute $\int_\Sigma \phi^*\omega$.

Since ϕ pull back $T\mathbb{S}^2$ to $T\Sigma$.
So $e(T\Sigma) = e(\phi^*T\mathbb{S}^2) = \phi^*e(T\mathbb{S}^2)$.
Therefore $-4 = \chi(\Sigma) = \chi(\mathbb{S}^2) \cdot \deg(f) = 2\deg(f)$.
Hence $\int_\Sigma \phi^*\omega = \deg(f) \cdot \int_{\mathbb{S}^2} = -8\pi$.

Remark 8.1. You can also compute it by Gauss-Bonnet theorem directly.

Exer 8.6. *Let X be a compact orientable surface of genus 2, and let $\phi : X \to X$ be a fixed-point-free homeomorphism of finite order.*

(1) Show that ϕ is of order 2 and orientation-reversing.

(2) Show that such homeomorphism of surfaces of genus 2 do exist.

(1) Let k be the order of ϕ.

Then it induce a k-sheeted covering map $\pi : X \to Y$.

So $k\chi(Y) = \chi(X) = -2$, *i.e.* $k = 2$ and $\chi(Y) = -1$.

Hence Y is non-orientable, *i.e.* ϕ is orientation-reversing.

(2) Since $X = \mathbb{T}^2 \# \mathbb{T}^2$.

So ϕ can be given by the reflection across the circle that swap two torus.

Exer 8.7. *Compute the degree of the self-map $g \to g^q$ defined on the group* SU(2)*. q here is an arbitrary (including negative) integer.*

Since SU(2) is homeomorphic to $\mathbb{S}^3$ by

$$\begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix} \mapsto x_1 + ix_2 + jx_3 + kx_4$$

and the group structure is given by quaternion multiplication.

Let $x = \cos\theta + v\sin\theta \in \mathbb{S}^3$ with $v^2 = -1$.
If $q > 0$, then $x^q = \cos(q\theta) + v\sin(q\theta)$.
So $f^{-1}(x)$ has q different points $\cos\phi + v\sin\phi$ with $q\phi = \theta$.
And since quaternion multiplication is orientation-preserving.
Therefore $\deg f = q$.
If $q = -1$, then $x^q = \cos(-\theta) + v\sin(-\theta)$.
So $f^{-1}(x) = \{f(x)\}$ and f is orientation-reversing.
Therefore $\deg f = -1 = q$.
Hence $\deg f = q$ for any $q \in \mathbb{Z}$.

Exer 8.8. *Let M be a smooth compact connected n-manifold (without boundary), and let $\Phi : M \to M$ be a smooth map that is smoothly homotopic to identity. Show that Φ must be surjective.*

Suppose Φ is not surjective with $f^{-1}(p) = \varnothing$.
Since $H_n(M; \mathbb{Z}/2\mathbb{Z}) \cong H_n(M, M \backslash \{p\}; \mathbb{Z}/2\mathbb{Z})$ is generated by the fundamental class.
So $H_n(M \backslash \{p\}; \mathbb{Z}/2\mathbb{Z}) = 0$.
But $\Phi_* : H_n(M) \to H_n(M)$ must be identity, contradiction!

9 Poincaré duality theorem and other duality theorems

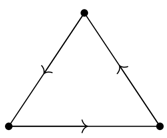
Exer 9.1. *Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$ and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.*

Let $H_1(M) = \mathbb{Z}[a, b]$ with a represent $\pi(\{x = 0\})$ and b represent $\pi(\{y = 0\})$.
Then a corresponds to $\mathrm{d}x$ and b corresponds to $-\mathrm{d}y$.
And since $S = \pi(\{3x - 7y = 0\})$ is represented by $3a + 7b$.
So it corresponds to $3\mathrm{d}x - 7\mathrm{d}y$.

Exer 9.2. *(1) Show that $\mathbb{CP}^{2n}$ cannot be the boundary of a compact manifold.*

(2) Show that $\mathbb{CP}^3$ is the boundary of some compact manifold.

(3) Let X be the topological space obtained by identifying all three sides of a triangle as shown in the diagram



Compute the homology groups of X with coefficients in $\mathbb{Z}$ and with coefficients in $\mathbb{Z}/3\mathbb{Z}$. Is X a closed manifold?

(1) By exercise 5.7, $\chi(\mathbb{CP}^{2n}) = 2n+1$ must be even, which is impossible.

(2) Consider fibration $\mathbb{S}^2 \to \mathbb{CP}^3 \to \mathbb{HP}^1$.

Then the $\mathbb{D}^3$-bundle associated to the above $\mathbb{S}^2$-bundle on $\mathbb{HP}^1$ has boundary $\mathbb{CP}^3$.

(3) By cellular homology,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So $H_0(X;\mathbb{Z}) = \mathbb{Z}, H_1(X;\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}, H_2(X;\mathbb{Z}) = 0$.

And by universal coefficient theorem, $H_0(X;\mathbb{Z}/3\mathbb{Z}) = H_1(X;\mathbb{Z}/3\mathbb{Z}) = H_2(X;\mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$.

Exer 9.3. *Let M, N be closed, connected, oriented 3-manifolds with fundamental groups $\pi_1(M) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(N) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Find all integral homology groups of M, N and rational homology of $M \times N$.*

Since $H_1(M;\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, H_1(N;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
By universal coefficient theorem, $H^1(M;\mathbb{Z}) = 0, H^1(N;\mathbb{Z}) = \mathbb{Z}$.
So by Poincaré duality, $H_2(M;\mathbb{Z}) = 0, H_2(N;\mathbb{Z}) = \mathbb{Z}$.
And since M, N are orientable.
Therefore $H_0(M;\mathbb{Z}) = H_3(M;\mathbb{Z}) = H_0(N;\mathbb{Z}) = H_3(N;\mathbb{Z}) = \mathbb{Z}$.
Moreover, $H_0(M\times N;\mathbb{Q}) = \mathbb{Q}, H_1(M\times N;\mathbb{Q}) = \mathbb{Q}, H_2(M\times N;\mathbb{Q}) = \mathbb{Q}, H_3(M\times N;\mathbb{Q}) = \mathbb{Q}^2$, $H_4(M\times N;\mathbb{Q}) = \mathbb{Q}, H_5(M\times N;\mathbb{Q}) = \mathbb{Q}, H_6(M\times N;\mathbb{Q}) = \mathbb{Q}$.

Exer 9.4. *Let X and Y be compact connected oriented 3-manifolds, with $\pi_1(X) = \mathbb{Z}/5\mathbb{Z}$ and $\pi_1(Y) = \mathbb{Z}/10\mathbb{Z}$. Find $H_n(X\times Y;\mathbb{Z})$ for all $n \geqslant 0$.*

Similar as last exercise, $H_1(M;\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}, H_1(N;\mathbb{Z}) = \mathbb{Z}/10\mathbb{Z}, H_2(M;\mathbb{Z}) = H_2(N;\mathbb{Z}) = 0$, $H_0(M;\mathbb{Z}) = H_3(M;\mathbb{Z}) = H_0(N;\mathbb{Z}) = H_3(N;\mathbb{Z}) = \mathbb{Z}$.
So $H_0(X\times Y;\mathbb{Z}) = \mathbb{Z}, H_1(X\times Y;\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}\oplus\mathbb{Z}/10\mathbb{Z}, H_2(X\times Y;\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}, H_3(X\times Y;\mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/5\mathbb{Z}, H_4(X\times Y;\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}\oplus\mathbb{Z}/10\mathbb{Z}, H_5(X\times Y;\mathbb{Z}) = 0, H_6(X\times Y;\mathbb{Z}) = \mathbb{Z}$.

Exer 9.5. *Let M be a compact orientable manifold of dimension $4n+2$, then $\dim H_{2n+1}(M;\mathbb{R})$ is even.*

Poincaré duality gives a nondegenerated anti-symmetric bilinear form on $H_{2n+1}(M;\mathbb{R})$.
Let A be the represented matrix.
Then $|A| = (-1)^{\dim H_{2n+1}(M;\mathbb{R})}|-A^T| = (-1)^{\dim H_{2n+1}(M;\mathbb{R})}|A| \neq 0$.
So $\dim H_{2n+1}(M;\mathbb{R})$ is even.

Exer 9.6. *(1) Draw the universal cover $\tilde{X}$ of the space $X = \mathbb{S}^2 \cup C$ where C is the chord joining the North to the South pole.*

(2) What is the relation of $\pi_2(X)$ and $\pi_2(\tilde{X})$?

(3) Compute $\pi_2 X$.

(1) $\tilde{X}$ is given by a line wedging a sphere on each integer point.

(2) $\pi_2(X) = \pi_2(\tilde{X})$ since $\tilde{X}$ is the covering space of X

(3) By Hurewicz theorem, $\pi_2(X) = H_2(\tilde{X}) = \mathbb{Z}^\infty$.

Exer 9.7. *Find an example of a compact 4-manifold M such that*

$$\dim_{\mathbb{Q}} H_1(M;\mathbb{Q}) \neq \dim_{\mathbb{Q}} H_3(M;\mathbb{Q}).$$

Take $M = \mathbb{RP}^2 \times \mathbb{T}^2$.
Then by Kunneth theorem, $H_1(M;\mathbb{Q}) = \mathbb{Q}^2, H_3(M;\mathbb{Q}) = 0$.

Exer 9.8. *Let $f : M \to N$ be continuous map between two closed, oriented, connected manifolds. Suppose $\deg(f) \neq 0$. Show $b_i(M) \geqslant b_i(N)$ for all i.*

For $a \in H_i(N;\mathbb{Q})$, by Poincaré duality $a = \alpha \frown [N]$ for some $\alpha \in H^i(N;\mathbb{Q})$, so

$$f_*\left(\frac{1}{\deg(f)} f^*(\alpha) \frown [M]\right) = \frac{1}{\deg(f)}\alpha \frown f_*[M] = \alpha \frown [N] = a.$$

Therefore $f_* : H_i(M;\mathbb{Q}) \to H_i(N;\mathbb{Q})$ is surjective, *i.e.* $b_i(M) \geqslant b_i(N)$.

Exer 9.9. *Let M be a closed, connected $(n-1)$-dimensional manifold with a smooth embedding $M \hookrightarrow \mathbb{S}^n$. Show that M is orientable and that $\mathbb{S}^n \backslash M$ has exactly two components. Does this result hold if we allow M to have a boundary?*

By Alexander duality, $H^{n-1}(M;\mathbb{Z}) = \tilde{H}_0(\mathbb{S}^n\backslash M;\mathbb{Z})$ is torsion free.
So by universal coefficient theorem, $H_{n-2}(M;\mathbb{Z})$ is torsion free.
And since $H_{n-1}(M;\mathbb{Z}/2\mathbb{Z}) = (H_{n-1}(M;\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(H_{n-2}(M;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.
Hence $H_{n-1}(M;\mathbb{Z}) = \mathbb{Z}$, *i.e.* M is orientable and $\mathbb{S}^n\backslash M$ has exactly two components.
If M have boundary, let M be the Mobius band with $n = 3$.
Then M is non-orientable and $\mathbb{S}^n\backslash M$ is connected.

10 Basic properties of higher homotopy groups

Exer 10.1. *Determine all of the possible degrees of maps $\mathbb{S}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$.*

By exercise 8.2, $f : \mathbb{S}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ is null-homotopic, *i.e.* $\deg f = 0$.

Remark 10.1. It can also be done by cohomology since f^* must be 0.

Exer 10.2. *(1) Compute the cohomology ring of the unitary group $U(n)$.*

(2) Compute π_1 and π_2 of $U(n)$.

Consider fiber bundle $U(n-1) \to U(n) \to \mathbb{S}^{2n-1}$.

(1) By Leray-Hirsch theorem, $H^*(U(n)) = H^*(U(n-1)) \otimes H^*(\mathbb{S}^{2n-1})$.
So $H^*(U(n)) = \Lambda_{\mathbb{Z}}[x_1, \ldots, x_n]$ with $|x_i| = 2i - 1$.

(2) There is a long exact sequence

$$\cdots \longrightarrow \pi_{k+1}(\mathbb{S}^{2n-1}) \longrightarrow \pi_k(U(n-1)) \longrightarrow \pi_k(U(n)) \longrightarrow \pi_k(\mathbb{S}^{2n-1}) \longrightarrow \cdots$$

So $\pi_1(U(n)) \cong \pi_1(U(n-1))$ for $n > 1$, $\pi_2(U(n)) \cong \pi_2(U(n-1))$ for $n > 2$.
Therefore $\pi_1(U(n)) \cong \pi_1(U(1)) = \mathbb{Z}, \pi_2(U(1)) = 0$ and $\pi_2(U(n)) \cong \pi_2(U(2))$ for $n \geqslant 2$.
And by the long exact sequence, $\pi_2(U(2)) = 0$, *i.e.* $\pi_2(U(n)) = 0$ for $n \geqslant 2$.

Exer 10.3. *Let Δ^4 be the standard 4-simplex:*

$$\Delta^4 = \left\{(x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 \middle| x_i \geqslant 0, \sum x_i = 1\right\}.$$

Let X be its 2-skeleton:

$$X = \{(x_0, x_1, x_2, x_3, x_4) \in \Delta^4 | x_i = x_j = 0 \text{ for some } 0 \leqslant i < j \leqslant 4\}.$$

Compute $\pi_1(X)$ and $\pi_2(X)$.

The 1-skeleton of X has 5 vertices and 10 edges, *i.e.* it is homotopy to $\bigvee^6 \mathbb{S}^1$.
And since X has six 2-cells each attach to 3 different edges.
So X is simply connected, *i.e.* $\pi_1(X) = 1$.
By Hurewicz theorem, $\pi_2(X) = H_2(X) = \mathbb{Z}^4$ by exercise 4.8.

Exer 10.4. *Find the homotopy groups $\pi_k(\mathbb{CP}^n)$ of complex projective n-space*

(1) for $k \leqslant 2n+1$;

(2) for $k = 2n+2$.

Consider fiber bundle $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{CP}^n$.
Then we have long exact sequence

$$\cdots \longrightarrow \pi_k(\mathbb{S}^1) \longrightarrow \pi_k(\mathbb{S}^{2n+1}) \longrightarrow \pi_k(\mathbb{CP}^n) \longrightarrow \pi_{k-1}(\mathbb{S}^1) \longrightarrow \cdots$$

So $\pi_k(\mathbb{CP}^n) \cong \pi_k(\mathbb{S}^{2n+1})$ for $k > 2$ and $\pi_1(\mathbb{CP}^n) = 0, \pi_2(\mathbb{CP}^n) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.
Hence $\pi_k(\mathbb{CP}^n) = 0$ for $2 \leqslant k \leqslant 2n$ and $\pi_{2n+1}(\mathbb{CP}^n) \cong \pi_{2n+1}(\mathbb{S}^{2n+1}) = \mathbb{Z}$.
Moreover, $\pi_{2n+2}(\mathbb{CP}^n) \cong \pi_{2n+2}(\mathbb{S}^{2n+1}) \cong \pi_4(\mathbb{S}^3) = \mathbb{Z}/2\mathbb{Z}$.

11 Fiber bundles and long exact sequence of homotopy groups

Exer 11.1. *Classify all vector bundles over the circle $\mathbb{S}^1$ up to isomorphism.*

Since $\mathbb{S}^1 \cong [0,1]/\{0,1\}$ and vector bundles on $[0,1]$ are all trivial.
So we only need to consider how the vector bundle identify on 0 and 1.
And since the gluing data is given by an element in $\mathrm{GL}(n,\mathbb{R})$, which has two component.
Suppose E^A, E^B are isomorphism by f, where $A, B \in \mathrm{GL}(n,\mathbb{R})$.
Then $f_0 = \mathrm{Id}$ and $f_1 = B^{-1}A$.
So f give a path from Id to $B^{-1}A$, *i.e.* A and B are path connected.
Hence there are two different vector bundle up to isomorphism.

Exer 11.2. *Are the following statements true:*

(1) Any map $\mathbb{S}^{2024} \to \mathbb{RP}^{2024}$ is null-homotopic.

(2) Any map $\mathbb{S}^{2024} \to \mathbb{CP}^{1012}$ is null-homotopic.

(1) Consider the quotient map $\pi : \mathbb{S}^{2024} \to \mathbb{RP}^{2024}$.
Then π is not null-homotopic since $f^* : H_{2024}(\mathbb{S}^{2024}; \mathbb{Z}/2\mathbb{Z}) \to H_{2024}(\mathbb{RP}^{2024}; \mathbb{Z}/2\mathbb{Z}), a \mapsto a$.

(2) By exercise 10.4, $\pi_{2024}(\mathbb{CP}^{1012}) = 0$.
So the map $\mathbb{S}^{2024} \to \mathbb{CP}^{1012}$ must be null-homotopic.

Exer 11.3. *Let M be a smooth closed manifold. Show that M is a fiber bundle over $\mathbb{S}^1$ if and only if there is a closed, nowhere vanishing 1-form on M.*

If M is a fiber bundle over $\mathbb{S}^1$ with $\pi : M \to \mathbb{S}^1$.
Then $\pi^*(\mathrm{d}\theta)$ gives a closed, nowhere vanishing 1-form on M.
Conversely, Consider a closed, nowhere vanishing 1-form ω on M.
Consider the generators $x_1, \ldots, x_n$ of $H_1(M;\mathbb{R})$ and let $\omega = c_1x_1 + \cdots + c_nx_n + \mathrm{d}f$ where f is a differentiable function.
Since we can approx c_i by rational numbers.
So WLOG, we may assume c_i are rational and ω is still nowhere vanishing.

Moreover, we assume c_i are integers by multiplying their common denominator.
Now we take a point $p \in M$ and let

$$f : M \to \mathbb{S}^1, q \mapsto \int_p^q \omega \mod 1.$$

This is well-defined and has no singular points.
So M is a fiber bundle over $\mathbb{S}^1$.

Exer 11.4. *Let X be a compact connected manifold and $M \subset X$ a compact connected embedded hypersurface (*i.e.* $\dim M = \dim X - 1$). Prove that*

(1) If the normal bundle of M in X is not orientable, then $\pi_1(X) \neq 0$.

(2) If M is non-empty, orientable and $X \backslash M$ is connected then $H^1_{dR}(X) \neq 0$.

(1) Since $N(M)$ is not orientable.

So there exists a loop l that intersects with M only once.

Therefore the cup product of Poincaré dual of $[M]$ and $[l]$ is $[X]$ with $\mathbb{Z}/2\mathbb{Z}$-coefficient.

Hence $[l] \in H_1(X)$ is nontrivial, *i.e.* $\pi_1(X) \neq 0$.

(2) Since $X \backslash M$ is connected.

So there exists a loop l that intersects with M only once.

Similar as (1), $[M] \in H_{n-1}(X)$ is nontrivial, *i.e.* $H^1_{dR}(X) \neq 0$ by Poincaré duality.

Exer 11.5. *Let M be a smooth compact manifold, and suppose that there is a smooth map $F : M \to \mathbb{S}^1$ whose derivative is non-zero at every point. Prove that the de Rham cohomology group $H^1_{dR}(M)$ is non-zero.*

Let $\omega = F^*(\mathrm{d}\theta)$.
Then ω is a closed, nowhere vanishing 1-form on M.
Suppose $\omega = \mathrm{d}f$ where f is a differentiable function.
Then $\omega = 0$ at the maximum point of f, contradiction!
Hence ω is nontrivial closed 1-form, *i.e.* $H^1_{dR}(M)$ is non-zero.

Exer 11.6. *Let X be a compact n-dimensional manifold, and $Y \subset X$ a closed submanifold of dimension m. Show that the Euler characteristic $\chi(X \backslash Y)$ of the complement of Y in X is given by*

$$\chi(X \backslash Y) = \chi(X) + (-1)^{n-m-1}\chi(Y).$$

Does the same result hold if we do not assume that X is compact, but only that the Euler characteristic of X and Y are finite?

Let T be the tubular bundle of Y.
Then by MV-sequence,

$$\cdots \longrightarrow H_{k+1}(X) \longrightarrow H_k(T \backslash X) \longrightarrow H_k(T) \oplus H_k(X \backslash Y) \longrightarrow H_k(X) \longrightarrow \cdots$$

So $\chi(X \backslash Y) + \chi(T) = \chi(X) + \chi(T \backslash Y)$.
And since $T \backslash Y$ is the trivial $\mathbb{S}^{n-m-1}$-bundle of Y.
So $\chi(X \backslash Y) = \chi(X) + (-1)^{n-m-1}\chi(Y)$.

12 Whitehead theorem, Hurewicz theorem, CW approximation theorem

Exer 12.1. *Let M be a simply-connected, closed 3-manifold. Show that M is homotopy equivalent to $\mathbb{S}^3$.*

Since M is simply connected.
So M is orientable and $H^1(X;\mathbb{Z}) = 0$ by universal coefficient theorem.
And by Poincaré duality, $H_2(X;\mathbb{Z}) = 0$.
By Hurewicz theorem, $\pi_2(X) = 0$ and $\pi_3(X) = \mathbb{Z}$.
So there exists a map $f : \mathbb{S}^3 \to M$ that generates $\pi_3(X)$.
Moreover, by relative Hurewicz theorem, $\pi_n(C_f, \mathbb{S}^3) = 0$ where C_f is the mapping cylinder.
Hence by Whitehead theorem, $M \simeq \mathbb{S}^3$.

Exer 12.2. *Let M be a connected, orientable manifold of dimensional $n \geqslant 2$ and let $f : \mathbb{S}^n \to M$ be a map of degree 1. Show that f must be a homotopy equivalence.*

Suppose $a \in H^k(M;\mathbb{Z})$ is nontrivial.
Then there exist $b \in H^{n-k}(M;\mathbb{Z})$ such that $\langle a \smile b, [M] \rangle = 1$.
But $f*(a) = f^*(b) = 0$, *i.e.* $f^*(a \smile b) = 0$, contradiction!
So $H_k(M;\mathbb{Z})$ are trivial for $k \neq 0, n$.
Hence similar as the last exercise, f is a homotopy equivalence.

Exer 12.3. *Consider the space $X = \mathbb{R}^3 \backslash (\{(0,0,z) | z \in \mathbb{R}\} \cup \{(1,0,0)\})$. Compute $\pi_2(X)$.*

$X \simeq \mathbb{S}^1 \times \mathbb{S}^1 \underset{\mathbb{S}^1 \times \{0\}}{\cup} \mathbb{D}^1 \simeq \mathbb{S}^1 \times \mathbb{S}^1 / (\mathbb{S}^1 \times \{0\}) \simeq \mathbb{S}^2/(p \sim q) \simeq \mathbb{S}^2 \vee \mathbb{S}^1$.
So its universal covering space is a line wedging a sphere on each integer point.
Hence $\pi_2(X) = \pi_2(\tilde{X}) \cong H_2(\tilde{X}) = \mathbb{Z}^\infty$.

Exer 12.4. *For $k \geqslant 2$, show that $\pi_k(\mathbb{S}^k \vee \mathbb{S}^k) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Since $\mathbb{S}^k \vee \mathbb{S}^k$ is simply connected and $\tilde{H}_i(\mathbb{S}^k \vee \mathbb{S}^k) = 0$ for $i < k$.
By Hurewicz theorem, $\pi_k(\mathbb{S}^k \vee \mathbb{S}^k) \cong H_k(\mathbb{S}^k \vee \mathbb{S}^k) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Exer 12.5. *Let X be a CW complex such that the reduced homology $\tilde{H}_*(X;\mathbb{Z}) = 0$. Consider the suspension*

$$\Sigma X = ([0,1] \times X)/\sim,$$

where $\sim$ is generated by

$$(0,x) \sim (0,x') \text{ and } (1,x) \sim (1,x').$$

Show that ΣX is contractible.

By exercise 4.5, $\tilde{H}_n(\Sigma X) = \tilde{H}_{n-1}(X) = 0$.
And since $\Sigma X = CX \underset{X}{\cup} CX$ is simply connected by Van Kampen theorem.
Hence by Hurewicz theorem, $\pi_n(\Sigma X) = 0$ for all n, *i.e.* ΣX is contractible.

Exer 12.6. *Let $f : X \to Y$ be a maps between connected CW complexes. Prove the following two results:*

(1) Suppose f induces isomorphism on $\pi_1(-)$ and isomorphism on $H_k(-;\mathbb{Z})$ for all k. Then f is a homotopy equivalence.

(2) Suppose X, Y are both n-dimensional. And f induces isomorphism on $\pi_k(-)$ for all $k \leqslant n$. Then f is a homotopy equivalence.

(1) Since $\pi_1(C_f, X) = 0$ and $H_n(C_f, X) = 0$ for all n.

By relative Hurewicz theorem, $\pi_n(C_f, X) = 0$ where C_f is the mapping cylinder of f.

So $f_* : \pi_k(X) \to \pi_k(Y)$ is isomorphism for all k.

Hence by Whitehead theorem, f is homotopy equivalence.

(2) Since $\pi_k(C_f, X) = 0$ for $k \leqslant n$ and X, Y are n-dimensional.

So $H_k(C_f, X) = 0$ for all k.

By (1), f is homotopy equivalence.

13 Inverse function theorem, implicit function theorem, submanifolds

Exer 13.1. *Suppose $\pi : M_1 \to M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_pM_1 \to T_{\pi(p)}M_2$ is a vector space isomorphism.*

(1) Show that if M_1 is compact, then π is a covering space projection.

(2) Given an example where M_2 is compact but $\pi : M_1 \to M_2$ is not a covering space (but has the π_* isomorphism property).

(1) Take $p \in M_2$ and $\tilde{p} \in \pi^{-1}(p)$.

By inverse function theorem, there is a neighborhood $U_{\tilde{p}}$ of $\tilde{p}$ such that $\pi\big|_{U_{\tilde{p}}} : U_{\tilde{p}} \to \pi(U_{\tilde{p}})$ is a diffeomorphism.

So $\pi^{-1}(p) \cap U_{\tilde{p}} = \{\tilde{p}\}$.

And since M_1 is compact.

Therefore $\pi^{-1}(p)$ must be finite, and we can take

$$U_p = \bigcap_{\tilde{p} \in \pi^{-1}(p)} U_{\tilde{p}}.$$

Then every component of U_p maps to $\pi(U_p)$ diffeomorphically.

Hence π is a covering space projection.

(2) take $M_1 = (0, 2), M_2 = \mathbb{S}^1$ with $\pi : M_1 \to M_2, t \mapsto e^{2\pi i t}$.

Then π_* is isomorphism but π is not covering map since $\pi^{-1}(1) = \{1\}$.

Exer 13.2. *Let M be a compact smooth manifold of dimension d. Prove that there exists some $n \in \mathbb{Z}_+$ such that M can be regularly embedded in the Euclidean space $\mathbb{R}^n$.*

Take a atlas of M consists of k charts U_i with coordinates $(x_1^i, \ldots, x_d^i)$, $1 \leqslant i \leqslant k$.
Consider a partition of unit $\{\rho_i\}$.
Let $f = (\rho_1, \rho_1 x_1^1, \ldots, \rho_1 x_d^1, \ldots, x_k x_d^k)$.
Then $f : M \to \mathbb{R}^{(d+1)k}$ is a regularly embedded.

Exer 13.3. *Let M be a smooth connected manifold and $f : M \to M$ be an smooth map such that $f \circ f = f$. Show that the image set $f(M)$ is a smooth submanifold in M.*

Since $(\mathrm{d}f)^2 = \mathrm{d}f$ in $f(M)$.
So $\mathrm{d}f$ is diagonalizable and its eigenvalues are 1 and 0.
Therefore $\mathrm{tr}(\mathrm{d}f) = \mathrm{rank}(\mathrm{d}f) \in \mathbb{Z}$.
And since $\mathrm{tr}(\mathrm{d}f)$ is connected in M.
Hence $\mathrm{rank}(\mathrm{d}f)$ is constant in $f(M)$ and by rank theorem $f(M)$ is a smooth submanifold.

Exer 13.4. *Let $\mathbb{T}^2 = \{(z,w) \in \mathbb{C}^2 | |z| = |w| = 1\}$ be the torus. Define a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ by $f(z,w) = (zw^3, w)$. Prove that f is a diffeomorphism.*

f is a bijection with inverse $f^{-1}(z,w) = (zw^{-3}, w)$.
Moreover, the local coordinate of $\mathbb{T}^2$ is given by $(\theta_1, \theta_2) \mapsto \left(e^{i\theta_1}, e^{i\theta_2}\right)$.
Then locally, $f : (\theta_1, \theta_2) \mapsto (\theta_1 + 3\theta_2, \theta_2)$.
So $\mathrm{d}f$ is locally represented as

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Hence f is a diffeomorphism.

Exer 13.5. *The unit tangent bundle of $\mathbb{S}^2$ is the subset*

$$T^1(\mathbb{S}^2) = \left\{(p,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \middle| \|p\| = \|v\| = 1, (p,v) = 0\right\}.$$

Show that it is a smooth submanifold of the tangent bundle $T\mathbb{S}^2$ of $\mathbb{S}^2$ and $T^1(\mathbb{S}^2)$ is diffeomorphic to $\mathbb{RP}^3$.

$T\mathbb{S}^2 = \{(p,v) \in \mathbb{R}^3 \times \mathbb{R}^3 | \|p\| = 1, (p,v) = 0\}$.
Consider $f : T\mathbb{S}^2 \to \mathbb{R}, (p,v) \mapsto |v|$.
Then $\mathrm{d}f = 2v_1 \mathrm{d}v_1 + 2v_2 \mathrm{d}v_2 + 2v_3 \mathrm{d}v_3$ at (p,v) with $v = (v_1, v_2, v_3)$.
So for $(p,v) \in T^1(\mathbb{S}^2)$, $\mathrm{d}f \neq 0$, *i.e.* $T^1(\mathbb{S}^2) = f^{-1}(1)$ is a smooth submanifold.
Moreover, $(p, v, p \times v)$ forms an element in $\mathrm{SO}(3)$.
Hence $t^1(\mathbb{S}^2)$ is diffeomorphic to $\mathrm{SO}(3) \cong \mathbb{RP}^3$.

Exer 13.6. *Show that*

$$Q^n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \middle| \sum_{i=1}^n x_i^4 = 1\right\}$$

is a differentiable manifold.

Consider

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i^4.$$

Then its derivative

$$\mathrm{d}f = \sum_{i=1}^n 4x_i^3 \mathrm{d}x_i.$$

So for $(x_1, \ldots, x_n) \in Q^n$, $\mathrm{d}f \neq 0$, *i.e.* $Q^n = f^{-1}(1)$ is a smooth manifold.

Exer 13.7. *Let M be a compact, simply connected smooth manifold of dimension n, prove that there is no smooth immersion $f : M \to \mathbb{T}^n$, where $\mathbb{T}^n$ is n-torus.*

Since M is simply connected.
So f can lift to a smooth immersion $f : M \to \mathbb{R}^n$, let $f = (f_1, \ldots, f_n)$.
Then at the maximum point p of f_1, $\mathrm{d}f_1 = 0$.
Hence $\mathrm{d}f$ is not isomorphism at p, contradiction!

Exer 13.8. *Consider the manifold*

$$M = \left\{((x_1, \ldots, x_n), [y_1 : \cdots : y_n]) \in \mathbb{R}^n \times \mathbb{RP}^{n-1} \middle| x_i y_j = x_j y_i \text{ for all } i, j\right\},$$

and the projection map $\pi : M \to \mathbb{R}^n$ onto its $\mathbb{R}^n$-factor. Determine whether or not π is a submersion.

$\pi^{-1}(x_1, \ldots, x_n) = \{((x_1, \ldots, x_n), [x_1 : \cdots : x_n])\}$ for $(x_1, \ldots, x_n) \neq 0$.
And $\pi^{-1}(0) = \left\{(0, [y_1 : \cdots : y_n]) | [y_1 : \cdots : y_n] \in \mathbb{RP}^{n-1}\right\}$.
So $\mathrm{d}\pi(v) = 0$ at $(0, [y])$ for $v \in T_{[y]}\mathbb{RP}^{n-1}$.
Hence π is not submersion.

Exer 13.9. *Suppose Σ is a smooth compact embedded hypersurface (*i.e.* a codimension 1 submanifold) without boundary in $\mathbb{R}^n$ for $n \geqslant 3$. Show that Σ is orientable.*

Suppose Σ is non-orientable.
Then take a tubular neighborhood of Σ and so it is non-orientable.
By exercise 11.4, contradiction!

Exer 13.10. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = x^2 + y^2 - 1$.*

(1) Prove that $M = f^{-1}(0)$ is a two-dimensional embedded submanifold of $\mathbb{R}^3$.

(2) For $a, b, c \in \mathbb{R}$, consider the vector field

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}.$$

For which values of a, b, c is X tangent to M at the point $(1, 0, 1)$?

(1) $\mathrm{d}f = 2x\mathrm{d}x + 2y\mathrm{d}y \neq 0$ in M.

So M is a two-dimensional embedded submanifold of $\mathbb{R}^3$.

(2) For Since X is tangent to M.

So $0 = \mathrm{d}f(X) = 2xa + 2yb$, *i.e.* $a = 0$ and b, c can be arbitrary real number.

Exer 13.11. *Let M be a smooth manifold with smooth boundary ∂M and N be a smooth manifold without boundary. Assume that $f : M \to N$ is smooth (this includes smoothness at points of ∂M) so that the tangent map $\mathrm{d}f_x : T_x M \to T_{f(x)}N$ is well-defined (including at points of ∂M). Let $y \in N$ be a regular value for both f and $f\big|_{\partial M}$.*

(1) Show that $M_1 = f^{-1}(y)$, if not empty, is a smooth submanifold with boundary in M such that the boundary $\partial M_1 = \left(f\big|_{\partial M}\right)^{-1}(y) = M_1 \cap \partial M$ is a submanifold of ∂M.

(2) If we only assume that y is a regular value for f but not for $f\big|_{\partial M}$, does the conclusion of (1) still hold?

(1) Since y is a regular value for $f\big|_{\mathring{M}}$ and $f\big|_{\partial M}$.

So $M_1 \cap \mathring{M}$ is a smooth submanifold in $\mathring{M}$ and $M_1 \cap \partial M$ is a smooth submanifold in ∂M.

And for $p \in M_1 \cap \partial M$, consider a chart $U \cong \mathbb{H}^m = \{(x_1, \ldots, x_m) | x_1 \geqslant 0\}$ such that $f(U) \cong \mathbb{R}^n$ and $p = 0$ in the coordinates.

Let F be the smooth extension of $f\big|_U$ to an open set of $\mathbb{R}^m$.

Then $\mathrm{d}F_p$ has full rank and moreover, it has full rank when $x_1 = 0$.

So by shuffle the coordinates $x_2, \ldots, x_m$ and implicit function theorem, M_1 is locally given by $\{(x_1, \ldots, x_{m-n}, g(x_1, \ldots, x_{m-n})) | x_1 \geqslant 0\}$ for some $g : \mathbb{R}^{m-n} \to \mathbb{R}^n$.

Hence M_1 is a smooth submanifold with boundary in M.

(2) Take $M = \mathbb{D}^2, N = \mathbb{R}$ and $f : M \to N, (x, y) \mapsto x$.

Then $M_1 = f^{-1}(1) = \{(1, 0)\}$ is not a smooth submanifold with boundary in M.

Exer 13.12. *Let $V_k(\mathbb{R}^n)$ denote the space of k-tuples of orthonormal vectors in $\mathbb{R}^n$. Show that $V_k(\mathbb{R}^n)$ is a manifold and compute its dimension.*

Consider $f : \mathbb{R}^{n\times k} \to S(k), X \mapsto X^T X$ where $S(k)$ is the space of $k \times k$ symmetric matrix.
Then $f^{-1}(I) = V_k(\mathbb{R}^n)$.
And since $(\mathrm{d}f)(Z) = Z^T X + X^T Z$ at $X \in V_k(\mathbb{R}^n)$.
So for $A \in S(k)$, $(\mathrm{d}f)\left(\frac{1}{2}XA\right) = \frac{1}{2}(A^T + A) = A$.
Therefore $\mathrm{d}f$ is surjective, *i.e.* I is a regular value.
Hence $V_k(\mathbb{R}^n)$ is a manifold and its dimension is $nk - \frac{k(k+1)}{2}$.

Exer 13.13. *The Grassmannian* $\mathrm{Gr}(k, n)$ *is the set of all k-dimensional subspaces of $\mathbb{R}^n$. Explicitly construct the structure of a smooth manifold on* $\mathrm{Gr}(k, n)$ *using atlases. What is its dimension?*

Let V be a k-dimensional subspace of $\mathbb{R}^n$ and $X = (v_1, \ldots, v_k)$ is its basis.
Then for any $A \in \mathrm{GL}_k(\mathbb{R})$, XA is a basis of V.
Since $\mathrm{rank}(X) = k$.
So there exists indices $J = (i_1, \ldots, i_k)$ of rows such that X_J is invertible.
Consider $U_J = \{V | V$ has a basis X such that X_J is invertible$\}$ and

$$\varphi_J : U_J \to \mathbb{R}^{(n-k)\times k}, V \mapsto X_{J^c}$$

where X is the basis of V such that $X_J = I$.
Hence (U_J, φ_J) form an atlas of $\mathrm{Gr}(k, n)$, and the dimension is $(n - k)k$.

Exer 13.14. *Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?*

Let $\hat{n}$ be the unit normal vectors of the line and d be the signed distance from 0.
In other words, the line is given by $\{x | x \cdot \hat{n} = d\}$.
Then $M \cong \mathbb{S}^1 \times \mathbb{R}/\sim$ with $(d, \theta) \sim (-d, \theta + \pi)$.
Hence M is diffeomorphic to the open Mobius band, *i.e.* it is non-orientable.

14 Sard's theorem, transversality

Exer 14.1. *Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n - 2$. Prove that the complement $\mathbb{R}^n - M$ is connected and simply connected.*

Consider a map $f : \mathbb{S}^1 \to \mathbb{R}^n \backslash M$, we want to show it is null-homotopic.
WLOG, assume f is smooth.
Since f is null-homotopic in $\mathbb{R}^n$, *i.e.* it can extend to $H : \mathbb{D}^2 \to \mathbb{R}^n$.
Let $F : \mathbb{R}^n \times \mathbb{D}^2 \to \mathbb{R}^n, (a, x) \mapsto H(x) + a$.
Then F is surjective and $\mathrm{d}F$ is surjective everywhere.
By implicit function theorem, $F^{-1}(M)$ is a smooth submanifold with dimension $m + 2 < n$.
Consider the projection $\pi : F^{-1}(M) \to \mathbb{R}^n, (a, x) \mapsto a$.
By Sard theorem, $a \in \mathbb{R}^n$ is regular value almost everywhere.
So $\pi(F^{-1}(M))$ must be zero measure.
Take small $a \notin \pi(F^{-1}(M))$ such that $\{ta + f(x) | t \in [0, 1], x \in \mathbb{S}^1\} \cap M = \varnothing$.
Then $f \simeq f + a \overset{H+a}{\simeq} C$ in $\mathbb{R}^n \backslash M$, where C is the constant map at $H(0) + a$.
Hence f is null-homotopic, *i.e.* $\mathbb{R}^n \backslash M$ is simply connected.

Exer 14.2. *Let $M^m \subset \mathbb{R}^n - \{0\}$ be a compact smooth submanifold of dimension m. Show that M is transverse to almost all k-dimensional linear subspaces in $\mathbb{R}^n$. (Here "almost all" means that the set of subspaces that are not transverse to M has measure zero.)*

Let $E = \{(v, V) \in \mathbb{R}^n \times \mathrm{Gr}(n,k) | v \in V^\perp\}, Z = \{(0, V)\} \subset E$.
Consider $f : M \times \mathrm{Gr}(n,k) \to E, (p, V) \mapsto (\pi_{V^\perp}(p), V)$.
Then $f^{-1}(Z) = \{(p, V) | p \in M \cap V\}$.
So $T_{f(p,V)}Z + \mathrm{d}f_{(p,V)}\left(T_{(p,V)}(M \times \mathrm{Gr}(n,k))\right) = T_V\mathrm{Gr}(n,k) + V^\perp = T_{f(p,V)}E$.
Therefore f is transverse to Z.
By parametric transversality theorem, $f_V : M \to E$ transverse to Z for almost all V.
So $f_V : M \to V^\perp$ transverse to $\{0\}$ for almost all V.
Thus for $p \in f_V^{-1}(0) = M \cap V$, $\mathrm{d}f_V$ is surjective at p.
Hence $T_pM + V = \mathbb{R}^n$, *i.e.* M is transverse to almost all k-dimensional linear subspaces.

Exer 14.3. *Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that, for almost all $a \in \mathbb{R}^n$, the translate $X + a = \{x + a | x \in X\}$ intersects Y transversely.*

Let $f : X \times \mathbb{R}^n \to \mathbb{R}^n, (x, a) \mapsto x + a$.
Then $\mathrm{d}f(v, w) = v + w$ is surjective everywhere.
By implicit function theorem, $f^{-1}(Y) = (X + a) \cap Y$ is a smooth submanifold.
Consider $\pi : f^{-1}(Y) \to \mathbb{R}^n, (x, a) \mapsto a$.
By Sard theorem, $a \in \mathbb{R}^n$ is regular value almost everywhere.
For regular value a and $x \in f^{-1}(Y)$, $\mathrm{d}\pi : T_{(x,a)}f^{-1}(Y) \to T_a\mathbb{R}^n, (v, w) \mapsto w$ is surjective.
So any $w \in \mathbb{R}^n$, there exists $v \in T_xX$ such that $v + w \in T_{x+a}Y$.
Hence $T_xX + T_{x+a}Y = \mathbb{R}^n$, *i.e.* $X + a$ intersects Y transversely.

Exer 14.4. *Let $M^m \subset \mathbb{R}^n$ be a closed connected submanifold of dimension M.*

(1) Show that $\mathbb{R}^n - M^m$ is connected when $m \leqslant n - 2$.

(2) When $m = n - 1$ show that $\mathbb{R}^n - M^m$ is disconnected.

(1) Consider two points $p, q \in \mathbb{R}^n \backslash M$, let C be a circle in $\mathbb{R}^n$ containing p, q.

Then by the last exercise, $C + a$ intersects M transversely for almost all $a \in \mathbb{R}^n$.

And since $\dim C = 1, \dim M \leqslant n - 2$.

So $(C + a) \cap M = 0$.

Take small enough a such that the lines p to $p + a$ and q to $q + a$ are disjoint with M.

Hence p, q is connected.

(2) Consider a map $f : \mathbb{S}^1 \to \mathbb{R}^n$ such that its image $C = f(\mathbb{S}^1)$ intersecting M transversely.

Since the fundamental class of C and M in $\mathbb{R}^n$ are trivial.

So the mod 2 intersection number $I_2(C, M) = 0$.

Fix a point $p \in \mathbb{R}^n \backslash M$ and $\mathbb{R}^n \backslash M$ can be separated into

$$U_1 = \{q \in \mathbb{R}^n \backslash M | \text{a path from } p \text{ to } q \text{ meets } M \text{ odd times}\},$$

$$U_2 = \{q \in \mathbb{R}^n \backslash M | \text{a path from } p \text{ to } q \text{ meets } M \text{ even times}\}.$$

For two paths γ_1, γ_2 from p to q, $\gamma_1 * \gamma_2^{-1}$ form a map $\mathbb{S}^1 \to \mathbb{R}^n$.

So $I_2(\gamma_1, M) = I_2(\gamma_2, M)$ in mod 2 sense.

Hence $\mathbb{R}^n \backslash M$ is disconnected.

15 Lie groups, Lie algebras, left/right-invariant vector fields

Exer 15.1. *Let* $\mathrm{SO}(3)$ *be the set of all* 3×3 *real matrices* A *with determinant* 1 *and satisfying* $A^T A = I$*, where* I *is the identity matrix and* A^T *is the transpose of* A*. Show that* $\mathrm{SO}(3)$ *is a smooth manifold, and find its fundamental group. You need to prove your claims.*

By exercise 13.12, $\mathrm{SO}(3)$ is a component of manifold $V_3(\mathbb{R}^3)$.
And since it is homeomorphic to $\mathbb{RP}^3$ (we have proven this several times previous).
So $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$.

Exer 15.2. *Let* $\mathrm{U}(n)$ *be the group of* $n \times n$ *unitary matrices, and* $\mathrm{O}(n)$ *be the group of* $n \times n$ *orthogonal matrices. Let* $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) | \det A = 1\}$ *be the special unitary group and* $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) | \det A = 1\}$ *be the special orthogonal group. All* $\mathrm{U}(n), \mathrm{SU}(n), \mathrm{O}(n), \mathrm{SO}(n)$ *are Lie groups with natural manifold structures.*

(1) Compute the dimensions of $\mathrm{SU}(n)$ *and* $\mathrm{SO}(n)$*.*

(2) Compute the fundamental groups of $\mathrm{SU}(n)$ *and* $\mathrm{SO}(n)$ *(*$n > 2$*).*

(1) Since $\mathrm{SU}(n)$ acts on $\mathbb{S}^{2n-1}$ transitively with stabilizer $\mathrm{SU}(n-1)$.

So $\dim \mathrm{SU}(n) = \dim \mathrm{SU}(n-1) + 2n - 1 = 3 + 5 + \cdots + 2n - 1 = n^2 - 1$.

Similarly, $\dim \mathrm{SO}(n) = \dim \mathrm{SO}(n-1) + n - 1 = 1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2}$.

(2) Since we have fiber bundle $\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to \mathbb{S}^{2n-1}$.

so there is a long exact sequence

$$\cdots \longrightarrow \pi_2(\mathbb{S}^{2n-1}) \longrightarrow \pi_1(\mathrm{SU}(n-1)) \longrightarrow \pi_1(\mathrm{SU}(n)) \longrightarrow \pi_1(\mathbb{S}^{2n-1}) \longrightarrow 0$$

Hence $\pi_1(\mathrm{SU}(n)) \cong \pi_1(\mathrm{SU}(n-1)) \cong \cdots \cong \pi_1(\mathrm{SU}(1)) = 0$ since $\dim \mathrm{SU}(1) = 0$.

Similarly, $\pi_1(\mathrm{SO(n)}) \cong \pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geqslant 3$

And since $\mathrm{SO}(2) \cong \mathbb{S}^1, \dim \mathrm{SO}(1) = 0$.

So $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$ and $\pi_1(\mathrm{SO}(1)) = 0$.

Exer 15.3. *Prove* $\mathbb{S}^{2n}$*,* $n \geqslant 1$ *is not a Lie group.*

Since $\chi(\mathbb{S}^{2n}) = 1 + (-1)^{2n} = 2$.
By Poincaré-Hopf theorem, $\mathbb{S}^{2n}$ has no nowhere vanishing vector space.
So $\mathbb{S}^{2n}$ is not a Lie group.

Exer 15.4. *Consider the matrix group*

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Prove that its exponential map from its Lie algebra

$$\exp : \mathfrak{g} \to G$$

is a diffeomorphism. Provide a generalization and justification of the statement.

By definition,

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & a & d \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, d, c \in \mathbb{R} \right\}$$

And the exponential map

$$\exp\left(\begin{bmatrix} 0 & a & d \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & a & d + \frac{ac}{2} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

So $\exp : \mathfrak{g} \to G$ is linear and bijective.
Hence exp is a diffeomorphism.

Generalization: for any simply connected nilponent Lie group G with Lie algebra $\mathfrak{g}$, its exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism.

Proof. Since $\mathfrak{g}^{(n)} = 0$ for some n.
So the BCH formula $\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \cdots\right)$ has finite terms.
Therefore we can define a Lie group structure on $\mathfrak{g}$ with $X \cdot Y = \mathrm{BCH}(X, Y)$.
And by Lie second theorem, $\mathfrak{g}_{BCH}$ is isomorphic to G as Lie group.
Hence $\exp : \mathfrak{g} \to G$ is diffeomorphism. □

Exer 15.5. *Let $X \in \mathfrak{k} := \mathrm{Lie}(K)$ be a real vector field on a compact connected smooth manifold M with an effective action of a compact real Lie group K. By choosing a K-invariant real symplectic form ω on M, assume $f \in C^\infty(M)_\mathbb{R}$ is such that $\mathrm{d}f = i_X\omega$. Show that the value $\max_M f - \min_M f$ is independent of the choice of ω as far as ω defines the same de Rham cohomology class $[\omega]$.*

Take $\omega' \in [\omega]$ with $\omega - \omega' = \mathrm{d}\eta$ and let $\mathrm{d}g = i_X\omega'$.
WLOG, assume η is K-invariant.
Consider a critical point p of f.
Then $\mathrm{d}f = i_X\omega = 0$ at p, *i.e.* $X_p = 0$ since ω is nondegenerate.
So $\mathrm{d}g = i_X\omega' = 0$ at p, *i.e.* the critical points of f is the same as those of g.
Moreover, $\mathrm{d}(f - g) = i_X\mathrm{d}\eta = L_X(\eta) - \mathrm{d}(\eta(X)) = -\,\mathrm{d}(\eta(X))$.
Therefore $f - g = -\eta(X) + C = -\eta(0) + C = C$ at critical points, where C is a constant.
Hence $\max_M f - \min_M f = \max_M g - \min_M g$.

Exer 15.6. *Suppose G is a compact Lie group with Lie algebra $\mathfrak{g}$. Consider an element $g \in G$, and let $\mathfrak{c} \subset \mathfrak{g}$ be the subalgebra $\mathfrak{c} = \{X | \mathrm{Ad}_g(X) = X\}$. Show there exists some $\varepsilon > 0$ such that for all $X \in \mathfrak{g}$ with $|X| < \varepsilon$, there exists $Y \in \mathfrak{c}$ such that $g\exp(X)$ is conjugate to $g\exp(Y)$.*

Let $f : G \times \mathfrak{c} \to G, (h, Y) \mapsto g^{-1}hg\exp(Y)h^{-1}$.
Then we want to show $\mathrm{d}f$ is surjective at $(e, 0)$.
At $(e, 0)$,

$$\begin{aligned} \mathrm{d}f(X, Y) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(g^{-1}\exp(tX)g\exp(0)e + g^{-1}eg\exp(tY)e + g^{-1}eg\exp(0)\exp(-tX)\right) \\ &= g^{-1}Xg + Y - X \end{aligned}$$

Since G is a compact Lie group, *i.e.* it has a bi-invariant metric.
So Ad_g is orthogonal, *i.e.* it is diagonalizable.
Therefore $\mathrm{rank}(\mathrm{Id} - \mathrm{Ad}_g) = \mathrm{rank}(\mathrm{Id} - \mathrm{Ad}_g)^2$.
Thus $\{\mathrm{d}f(X, Y) | X \in \mathfrak{g}, Y \in \mathfrak{c}\} = \ker(\mathrm{Id} - \mathrm{Ad}_g) + \mathrm{im}(\mathrm{Id} - \mathrm{Ad}_g) = \mathfrak{g}$.
Hence $\mathrm{d}f$ is surjective.

Exer 15.7. *Define the set*

$$H = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

(1) Equip H with a C^∞ differentiable structure so that it is diffeomorphic to $\mathbb{R}^2$.

(2) Show that H is a Lie group under matrix multiplication.

(3) Show that

$$\left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \right\}$$

forms a basis of left-invariant vector fields of the associated Lie algebra.

(1) just map (x, y) to the corresponding matrix.

(2)

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_2 & y_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_1 + x_2 & y_1 + y_2 + x_1 x_2 \\ 0 & 1 & x_1 + x_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -x & x^2 - y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}$$

So the multiplication is given by

$$(x_1, y_1, x_2, y_2) \mapsto (x_1 + x_2, y_1 + y_2 + x_1 x_2)$$

and inverse is given by

$$(x, y) \mapsto (-x, x^2 - y),$$

these maps are obviously smooth.

Hence H is a Lie group.

(3) Let $g = (x, y), h = (a, b)$.

Since

$$\left(\frac{\partial}{\partial y}\right)_h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So

$$(L_g)_* \left(\frac{\partial}{\partial y}\right)_h = \begin{bmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left(\frac{\partial}{\partial y}\right)_{gh}.$$

And

$$\left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)_h = \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$(L_g)_* \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)_h = \begin{bmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & a + x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)_{gh}.$$

Hence $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ are left-invariant.

Exer 15.8. *Let*

$$G = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 1 \end{bmatrix} : x, y \in \mathbb{R}_+, z \in \mathbb{R} \right\}$$

(1) Show that G is a Lie subgroup of $\mathrm{GL}_3(\mathbb{R})$.

(2) Prove that

$$\left\{ x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, y\frac{\partial}{\partial z} \right\}$$

forms a basis for the left-invariant vector fields on G.

(3) Determine the structure of the Lie algebra $\mathfrak{g}$ of G in terms of the basis of (b).

(1)

$$\begin{bmatrix} x_1 & 0 & 0 \\ 0 & y_1 & z_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 & 0 & 0 \\ 0 & y_2 & z_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1x_2 & 0 & 0 \\ 0 & y_1y_2 & y_1z_2 + z_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & -y^{-1}z \\ 0 & 0 & 1 \end{bmatrix}$$

And since G is obviously closed.

So by closed-subgroup theorem, G is a Lie subgroup of $\mathrm{GL}_3(\mathbb{R})$.

(2) Let $g = (x, y, z), h = (a, b, c)$.

Since

$$\left(x\frac{\partial}{\partial x}\right)_h = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \left(y\frac{\partial}{\partial y}\right)_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}, \left(y\frac{\partial}{\partial z}\right)_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}.$$

So

$$(L_g)_* \left(x\frac{\partial}{\partial x}\right)_h = \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} xa & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left(x\frac{\partial}{\partial x}\right)_{gh}$$

$$(L_g)_* \left(y\frac{\partial}{\partial y}\right)_h = \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left(y\frac{\partial}{\partial y}\right)_{gh}$$

$$(L_g)_* \left(y\frac{\partial}{\partial z}\right)_h = \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & yb \\ 0 & 0 & 0 \end{bmatrix} = \left(y\frac{\partial}{\partial z}\right)_{gh}$$

And since dimension of left-invariant vector fields is $\dim G = 3$.
Hence $\left\{ x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, y\frac{\partial}{\partial z} \right\}$ is a basis of left-invariant vector fields.

(3)

$$\left[x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right] = x\frac{\partial y}{\partial x}\frac{\partial}{\partial y} - y\frac{\partial x}{\partial y}\frac{\partial}{\partial x},$$

$$\left[x\frac{\partial}{\partial x}, y\frac{\partial}{\partial z}\right] = x\frac{\partial y}{\partial x}\frac{\partial}{\partial z} - y\frac{\partial x}{\partial z}\frac{\partial}{\partial x},$$

$$\left[y\frac{\partial}{\partial y}, y\frac{\partial}{\partial z}\right] = y\frac{\partial y}{\partial y}\frac{\partial}{\partial z} - y\frac{\partial y}{\partial z}\frac{\partial}{\partial y} = y\frac{\partial}{\partial z}.$$

$\mathfrak{g}$ is span by X, Y, Z with $[X, Y] = [X, Z] = 0, [Y, Z] = Z$.

Exer 15.9. *Prove that the tangent bundle of* $\mathrm{O}(n)$ *is trivial.*

$\mathrm{O}(n)$ is a Lie group with $\mathfrak{o}(n) = \{X | X + X^T = 0\}$.
So its tangent bundle $T\mathrm{O}(n) \cong \mathfrak{o}(n) \times \mathrm{O}(n)$ is trivial.

Exer 15.10. *(1) Show that the Lie group* $\mathrm{SL}_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) | \det(A) = 1\}$ *is diffeomorphic to* $\mathbb{S}^1 \times \mathbb{R}^2$.

(2) Show that the Lie group $\mathrm{SL}_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) | \det(A) = 1\}$ *is diffeomorphic to* $\mathbb{S}^3 \times \mathbb{R}^3$.

(1) Let e_1, e_2 be the standard basis of $\mathbb{R}^2$.
Consider $f : \mathrm{SL}_2(\mathbb{R}) \to (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}, A \mapsto (v_1, \pi_{v_1}(v_2))$ where $v_i = Ae_i$.
Then $f^{-1}(v_1, t) = A$ with $A(e_1) = v_1, A(e_2) = v_1^{\perp} + tv_1$, where $v_1^{\perp} \perp v_1$ and $\left|v_1^{\perp}\right| = |v_1|^{-1}$.
So $\det(A) = 1$, *i.e.* $A \in \mathrm{SL}_2(\mathbb{R})$.
Hence f is a well-defined diffeomorphism $\mathrm{SL}_2(\mathbb{R}) \cong (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R} \cong \mathbb{S}^1 \times \mathbb{R}^2$.

(2) Similarly, consider $\mathrm{SL}_2(\mathbb{C}) \to (\mathbb{C}^2 \backslash \{0\}) \times \mathbb{C}, A \mapsto (v_1, \pi_{v_1}(v_2))$.
Then f is a well-defined diffeomorphism $\mathrm{SL}_2(\mathbb{C}) \cong (\mathbb{C}^2 \backslash \{0\}) \times \mathbb{C} \cong \mathbb{S}^3 \times \mathbb{R}^3$

16 Integral curves and flows, Lie derivatives, Lie brackets, Frobenius theorem

Exer 16.1. *Let M be a smooth manifold of dimension n, and $X_1, \ldots, X_k$ be k everywhere linearly independent smooth vector fields on an open set $U \subset M$ satisfying that $[X_i, X_j] = 0$ for $1 \leqslant i, j \leqslant k$. Prove that for any point $p \in U$ there is a coordinate chart (V, y^i) with $p \in V \subset U$ and coordinates $\{y^1, \ldots, y^n\}$ such that $X_i = \frac{\partial}{\partial y^i}$ on V for each $1 \leqslant i \leqslant k$.*

Take a chart (V, φ, x^i) around p such that $\varphi(p) = 0$ and $(X_i)_p = \frac{\partial}{\partial x^i}\big|_0$.
Let θ_t^i be the flow of X_i and consider the map

$$f : \mathbb{R}^n \to \mathbb{R}^n, (y^1, \ldots, y^n) \mapsto \theta_{y^1}^1 \circ \theta_{y^2}^2 \circ \cdots \circ \theta_{y^k}^k \left(0, \ldots, 0, y^{k+1}, \ldots, y^n\right).$$

Notice that the order of flow can be adjusted freely since $[X_i, X_j] = 0$.
So for $i \leqslant k$,

$$\begin{aligned}
\left.\frac{\partial f}{\partial y^i}\right|_0 &= \frac{\partial}{\partial y^i}\left(\theta_{y^1}^1 \circ \theta_{y^2}^2 \circ \cdots \circ \theta_{y^k}^k\right)\left(0, \ldots, 0, y^{k+1}, \ldots, y^n\right)\Big|_0 \\
&= \frac{\partial}{\partial y^i}\theta_{y^i}^i \left(\theta_{y^1}^1 \circ \cdots \circ \hat{\theta}_{y^i}^i \circ \cdots \circ \theta_{y^k}^k \left(0, \ldots, 0, y^{k+1}, \ldots, y^n\right)\right)\Big|_0 \\
&= \left.\frac{\partial}{\partial y^i}\theta_{y^i}^i\right|_0 = (X_i)_p = \left.\frac{\partial}{\partial x^i}\right|_0
\end{aligned}$$

And for $i > k$,

$$\begin{aligned}
\left.\frac{\partial f}{\partial y^i}\right|_0 &= \frac{\partial}{\partial y^i}\left(\theta_{y^1}^1 \circ \theta_{y^2}^2 \circ \cdots \circ \theta_{y^k}^k\right)\left(0, \ldots, 0, y^{k+1}, \ldots, y^n\right)\Big|_0 \\
&= \left.\frac{\partial}{\partial y^i}(0, \ldots, 0, y^i, 0, \ldots, 0)\right|_0 = \left.\frac{\partial}{\partial x^i}\right|_0
\end{aligned}$$

Therefore $\text{Jac}(f)_0 = I$.
By inverse function theorem, f is diffeomorphism locally around 0.
Moreover, $X_i = \frac{\partial}{\partial y^i}$ in this coordinate.

Exer 16.2. *Let $F : M \to N$ be a smooth map between two manifolds. Let X_1, X_2 be smooth vector fields on M and let Y_1, Y_2 be smooth vector fields on N. Prove that if $Y_1 = F_* X_1$ and $Y_2 = F_* X_2$, then $F_*[X_1, X_2] = [Y_1, Y_2]$, where $[\ ,\ ]$ is the Lie bracket.*

$$\begin{aligned}
[X_1, X_2](f \circ F) &= X_1 X_2 (f \circ F) - X_2 X_1 (f \circ F) \\
&= X_1((F_* X_2)(f) \circ F) - X_2((F_* X_1)(f) \circ F) \\
&= (F_* X_1)(F_* X_2)(f) \circ F - (F_* X_2)(F_* X_1)(f) \circ F \\
&= [Y_1, Y_2](f) \circ F
\end{aligned}$$

So $F_*[X_1, X_2] = [Y_1, Y_2]$.

Exer 16.3. *Let X and Y be smooth vector fields on a smooth manifold. Prove that the Lie derivative satisfies the identity $\mathcal{L}_X Y = [X, Y]$.*

Let θ_t be the flow of X, then

$$\begin{aligned}
\mathcal{L}_X Y(f) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\theta_{-t})_*(Yf) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 Y(f \circ \theta_{-t}) \circ \theta_t \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 Y(f \circ \theta_{-t}) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (Yf) \circ \theta_t \\
&= -YX(f) + X(Yf) = [X, Y](f)
\end{aligned}$$

Exer 16.4. *(1) Let f be a diffeomorphism group of a circle $\mathbb{S}^1$, assume f has no fixed point and it is generated by a smooth vector field, show that f must be conjugate to a rotation.*

(2) Show that there is a diffeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$, such that f can not be generated by a smooth vector field but it is arbitrarily closed the identity map $i : \mathbb{S}^1 \to \mathbb{S}^1$ in C^∞-topology.

(1) Let V be the vector field that generate f and θ_t be the flow of V.
Then $f(p) = \theta_t(p)$ for some $t \in \mathbb{R}$.
Since f has no fixed point, *i.e.* $\theta_t(p) \neq p$.
So V is nowhere vanishing.
Fixing p_0, the map $\mathbb{R} \to \mathbb{S}^1, t \mapsto \theta_t(p_0)$ is a covering map since $\theta_s \circ \theta_t = \theta_{s+t}$.
Therefore $\theta_t(p_0) = \theta_{t+T}(p_0)$ for some period T.
Let $g : \mathbb{S}^1 \to \mathbb{S}^1, p \mapsto \frac{t}{T}$ where $p = \theta_t(p_0)$.
Then $g(f(g^{-1}(x))) = g\left(\theta_t\left(\theta_{xT}(p_0)\right)\right) = g(\theta_{t+xT}(p_0)) = x + \frac{t}{T}$.
So $g \circ f \circ g^{-1}$ is a rotation.

(2) Take $f(x) = x + \frac{1}{n} + \varepsilon \sin^2(n\pi x)$ for large n ans small ε.
So f can be arbitrarily closed to the identity map, but it is not generated by a vector field.

Exer 16.5. *Let a_{ij} for $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n$ be real constants. For $1 \leqslant i \leqslant n-1$, consider the vector field*

$$X_i = \left(0, \ldots, 0, 1, 0, \ldots, 0, a_{ij}x^j\right) \text{ (1 is the } i\text{-th position)}$$

on $\mathbb{R}^n$ (with coordinates $x^1, \ldots, x^n$). Let Π be the distribution of the tangent subspace of dimension $n-1$ in $\mathbb{R}^n$ spanned by $X_1, \ldots, X_{n-1}$. Determine the necessary and sufficient condition for Π to be integrable. Express the condition in terms of symmetry properties of the $(n-1) \times (n-1)$ matrix $(a_{ij})_{1 \leqslant i,j \leqslant n-1}$ and the relation among the ratios $\frac{a_{ik}}{a_{jk}}$ for $1 \leqslant 1 < j \leqslant n-1$ and $1 \leqslant k \leqslant n$.

$$[X_i, X_j] = \left(a_{ji} - a_{ij} - a_{ik}x^k a_{jn} + a_{jk}x^k a_{in}\right) \frac{\partial}{\partial x^n}$$

And since $X_1, \ldots, X_{n-1}$ is linearly independent with $\frac{\partial}{\partial x^n}$.

So by Frobenius theorem,

$$a_{ji} - a_{ij} + \left(a_{jk}a_{in} - a_{ik}a_{jn}\right) x^k = 0.$$

Hence $(a_{ij})_{1 \leqslant i,j \leqslant n-1}$ is symmetric and $\frac{a_{jk}}{a_{ik}} = \frac{a_{jn}}{a_{in}}$.

Exer 16.6. *Let G be an open subset of $\mathbb{R}^n$. For $1 \leqslant p \leqslant n-1$ denote by $\bigwedge^p T_G$ the exterior product of p copies of the tangent bundle T_G of G. For $1 \leqslant j \leqslant m$ let η_j be a C^∞ section of $\bigwedge^p T_G$ over G. For a C^∞ vector field ξ on an open subset of G, denote by $\mathcal{L}_\xi \eta_j$ with respect to ξ, which means that if $\varphi_{\xi,t}$ is the local diffeomorphism defined by ξ so that the tangent vector $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{\xi,t}$ equals the value of ξ at $\varphi_{\xi,t}$, then*

$$\mathcal{L}_\xi \eta_j = \lim_{t \to 0} \frac{1}{t} \left((\varphi_{\xi,t})_* \eta_j - \eta_j\right)$$

where $(\varphi_{\xi,t})_* \eta_j$ is the pushforward of η_j under $\varphi_{\xi,t}$. Let $\Phi_{\eta_j} : T_G \to \bigwedge^{p+1} T_G$ be defined by exterior product with η_j. Assume that the intersection $\cap_{j=1}^m \ker \Phi_{\eta_j}$ is a subbundle of T_G of rank q over G. Suppose for any C^∞ tangent vector field ζ in any open subset W there exist C^∞ functions $g_{j,k,\zeta}$ on W for $1 \leqslant j,k \leqslant m$ such that

$$\mathcal{L}_\zeta \eta_j = \sum_{k=1}^m g_{j,k,\zeta} \eta_k$$

on W. Prove that for every point x of G, there exists some open neighborhood U_x of x in G and C^∞ functions $f_1, \ldots, f_{n-q}$ on U_x such that the fiber of $\cap_{j=1}^m \ker \Phi_{\eta_j}$ at y is equal to $\cap_{k=1}^{n-q} \ker \mathrm{d} f_k$ at y for $y \in U_x$.

By Frobenius theorem, we need to prove Lie bracket is closed in $\cap_{j=1}^m \ker \Phi_{\eta_j}$.

Let ξ_1, ξ_2 be two smooth sections of $\cap_{j=1}^m \ker \Phi_{\eta_j}$.

Then $\xi_i \wedge \eta_j = 0$ for any i, j.

And since

$$\begin{aligned}
[\xi_1, \xi_2] \wedge \eta_j &= L_{\xi_1}\xi_2 \wedge \eta_j \\
&= L_{\xi_1}(\xi_2 \wedge \eta_j) - \xi_2 \wedge L_{\xi_1}(\eta_j) \\
&= -\xi_2 \wedge \sum_{k=1}^m g_{j,k,\xi_1} \eta_k = 0
\end{aligned}$$

So $[\xi_1, \xi_2] \in \cap_{j=1}^m \ker \Phi_{\eta_j}$.

Exer 16.7. *Let $X = T^*\mathbb{C}^\times = \mathbb{C}^\times \times \mathbb{C}$, where we write z, w for holomorphic coordinates on the base and fiber, respectively. Find all time-1 periodic orbits of the vector field $V = \operatorname{Re}\left(zw\frac{\partial}{\partial z}\right)$,* i.e. *all points $x \in X$ such that the time-1 flow of x under V is equal to x.*

Let $\theta_t(z,w) = \left(ze^{\frac{tw}{2}}, w\right)$ and $z = x + yi, w = u + vi$.

Then $\theta_0(z,w) = (z,w)$ and for $\gamma : t \mapsto \theta_t(z_0, w_0)$,

$$\begin{aligned}\gamma'(t) &= \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x}\bigg|_{(z,w)} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y}\bigg|_{(z,w)} \\ &= \mathrm{Re}\left(\frac{zw}{2}\right)\frac{\partial}{\partial x}\bigg|_{(z,w)} + \mathrm{Im}\left(\frac{zw}{2}\right) i\,\frac{\partial}{\partial y}\bigg|_{(z,w)} \\ &= \mathrm{Re}\left(zw\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}\right)\right)\bigg|_{(z,w)}\end{aligned}$$

where $(z,w) = \gamma(t)$.

So θ_t is the flow of V.

And $\theta_1(z,w) = \left(ze^{\frac{w}{2}}, w\right) = (z,w)$.

Hence time-1 periodic orbits of V is $\{(z, 4\pi ki) | k \in \mathbb{Z}\}$.

Exer 16.8. *Recall that there is a 1-1 correspondence between vector field* v *on a smooth manifold* M *and the derivation* $\mathcal{L}_v : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$. *For two vector fields* v, w, *define* $[v,w]$ *to be the vector field corresponding to the derivation*

$$\mathcal{L}_{[v,w]}(f) = \mathcal{L}_v \circ \mathcal{L}_w(f) - \mathcal{L}_w \circ \mathcal{L}_v(f),$$

where $f \in C^\infty(M;\mathbb{R})$.

(1) Show that the Jacobian identity holds for any three vector fields u, v, w

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$$

(2) Suppose on a local chart U, *we write*

$$v = \sum_i v^i \frac{\partial}{\partial x^i}, w = \sum_j w^j \frac{\partial}{\partial x^j}, [v,w] = \sum_k u^k \frac{\partial}{\partial x^k}.$$

Compute the formula of u^k *using* v^i *and* w^j.

(1)

$$\begin{aligned}[u,[v,w]] &= \mathcal{L}_u \circ (\mathcal{L}_v \circ \mathcal{L}_w(f) - \mathcal{L}_w \circ \mathcal{L}_v(f)) - (\mathcal{L}_v \circ \mathcal{L}_w - \mathcal{L}_w \circ \mathcal{L}_v) \circ \mathcal{L}_u(f) \\ &= (\mathcal{L}_u \circ \mathcal{L}_v \circ \mathcal{L}_w - \mathcal{L}_v \circ \mathcal{L}_w \circ \mathcal{L}_u)(f) + (\mathcal{L}_w \circ \mathcal{L}_v \circ \mathcal{L}_u - \mathcal{L}_u \circ \mathcal{L}_w \circ \mathcal{L}_v)(f)\end{aligned}$$

So the cycle summation is 0.

(2)

$$\begin{aligned}u^k \frac{\partial f}{\partial x^k} &= \mathcal{L}_{[v,w]}(f) = \mathcal{L}_v \circ \mathcal{L}_w(f) - \mathcal{L}_w \circ \mathcal{L}_v(f) \\ &= v^i \frac{\partial}{\partial x^i}\left(w^j \frac{\partial f}{\partial x^j}\right) - w^j \frac{\partial}{\partial x^j}\left(v^i \frac{\partial f}{\partial x^i}\right) \\ &= v^i \frac{\partial w^j}{\partial x^i}\frac{\partial f}{\partial x^j} + v^i w^j \frac{\partial^2 f}{\partial x^i \partial x^j} - w^j \frac{\partial v^i}{\partial x^j}\frac{\partial f}{\partial x^i} - w^j v^i \frac{\partial^2 f}{\partial x^j \partial x^i} \\ &= v^i \frac{\partial w^j}{\partial x^i}\frac{\partial f}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}\frac{\partial f}{\partial x^i} = \left(v^i \frac{\partial w^k}{\partial x^i} - w^j \frac{\partial v^k}{\partial x^j}\right)\frac{\partial f}{\partial x^k}\end{aligned}$$

So

$$u^k = v^i \frac{\partial w^k}{\partial x^i} - w^j \frac{\partial v^k}{\partial x^j}.$$

Exer 16.9. *Let $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ be two sets of distinct points in a connected smooth manifold M with $\dim(M) > 1$, and $v_1, v_2, \ldots, v_k$ and $w_1, w_2, \ldots, w_k$ be the corresponding two sets of non-zero tangent vectors at these points. Show that there is a diffeomorphism f os M such that $f(x_i) = y_i$ and $\mathrm{d}f_{x_i}(v_i) = w_i$ for $i = 1, \ldots, k$.*

Let γ_i be a closed curve passing x_i with $\gamma_i' = v_i$ at x_i and δ_i is similarly defined for y_i, w_i.
Then we can take γ_i, δ_i sufficiently small so that they are null-homotopic in M.
Let $f : [0,1] \times \bigsqcup^k \mathbb{S}^1 \to M$ such that on i-th $\mathbb{S}^1$, f gives a homotopy from (γ_i, x_i) to (δ_i, y_i).
By isotopy extension theorem, there exists a family of diffeomorphism $F_t : M \to M$ such that $F_0 = \mathrm{Id}$ and $F_t(f(0,x)) = f(t,x)$.
Hence $F_1(x_i) = y_i$ and $(\mathrm{d}F_1)_{x_i}(v_i) = w_i$.

Exer 16.10. *Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (*i.e.* , $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that $f_*(\mathcal{L}_Y\omega) = \mathcal{L}_X(f_*\omega)$ where ω is a 1-form on N. Here $\mathcal{L}$ denotes the Lie derivative.*

Let Z be a vector field on M.

$$\begin{aligned}
f_*(\mathcal{L}_Y\omega)_p(Z_p) &= (\mathcal{L}_Y\omega)_{f(p)}(\mathrm{d}f_p(Z_p)) \\
&= \mathcal{L}_Y\left(\omega(f_*Z)\right)_{f(p)} - \omega_{f(p)}\left(\mathcal{L}_Y(f_*Z)_{f(p)}\right) \\
&= Y\left(\omega(f_*Z)\right)_{f(p)} - \omega_{f(p)}[Y, f_*Z]_{f(p)} \\
&= X\left(\omega(f_*Z)\circ f\right)_p - \omega_{f(p)}\left(\mathrm{d}f_p[X,Z]_p\right) \\
&= X(f_*\omega(Z))_p - (f_*\omega)_p(L_XZ_p) \\
&= (\mathcal{L}_X(f_*\omega))_p(Z_p)
\end{aligned}$$

So $f_*(\mathcal{L}_Y\omega) = \mathcal{L}_X(f_*\omega)$.

Exer 16.11. *Let M be a smooth manifold and let $\omega \in \Omega^1(M)$ be a nowhere vanishing smooth 1-form. Prove that the following are equivalent.*

(1) $\ker(\omega)$ *is an integrable distribution.*

(2) $\omega \wedge \mathrm{d}\omega = 0$.

(3) There exists some smooth 1-form $\alpha \in \Omega^1(M)$ satisfying $\mathrm{d}\omega = \alpha \wedge \omega$.

(1)⇒(2): Let (U, x^i) be a chart around p in M such that $\ker(\omega)$ is span by $\frac{\partial}{\partial x^i}$ for $i \leqslant n-1$.
Then $\omega = f\mathrm{d}x^n$ for some smooth function f.
So $\omega \wedge \mathrm{d}\omega = f\mathrm{d}x^n \wedge \mathrm{d}f \wedge \mathrm{d}x^n = 0$.
(2)⇒(3): Let $\{\eta^1 = \omega, \eta^2, \ldots, \eta^n\}$ be a local coframe of M around p.
Let $\mathrm{d}\omega = \Omega_{ij}\eta^i \wedge \eta^j$.
Then $\omega \wedge \mathrm{d}\omega = \Omega_{ij}\eta^1 \wedge \eta^i \wedge \eta^j = 0$.
So $\Omega_{ij} = 0$ for $i, j \neq 1$, *i.e.* $\mathrm{d}\omega = \alpha \wedge \omega$ around p.
And by partition of unity, $\mathrm{d}\omega = \alpha \wedge \omega$ for a smooth 1-form α globally.
(3)⇒(1): Consider $X, Y \in \ker(\omega)$, then

$$\begin{aligned}
\omega[X,Y] &= X(\omega(Y)) - Y(\omega(X)) - \mathrm{d}\omega(X,Y) \\
&= -(\alpha \wedge \omega)(X,Y) \\
&= \alpha(Y)\omega(X) - \alpha(X)\omega(Y) = 0
\end{aligned}$$

So $\ker(\omega)$ is involutive.
By Frobenius theorem, $\ker(\omega)$ is an integrable distribution.

Exer 16.12. *Find two vector fields X and Y on $\mathbb{R}^3$ such that $X, Y, [X, Y]$ are everywhere linearly independent.*

Take $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$.
Then $[X, Y] = \frac{\partial x}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$.
So X, Y and $[X, Y]$ are linearly independent everywhere.

17 Poincaré-Hopf theorem

Exer 17.1. *Let M be a smooth, compact, oriented n-dimensional manifold. Suppose that the Euler characteristic of M is zero. Show that M admits a nowhere vanishing vector field.*

Let V be a vector field on M with isolated zeros $x_1, \dots, x_k$.
By exercise 16.9, we can assume that $x_1, \dots, x_k$ are contained in a chart $U \cong \mathbb{D}^n$.
Consider $u : \partial U \to \mathbb{S}^{n-1}, x \mapsto \frac{V(x)}{|V(x)|}$.
Then by Poincaré-Hopf theorem, $\deg u = \chi(M) = 0$.
So by Hopf theorem, u is null-homotopic.
Therefore there exists a nowhere vanishing vector field V_0 on $\bar{U}$ with $V_0\big|_{\partial U} = V\big|_{\partial U}$ and $u_0 : \bar{U} \to \mathbb{S}^{n-1}, x \mapsto \frac{V_0(x)}{|V_0(x)|}$ gives a homotopy from u to a constant map.
Hence extend V_0 to M by V, it becomes a nowhere vanishing vector field on M.

Exer 17.2. *Is $T\mathbb{S}^2$ diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}^2$? Verify your answer. Here $T\mathbb{S}^2$ is the total space of the tangent bundle of $\mathbb{S}^2$.*

Since $\chi(\mathbb{S}^2) = 1 + (-1)^2 = 2$.
So $\mathbb{S}^2$ has no nowhere vanishing vector field.
Hence $T\mathbb{S}^2$ is nontrivial bundle.

Exer 17.3. *Let M be an oriented Riemannian 4-manifold. A 2-form ω on M is said to be self-dual if $*\omega = \omega$, and anti-self-dual if $*\omega = -\omega$.*

(1) Show that every 2-form ω on M can be written uniquely as a sum of a self-dual form and an anti-self-dual form.

(2) On $M = \mathbb{R}^4$ with the Euclidean metric, determine the self-dual and anti-self dual forms in standard coordinates.

(1) Let $\omega_1 = \frac{1}{2}(\omega + *\omega), \omega_2 = \frac{1}{2}(\omega - *\omega)$.
Then $*\omega_1 = \frac{1}{2}(*\omega + \omega) = \omega_1, *\omega_2 = \frac{1}{2}(*\omega - \omega) = - * \omega_2$.
So $\omega = \omega_1 + \omega_2$ and ω_1, ω_2 are self-dual form and anti-self-dual form resp.

(2) $*(\mathrm{d}x^1 \wedge \mathrm{d}x^2) = \mathrm{d}x^3 \wedge \mathrm{d}x^4, *(\mathrm{d}x^1 \wedge \mathrm{d}x^3) = \mathrm{d}x^4 \wedge \mathrm{d}x^2, *(\mathrm{d}x^1 \wedge \mathrm{d}x^4) = \mathrm{d}x^2 \wedge \mathrm{d}x^3$.
So self-dual are span by $\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4, \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^4 \wedge \mathrm{d}x^2, \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3$.
anti-self-dual are span by $\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \mathrm{d}x^3 \wedge \mathrm{d}x^4, \mathrm{d}x^1 \wedge \mathrm{d}x^3 - \mathrm{d}x^4 \wedge \mathrm{d}x^2, \mathrm{d}x^1 \wedge \mathrm{d}x^4 - \mathrm{d}x^2 \wedge \mathrm{d}x^3$.

Exer 17.4. *Given a properly discontinuous action $F : G \times M \to M$ on a smooth manifold M, show that M/G is orientable if and only if M is orientable and $F(g, \cdot)$ preserves the orientation of M. Use this statement to show that the Möbius band is not orientable and $\mathbb{RP}^n$ is orientable if and only if n is odd.*

Since F is properly discontinuous.
So $\pi : M \to M/G$ is a covering map.
If M/G is orientable, then let ω be the nowhere vanishing n-form on M/G.
So $\pi^*\omega$ is a nowhere vanishing n-form on M, *i.e.* M is orientable.
Moreover, since π is orientation preserving and $\pi \circ F_g = \pi$.
Therefore F_g is orientation preserving.
If M is orientable and F_g is orientation preserving, then let $\mathcal{O}_M$ be the orientation of M.
For $p \in M/G$ and a chart (U, φ), there is an embedding $f : U \to M$ such that $\pi \circ f = \mathrm{Id}$.
So we define the orientation on U by $f^*\mathcal{O}_M$.
For another embedding $g : U \to M$ with $\pi \circ g = \mathrm{Id}$, there exists $h \in G$ such that $F_h \circ f = g$.
Therefore $g^*\mathcal{O}_M = f^* \circ F_h^*\mathcal{O}_M = f^*\mathcal{O}_M$ since F_h is orientation preserving.
Thus $f^*\mathcal{O}_M$ is well-defined and compatible with any other chart V.
Hence M/G is orientable.
Moreover, Möbius band $M = \mathbb{R} \times \mathbb{S}^1/(x, \theta) \sim (-x, \theta + \pi)$ and $\mathbb{RP}^n = \mathbb{S}^n/x \sim -x$.
So Möbius band is not orientable and $\mathbb{RP}^n$ is orientable iff n is odd.

Exer 17.5. *Prove the Cartan formulas:* $\mathcal{L}_X = \mathrm{d}i_X + i_X\mathrm{d}$ *and* $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$.

For p-form α and q-form β, suppose they both satisfy the Cartan magic formula, then

$$\begin{aligned}
\mathcal{L}_X(\alpha \wedge \beta) &= (\mathcal{L}_X\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X\beta) \\
&= (\mathrm{d}i_X\alpha + i_X\mathrm{d}\alpha) \wedge \beta + \alpha \wedge (\mathrm{d}i_X\beta + i_X\mathrm{d}\beta) \\
&= \mathrm{d}i_X\alpha \wedge \beta + (-1)^{p-1}i_X\alpha \wedge \mathrm{d}\beta + i_X\mathrm{d}\alpha \wedge \beta + (-1)^{p+1}\mathrm{d}\alpha \wedge i_X\beta \\
&+ \alpha \wedge \mathrm{d}i_X\beta + (-1)^p\mathrm{d}\alpha \wedge i_X\beta + \alpha \wedge i_X\mathrm{d}\beta + (-1)^p i_X\alpha \wedge \mathrm{d}\beta \\
&= \mathrm{d}(i_X\alpha \wedge \beta) + i_X(\mathrm{d}\alpha \wedge \beta) + (-1)^p\,\mathrm{d}(\alpha \wedge i_X\beta) + (-1)^p i_X(\alpha \wedge \mathrm{d}\beta) \\
&= \mathrm{d}i_X(\alpha \wedge \beta) + i_X\,\mathrm{d}(\alpha \wedge \beta)
\end{aligned}$$

So we only need to check the Cartan magic formula for 0-form and exact 1-form.
Notice that

$$\mathrm{d}i_X f + i_X\mathrm{d}f = \mathrm{d}f(X) = Xf = \mathcal{L}_X f,$$

$$\mathrm{d}i_X(\mathrm{d}f) + i_X\,\mathrm{d}(\mathrm{d}f) = \mathrm{d}X(f) = \mathrm{d}\mathcal{L}_X f = \mathcal{L}_X(\mathrm{d}f).$$

Hence $\mathcal{L}_X = \mathrm{d}i_X + i_X\mathrm{d}$.
Similarly, for p-form α and q-form β, suppose they both satisfy the second formula, then

$$\begin{aligned}
i_{[X,Y]}(\alpha \wedge \beta) &= (i_{[X,Y]}\alpha) \wedge \beta + (-1)^p\alpha \wedge (i_{[X,Y]}\beta) \\
&= (\mathcal{L}_X i_Y\alpha - i_Y\mathcal{L}_X\alpha) \wedge \beta + (-1)^p\alpha \wedge (\mathcal{L}_X i_Y\beta - i_Y\mathcal{L}_X\beta) \\
&= \mathcal{L}_X i_Y\alpha \wedge \beta + i_Y\alpha \wedge \mathcal{L}_X\beta - i_Y\mathcal{L}_X\alpha \wedge \beta - (-1)^p\mathcal{L}_X\alpha \wedge i_Y\beta \\
&+ (-1)^p\alpha \wedge \mathcal{L}_X i_Y\beta + (-1)^p\mathcal{L}_X\alpha \wedge i_Y\beta - (-1)^p\alpha \wedge i_Y\mathcal{L}_X\beta - i_Y\alpha \wedge \mathcal{L}_X\beta \\
&= \mathcal{L}_X(i_Y\alpha \wedge \beta) - i_Y(\mathcal{L}_X\alpha \wedge \beta) + (-1)^p\mathcal{L}_X(\alpha \wedge i_Y\beta) - i_Y(\alpha \wedge \mathcal{L}_X\beta) \\
&= (\mathcal{L}_X i_Y - i_Y\mathcal{L}_X)(\alpha \wedge \beta)
\end{aligned}$$

So we only need to check the second formula for 0-form and exact 1-form.
Notice that

$$[\mathcal{L}_X, i_Y]f = -i_Y X(f) = 0 = i_{[X,Y]}f,$$

$$[\mathcal{L}_X, i_Y]\mathrm{d}f = \mathcal{L}_X Y(f) - i_Y\mathrm{d}X(f) = X(Y(f)) - Y(X(f)) = i_{[X,Y]}\mathrm{d}f.$$

Hence $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$.

Exer 17.6. *Let* $\omega_1, \ldots, \omega_k$ *be one-forms. Show* $\omega_1, \ldots, \omega_k$ *are linearly independent if and only if* $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$.

Locally, let $(x^1, \dots, x^n)$ be a coordinate and $\omega_j = f_{j,i}\mathrm{d}x^i$, then

$$\omega_1 \wedge \cdots \wedge \omega_k = \sum_{i_1,\dots,i_k} f_{1,i_1} \cdots f_{k,i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Fixing the set $\{i_1, \dots, i_k\}$, the coefficient of $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ is the determinant of $k \times k$ submatrix of $(f_{j,i})$ given by rows $\{i_1, \dots, i_k\}$.

So $\omega_1 \wedge \cdots \wedge \omega_k \neq 0 \Leftrightarrow$ matrix $(f_{j,i})$ has full rank $\Leftrightarrow \omega_1, \dots, \omega_k$ are linearly independent.

Exer 17.7. *Consider the following* 2*-form on* $\mathbb{R}^3$*:* $\omega = x\mathrm{d}y \wedge \mathrm{d}z - y\mathrm{d}x \wedge \mathrm{d}z + z\mathrm{d}x \wedge \mathrm{d}y$*. In what follows,* $\mathbb{S}^2$ *is the unit* 2*-sphere in* $\mathbb{R}^3$*. Also,* $N = (0,0,1)$ *and* $S = (0,0,-1)$ *are the north and south poles, respectively.*

(1) Compute $\int_{\mathbb{S}^2} \omega\big|_{\mathbb{S}^2}$*, and show that* $\omega\big|_{\mathbb{S}^2}$ *is not exact.*

(2) Consider spherical coordinates $x = \sin\theta\cos\varphi$ *and* $y = \sin\theta\sin\varphi$ *and* $z = \cos\theta$ *for* $(x,y,z) \in \mathbb{S}^2 - \{N,S\}$ *(here,* $\theta \in [0,\pi]$ *and* $\varphi \in [0,2\pi)$*). Compute* $\omega\big|_{\mathbb{S}^2-\{N,S\}}$ *in terms of spherical coordinates* θ, φ*.*

(3) Show that the restriction $\omega\big|_{\mathbb{S}^2-\{N,S\}}$ *is exact.*

(1)

$$\int_{\mathbb{S}^2} \omega\bigg|_{\mathbb{S}^2} = \int_{\mathbb{D}^3} \mathrm{d}\omega = 3\int_{\mathbb{D}^3} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = 4\pi.$$

(2)

$$\begin{aligned}
\omega\big|_{\mathbb{S}^2-\{N,S\}} &= \sin\theta\cos\varphi\,\mathrm{d}(\sin\theta\sin\varphi) \wedge \mathrm{d}\cos\theta - \sin\theta\sin\varphi\,\mathrm{d}(\sin\theta\cos\varphi) \wedge \mathrm{d}\cos\theta \\
&\quad + \cos\theta\,\mathrm{d}(\sin\theta\cos\varphi) \wedge \mathrm{d}(\sin\theta\sin\varphi) \\
&= \sin\theta\cos\varphi(\cos\theta\sin\varphi\,\mathrm{d}\theta + \sin\theta\cos\varphi\,\mathrm{d}\varphi) \wedge (-\sin\theta)\mathrm{d}\theta \\
&\quad - \sin\theta\sin\varphi(\cos\theta\cos\varphi\,\mathrm{d}\theta - \sin\theta\sin\varphi\,\mathrm{d}\varphi) \wedge (-\sin\theta)\mathrm{d}\theta \\
&\quad + \cos\theta(\cos\theta\cos\varphi\,\mathrm{d}\theta - \sin\theta\sin\varphi\,\mathrm{d}\varphi) \wedge (\cos\theta\sin\varphi\,\mathrm{d}\theta + \sin\theta\cos\varphi\,\mathrm{d}\varphi) \\
&= -\sin^3\theta\cos^2\varphi\,\mathrm{d}\varphi \wedge \mathrm{d}\theta - \sin^3\theta\sin^2\varphi\,\mathrm{d}\varphi \wedge \mathrm{d}\theta \\
&\quad + \cos^2\theta\sin\theta\cos^2\varphi\,\mathrm{d}\theta \wedge \mathrm{d}\varphi - \cos^2\theta\sin\theta\sin^2\varphi\,\mathrm{d}\varphi \wedge \mathrm{d}\theta \\
&= \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi
\end{aligned}$$

(3) Let $\eta = -\cos\theta\mathrm{d}\varphi$.

Then $\mathrm{d}\eta = \sin\theta\mathrm{d}\theta \wedge \mathrm{d}\varphi = \omega\big|_{\mathbb{S}^2-\{N,S\}}$.

Exer 17.8. *Prove Cartan's lemma: Let* M *be a smooth manifold of dimension* n*. Fix* $1 \leqslant k \leqslant n$*. Let* ω^i *and* φ_i *be* 1*-forms on* M*. Suppose that the* $\{\omega^1, \dots, \omega^k\}$ *are linearly independent and that*

$$\sum_{i=1}^k \varphi_i \wedge \omega^i = 0.$$

Prove that there exists smooth function $h_{ij} = h_{ji} : M \to \mathbb{R}$ *such that for all* $i = 1, \dots, k$*,* $\varphi_i = h_{ij}\omega^j$*.*

Locally, extend $\{\omega^1, \dots, \omega^k\}$ to a local frame $\{\omega^1, \dots, \omega^n\}$ and let $\varphi_i = h_{ij}\omega^j$, then

$$\sum_{i=1}^k \varphi_i \wedge \omega^i = \sum_{i=1}^k \sum_{j=1}^n h_{ij}\omega^j \wedge \omega^i = \sum_{1 \leqslant i < j \leqslant k} (h_{ij} - h_{ji})\omega^j \wedge \omega^i + \sum_{i=1}^k \sum_{j=k+1}^n h_{ij}\omega^j \wedge \omega^i.$$

So $h_{ij} = h_{ji}$ for $1 \leqslant j \leqslant k$ and $h_{ij} = 0$ for $k+1 \leqslant j \leqslant n$.

Exer 17.9. *Let ω be a smooth 1-form on a manifold M and let X and Y be smooth vector fields on M. Use the Cartan formula for Lie derivatives to derive the following formula:*

$$\mathrm{d}\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

$$\begin{aligned}\mathrm{d}\omega(X,Y) &= i_X\mathrm{d}\omega(Y) = \mathcal{L}_X\omega(Y) - \mathrm{d}i_X\omega(Y)\\ &= \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_XY) - Y\omega(X)\\ &= X\omega(Y) - Y\omega(X) - \omega([X,Y])\end{aligned}$$

Exer 17.10. *Let ω be a closed 2-form on a smooth manifold M and let X, Y be smooth vector fields on M. Show that if $i_X\omega = i_Y\omega = 0$, then $i_{[X,Y]}\omega = 0$.*

$$\begin{aligned}i_{[X,Y]}\omega &= \mathcal{L}_X i_Y\omega - i_Y\mathcal{L}_X\omega\\ &= -i_Y(i_X\mathrm{d}\omega + \mathrm{d}i_X\omega)\\ &= 0\end{aligned}$$

18 Integration on manifolds, Stokes theorem

Exer 18.1. *Let M be an n-dimensional compact oriented Riemannian manifold with boundary and X a smooth vector field on M. If $\mathbf{n}$ is the inward unit normal vector of the boundary, show that:*

$$\int_M \mathrm{div}(X)\mathrm{d}V_M = \int_{\partial M} X\cdot\mathbf{n}\,\mathrm{d}V_{\partial M}.$$

Let $X^\perp = (X\cdot\mathbf{n})\mathbf{n}$ and $X^\top = X - X^\perp$, then

$$i^*_{\partial M}(i_{X^\perp}\mathrm{d}V_M) = (X\cdot\mathbf{n})i^*_{\partial M}(i_{\mathbf{n}}\mathrm{d}V_M) = X\cdot\mathbf{n}\,\mathrm{d}V_{\partial M}.$$

And since $X^\top$ is a vector field on ∂M.
So $i^*_{\partial M}(i_{X^\top}\mathrm{d}V_M) = 0$.
Hence by Stokes theorem,

$$\int_M \mathrm{div}(X)\mathrm{d}V_M = \int_M \mathrm{d}(i_X\mathrm{d}V_M) = \int_{\partial M} i^*_{\partial M}(i_X\mathrm{d}V_M) = \int_{\partial M} X\cdot\mathbf{n}\,\mathrm{d}V_{\partial M}.$$

Exer 18.2. *State and prove the Stokes theorem for oriented compact manifolds.*

Let M be an oriented compact manifolds with boundary, let ω be a smooth $(n-1)$-form on M, then

$$\int_M \mathrm{d}\omega = \int_{\partial M}\omega.$$

Proof. By partition of unity, we only need to prove this for compactly supported $(n-1)$-form in $\mathbb{R}^n$ and $\mathbb{H}^n$.

For $\mathbb{H}^n$, let $\omega = \omega_i\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^n$, then

$$\mathrm{d}\omega = \sum_{i=1}^n(-1)^{i-1}\frac{\partial\omega_i}{\partial x^i}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n.$$

So

$$\begin{aligned}\int_{\mathbb{H}^n} \mathrm{d}\omega &= \sum_{i=1}^{n-1}(-1)^{i-1}\int_{\mathbb{H}^{n-1}}\left(\int_{-\infty}^{+\infty}\frac{\partial\omega_i}{\partial x^i}\mathrm{d}x^i\right)\mathrm{d}x^1\cdots\widehat{\mathrm{d}x^i}\cdots\mathrm{d}x^n\\
&+(-1)^{n-1}\int_{\mathbb{R}^{n-1}}\left(\int_0^{+\infty}\frac{\partial\omega_n}{\partial x^n}\mathrm{d}x^n\right)\mathrm{d}x^1\cdots\mathrm{d}x^{n-1}\\
&=\sum_{i=1}^{n-1}(-1)^{i-1}\int_{\mathbb{H}^{n-1}}\omega_i(x)\Big|_{x^i=-\infty}^{x^i=+\infty}\mathrm{d}x^1\cdots\widehat{\mathrm{d}x^i}\cdots\mathrm{d}x^n\\
&+(-1)^{n-1}\int_{\mathbb{R}^{n-1}}\omega_n\Big|_{x^i=0}^{x^i=+\infty}\mathrm{d}x^1\cdots\mathrm{d}x^{n-1}\\
&=(-1)^n\int_{\mathbb{R}^{n-1}}\omega_n(x^1,\dots,x^{n-1},0)\mathrm{d}x^1\cdots\mathrm{d}x^{n-1}\\
&=\int_{\partial\mathbb{H}^n}\omega_n\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^{n-1}=\int_{\partial\mathbb{H}^n}\omega\end{aligned}$$

The last equation is because the pullback form of $\mathrm{d}x^n$ on $\partial\mathbb{H}^n$ is 0.
For $\mathbb{R}^n$, the similar computation shows that the integral of $\mathrm{d}\omega$ vanish. □

Exer 18.3. *On the Euclidean space $\mathbb{R}^n$, we consider an $n-1$ form α, which is of class C^1, such that both α and $\mathrm{d}\alpha$ are in L^1. Show that $\int_{\mathbb{R}^n}\mathrm{d}\alpha=0$.*

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\mathrm{d}\alpha\right| &= \lim_{r\to\infty}\left|\int_{B_r}\mathrm{d}\alpha\right| = \lim_{r\to\infty}\left|\int_{\partial B_r}\alpha\right|\\
&\leqslant \liminf_{r\to\infty}\int_{\partial B_r}|\alpha|\\
&=\liminf_{R\to\infty}\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_{B_r}|\alpha|\right)\Big|_R\\
&=0\end{aligned}$$

The last equation is because α is L^1.

Exer 18.4. *Let $\Omega = x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y$ and $\alpha>0$. Suppose D is the surface in $\mathbb{R}^3$ defined by $\left\{(x,y,z)|x^2+y^2+z^2=1, z\geqslant\alpha\sqrt{x^2+y^2}\right\}$.*

(1) Show that $\Omega\big|_D$ is an orientation form and makes D an oriented manifold with boundary.

(2) Evaluate $\int_D\Omega$. Your answer should be interms of α.

(1) Since $x^2+y^2+z^2=1$.
So $\Omega\big|_D$ is nowhere vanishing, *i.e.* it is an orientation form.

(2) Let $x=\sin\theta\cos\varphi, y=\sin\theta\sin\varphi, z=\cos\theta$.
Then $\Omega=\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi$ in $D\backslash\{(0,0,1)\}$.
Since $z=\cos\theta\geqslant\alpha\sqrt{x^2+y^2}=\alpha\sin\theta$.
So

$$\begin{aligned}\int_D\Omega=\int_{D\backslash\{(0,0,1)\}}\Omega&=\int_0^{\theta_0}\int_0^{2\pi}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\\
&=2\pi\int_0^{\theta_0}\mathrm{d}(-\cos\theta)\\
&=2\pi(1-\cos\theta_0)\end{aligned}$$

where $\cot\theta_0 = \alpha$.

Hence $\int_D \Omega = 2\pi\left(1 - \frac{\alpha}{\sqrt{1+\alpha^2}}\right)$

Exer 18.5. *Recall that a symplectic form ω on a smooth manifold M is a degree 2 differential form which is closed and non-degenerate. Here non-degeneracy means that if there is a point $x \in M$ and a tangent vector $u \in T_xM$ such that $\omega(u,v) = 0$ for any $v \in T_xM$, then $u = 0$. Let ω be a symplectic form on M.*

(1) Show that M is orientable.

(2) A vector field V on M is called a Lioville vector field with respect to ω if $\mathcal{L}_V\omega = \omega$. Here $\mathcal{L}_V$ denotes the Lie derivative with respect to V. Show that there isn't any Liouville vector field on M if M is a closed manifold.

(1) Locally, let $\omega = a_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$.

Since $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$.

So WLOG, we assume $A = (a_{ij})$ is anti-symmetric.

And since ω is non-degenerate, *i.e.* $\det A \neq 0$.

WLOG, we can assume $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \cdots + \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n}$ by linear algebra.

So $\omega^n = n!\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n}$ is nowhere vanishing.

Hence M is orientable.

(2) Suppose V is a Liouville vector field.

By Cartan magic formula, $\omega = \mathrm{d}i_V\omega + i_V\mathrm{d}\omega = \mathrm{d}i_V\omega$.

So by Stokes theorem,

$$\int_M \omega^n = \int_M (\mathrm{d}i_V\omega) \wedge \omega^{n-1} = \int_M \mathrm{d}\big(i_V\omega \wedge \omega^{n-1}\big) = 0.$$

This is impossible since ω^n is a volume form of M, contradiction!

Exer 18.6. *Let X be a C^∞ oriented compact manifold of dimension n. Let ξ and η be C^∞ volume forms on X in the sense that ξ and η are C^∞ n-forms on X which are strictly positive everywhere on X with respect to the given orientation X. Suppose the volume $\int_X \xi$ of X with respect to ξ is equal to the volume $\int_X \eta$ of X with respect to η. Prove that there exists a C^∞ diffeomorphism f of X such that the pull back of η by f equals ξ by implementing the following three steps.*

(1) There exists a C^∞ $(n-1)$-form σ on X such that $\mathrm{d}\sigma = \xi - \eta$ on X.

(2) For each fixed $t \in [0,1]$ there exists a C^∞ vector $v(t)$ on X such that the contraction (also known as the interior product) of $(1-t)\xi + t\eta$ with $v(t)$ is σ and the dependence of $v(t)$ on t is C^∞.

(3) Let Φ_t (for $t \geqslant 0$) be the t-parametrized C^∞ family of C^∞ diffeomorphisms of X such that Φ_0 is the identity map of X and for $x \in X$ the partial derivative of $\Phi_t(x)$ with respect to t is equal to the vector $v(t)$ of X at $\Phi_t(x)$. For each fixed $x \in X$, compute the derivative, with respect to t, of the pullback $\Phi_t^*((1-t)\xi + t\eta)$ of the n-form $(1-t)\xi + t\eta$ of X by Φ_t. Use Φ_t (and the result of the computation) to construct the required diffeomorphism f. Note that for fixed $x \in X$, when t is regarded as the time variable, the curve $t \mapsto \Phi_t$ is the trajectory of x for the time-dependent vector field $v(t)$ such that the velocity vector at time t is equal to $v(t)$ at $\Phi_t(x)$.

(1) By de Rham theorem, $[\xi] = [\eta]$ in $H^n_{dR}(X)$.
So $\xi - \eta = \mathrm{d}\sigma$ for some smooth $(n-1)$-form σ.

(2) Let $\xi = \xi_0 \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \eta = \eta_0 \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ and $\sigma = \sigma_i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$.
Then take

$$v(t) = \frac{1}{t\xi_0 + (1-t)\eta_0}\left(\sigma_1 \frac{\partial}{\partial x^1} - \sigma_2 \frac{\partial}{\partial x^2} + \cdots + (-1)^{n-1}\sigma_n \frac{\partial}{\partial x^n}\right).$$

So we have

$$i_{v(t)}((1-t)\xi + t\eta) = \sum_{i=1}^{n} (-1)^{i-1} \cdot (-1)^{i-1} \sigma_i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n = \sigma$$

(3)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^*((1-t)\xi + t\eta) &= \Phi_t^*\left(\mathcal{L}_{v(t)}((1-t)\xi + t\eta)\right) - \Phi_t^*(\eta - \xi) \\
&= \Phi_t^*\left(\mathrm{d}i_{v(t)}((1-t)\xi + t\eta)\right) + \Phi_t^*\mathrm{d}\sigma \\
&= \Phi_t^*\mathrm{d}\sigma - \Phi_t^*\mathrm{d}\sigma = 0
\end{aligned}$$

So $\Phi_t^*((1-t)\xi + t\eta)$ is constant, *i.e.* $\Phi_1^*\eta = \Phi_0^*\xi = \xi$.
Hence Φ is the required diffeomorphism.

Exer 18.7. *Let M be a closed n-dimensional manifold and let Ω be a volume form (*i.e.* a nonvanishing n-form) on M. Given a vector field X on M, its divergence* $\mathrm{div}(X)$ *is the smooth function on M defined by the identity:*

$$\mathcal{L}_X \Omega = \mathrm{div}(X)\Omega$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

(1) Prove that $\int_M \mathrm{div}(X)\Omega = 0$.

(2) Express $\mathrm{div}(X)$ *in local coordinates.*

(1)

$$\mathrm{div}(X)\Omega = \mathcal{L}_X\Omega = \mathrm{d}i_X\Omega.$$

So by Stokes theorem, $\int_M \mathrm{div}(X)\Omega = \int_M \mathrm{d}i_X\Omega = 0$.

(2) Let $\Omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, then

$$\mathrm{d}i_X\Omega = \mathrm{d}\sum_{i=1}^{n}(-1)^{i-1}X^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n = \frac{\partial X^i}{\partial x^i}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

So $\mathrm{div}(X) = \frac{\partial X^i}{\partial x^i}$.

Exer 18.8. *Let*

$$\omega = \frac{x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

be a 2-form defined on $\mathbb{R}^3 - \{0\}$ and $\mathbb{S}^2 \subset \mathbb{R}^3$ the unit sphere.

(1) Compute $\int_{\mathbb{S}^2} i^*\omega$, where $i : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion.

(2) Compute $\int_{\mathbb{S}^2} j^*\omega$, *where* $j : \mathbb{S}^2 \to \mathbb{R}^3$ *is defined by* $j(x, y, z) = (2x, 3y, 5z)$.

(1)

$$\begin{aligned}\int_{\mathbb{S}^2} i^*\omega &= \int_{\mathbb{S}^2} x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y \\ &= \int_{\mathbb{D}^3} 3\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= 4\pi\end{aligned}$$

(2) Since $\mathrm{d}\omega = 0$ in $\mathbb{R}^3 \backslash \{0\}$.

So the integral is invariant under homotopy.

Hence $\int_{\mathbb{S}^2} j^*\omega = \int_{\mathbb{S}^2} i^*\omega = 4\pi$.

19 De Rham cohomology

Exer 19.1. *Let* M *be a smooth* 4*-dimensional manifold. A symplectic form is a closed* 2*-form* ω *on* M *such that* $\omega \wedge \omega$ *is a nowhere vanishing* 4*-form.*

(1) Construct a symplectic form on $\mathbb{R}^4$.

(2) Show that there are no symplectic forms on $\mathbb{S}^4$.

(1) Let $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4$.

Then $\omega \wedge \omega = 2\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^4$.

(2) $H^2_{dR}(\mathbb{S}^4) = 0$.

So every closed 2-form on $\mathbb{S}^4$ is exact.

Let $\omega = \mathrm{d}\eta$ be a 2-form.

Then $\omega \wedge \omega = (\mathrm{d}\eta) \wedge \omega = \mathrm{d}(\eta \wedge \omega)$ is exact.

So by Stokes theorem, $\omega \wedge \omega$ is not nowhere vanishing.

Hence there is no symplectic form on $\mathbb{S}^4$.

Exer 19.2. *Let* $\mathcal{F}^k(M)$ *be the space of all* C^∞ *k-forms on a differentiable manifold* M*. Suppose* U *and* V *are open subsets of* M*.*

(1) Explain carefully how the usual exact sequence:

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0$$

arises.

(2) Write down the "long exact sequence" in de Rham cohomology associated to the short exact sequence in part (1) and descirbe explicitly how the map

$$H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(U \cap V)$$

arises.

(1) $f : \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V)$ is given by $\omega \mapsto \left(\omega\big|_U, \omega\big|_V\right)$.

And $g : \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is given by $(\omega_1, \omega_2) \mapsto \omega_1\big|_{U\cap V} - \omega_2\big|_{U\cap V}$.

So it is obvious that f is an injection and $\ker g = \operatorname{im} f = \left\{(\omega_1, \omega_2) \middle| \omega_1\big|_{U\cap V} = \omega_2\big|_{U\cap V}\right\}$.

And let $\{\rho_U, \rho_V\}$ be a partition of unity.

Then for $\omega \in \mathcal{F}(U \cap V)$, $g(\rho_V\omega, -\rho_U\omega) = \rho_V\omega + \rho_U\omega = \omega$.

Therefore g is surjective, *i.e.* the sequence is exact.

(2) The long exact sequence is

$$\cdots \longrightarrow H^k_{dR}(U \cup V) \xrightarrow{f^*} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{g^*} H^k_{dR}(U \cap V) \xrightarrow{\mathrm{d}^*} H^{k+1}_{dR}(U \cup V) \longrightarrow \cdots$$

And

$$\mathrm{d}^*[\omega] = \begin{cases} [\mathrm{d}(\rho_V\omega)] \text{ on } U \\ [-\mathrm{d}(\rho_U\omega)] \text{ on } V \end{cases}$$

This is well-defined since $\mathrm{d}(\rho_V\omega) = \mathrm{d}\omega - \mathrm{d}(\rho_U\omega) = -\mathrm{d}(\rho_U\omega)$.

For $\mathrm{d}^*[\omega] = 0$, let $\sigma = \rho_V\omega$ and $\tau = -\rho_U\omega$.

Then $g(\sigma, \tau) = \omega$ and $\mathrm{d}\sigma, \mathrm{d}\tau$ glue to a global form ξ such that $\mathrm{d}^*[\omega] = [\xi]$.

Let $\xi = \mathrm{d}\eta$.

So $\mathrm{d}\sigma = \mathrm{d}\eta\big|_U, \mathrm{d}\tau = \mathrm{d}\eta\big|_V$.

Therefore $g^*\left(\left[\sigma - \eta\big|_U\right], \left[\tau - \eta\big|_V\right]\right) = [\omega]$.

For $f^*[\xi] = 0$, let $\xi\big|_U = \mathrm{d}\sigma$, $\xi\big|_V = \mathrm{d}\tau$ and $\omega = g(\sigma, \tau)$.

Then $\mathrm{d}\omega = \mathrm{d}\sigma\big|_{U\cap V} - \mathrm{d}\tau\big|_{U\cap V} = 0$.

Since $g(\sigma, \tau) = g(\rho_V\omega, -\rho_U\omega) = \omega$.

So $\sigma - \rho_V\omega = \eta\big|_U, \tau + \rho_V\omega = \eta\big|_V$ for some $\eta \in \mathcal{F}^k(U \cup V)$.

Therefore $d^*[\omega] = [\xi - \mathrm{d}\eta] = [\xi]$.

Hence the the long exact sequence holds with d^*.

Exer 19.3. *Compute the de Rham cohomology of a punctured two-dimensional torus $\mathbb{T}^2 - \{p\}$ where $p \in \mathbb{T}^2$. If $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates $(x, y) \in \mathbb{R}^2$, then is the volume form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ exact?*

Since $\mathbb{T}^2 - \{p\} \simeq \mathbb{S}^1 \wedge \mathbb{S}^1$.
So de Rham cohomology is $H^0_{dR}(\mathbb{T}^2 - \{p\}) = \mathbb{R}, H^1_{dR}(\mathbb{T}^2 - \{p\}) = \mathbb{R}^2$ and $H^2_{dR}(\mathbb{T}^2 - \{p\}) = 0$.
Hence $[\omega] = 0 \in H^2_{dR}(\mathbb{T}^2 - \{p\})$, *i.e.* ω is exact.

Exer 19.4. *Consider the differential 1-form $\omega = x\mathrm{d}y - y\mathrm{d}x + \mathrm{d}z$ in $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.*

$$\begin{aligned} \mathrm{d}(f\omega) &= \mathrm{d}f \wedge \omega + f\mathrm{d}\omega \\ &= \left(\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z\right) \wedge (x\mathrm{d}y - y\mathrm{d}x + \mathrm{d}z) + f(\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}y \wedge \mathrm{d}x) \\ &= \left(\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y + 2f\right)\mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}x\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}y\right)\mathrm{d}x \wedge \mathrm{d}z \end{aligned}$$

So $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}x, \frac{\partial f}{\partial x} = -\frac{\partial f}{\partial z}y$.

Hence
$$\frac{\partial f}{\partial x}x+\frac{\partial f}{\partial y}y+2f=-\frac{\partial f}{\partial z}yx+\frac{\partial f}{\partial z}xy+2f=2f=0.$$
Contradiction!

Exer 19.5. *Let* Conf_n *be the following submanifold of* $\mathbb{C}^n$*:*
$$\mathrm{Conf}_n=\{(z_1,\dots,z_n)\in\mathbb{C}^n|z_i\neq z_j,\,\forall i\neq j\}.$$
For every pair (i,j) *with* $i\neq j$*, we define the complex valued* 1*-form*
$$\omega_{ij}=\frac{\mathrm{d}z_i-\mathrm{d}z_j}{z_i-z_j}.$$

(1) Show that for any $i\neq j$*,* ω_{ij} *represents a non-zero de Rham cohomology class in* $H^1(\mathrm{Conf}_n,\mathbb{C})$*.*

(2) Show that for any pair-wise distinct indices i,j,k*,*
$$\omega_{ij}\wedge\omega_{jk}+\omega_{jk}\wedge\omega_{ki}+\omega_{ki}\wedge\omega_{ij}=0.$$

(1)
$$\begin{aligned}\mathrm{d}\omega_{ij}&=\mathrm{d}\left(\frac{1}{z_i-z_j}\right)\wedge(\mathrm{d}z_i-\mathrm{d}z_j)\\&=\left(-\frac{1}{(z_i-z_j)^2}\mathrm{d}z_i+\frac{1}{(z_i-z_j)^2}\mathrm{d}z_j\right)\wedge(\mathrm{d}z_i-\mathrm{d}z_j)\\&=\frac{1}{(z_i-z_j)^2}\mathrm{d}z_i\wedge\mathrm{d}z_j+\frac{1}{(z_i-z_j)^2}\mathrm{d}z_j\wedge\mathrm{d}z_i\\&=0\end{aligned}$$

Fixing all z_k except $k=i$ and let $z_i=z_j+e^{i\theta}$, then
$$\int_0^{2\pi}\frac{\mathrm{d}\big(z_j+e^{i\theta}\big)-\mathrm{d}z_j}{e^{i\theta}}=\int_0^{2\pi}i\mathrm{d}\theta=2\pi i\neq 0$$

So $[\omega_{ij}]\neq 0$ in $H^1(\mathrm{Conf}_n,\mathbb{C})$.

(2) Notice that
$$\begin{aligned}&(\mathrm{d}z_i-\mathrm{d}z_j)\wedge(\mathrm{d}z_j-\mathrm{d}z_k)=(\mathrm{d}z_j-\mathrm{d}z_k)\wedge(\mathrm{d}z_k-\mathrm{d}z_i)=(\mathrm{d}z_k-\mathrm{d}z_i)\wedge(\mathrm{d}z_i-\mathrm{d}z_j)\\=&\,\mathrm{d}z_i\wedge\mathrm{d}z_j+\mathrm{d}z_k\wedge\mathrm{d}z_i+\mathrm{d}z_j\wedge\mathrm{d}z_k\end{aligned}$$

So
$$\begin{aligned}&\omega_{ij}\wedge\omega_{jk}+\omega_{jk}\wedge\omega_{ki}+\omega_{ki}\wedge\omega_{ij}\\=&\frac{(\mathrm{d}z_i-\mathrm{d}z_j)\wedge(\mathrm{d}z_j-\mathrm{d}z_k)}{(z_i-z_j)(z_j-z_k)}+\frac{(\mathrm{d}z_j-\mathrm{d}z_k)\wedge(\mathrm{d}z_k-\mathrm{d}z_i)}{(z_j-z_k)(z_k-z_i)}+\frac{(\mathrm{d}z_k-\mathrm{d}z_i)\wedge(\mathrm{d}z_i-\mathrm{d}z_j)}{(z_k-z_i)(z_i-z_j)}\\=&\frac{z_k-z_i+z_i-z-j+z_j-z_k}{(z_i-z_j)(z_j-z_k)(z_k-z_i)}(\mathrm{d}z_i\wedge\mathrm{d}z_j+\mathrm{d}z_k\wedge\mathrm{d}z_i+\mathrm{d}z_j\wedge\mathrm{d}z_k)\\=&\,0\end{aligned}$$

Exer 19.6. *Let* M *be a compact oriented manifold of (real) dimension* 4*. Consider the following symmetric bilinear form on* $H^2(M)$
$$H^2(M)\times H^2(M)\to\mathbb{R},([\alpha],[\beta])\mapsto\int_M\alpha\wedge\beta.$$
Let $\sigma(M)$ *be the signature of this bilinear form,* i.e. *the number of positive eigenvalues minus the number of negative eigenvalues. Compute* $\sigma(M)$ *for* $M=\mathbb{S}^4,\mathbb{CP}^2$ *and* $\mathbb{S}^2\times\mathbb{S}^2$*.*

The symmetric bilinear form is the same to the cup product in cohomology ring.
For $M = \mathbb{S}^4$, $H^*(M;\mathbb{R}) = \mathbb{R}[a]/(a^2)$ with $|a| = 4$, so $\sigma(M) = 0$.
For $M = \mathbb{CP}^2$, $H^*(M;\mathbb{R}) = \mathbb{R}[a]/(a^3)$ with $|a| = 2$, so $\sigma(M) = 1$.
For $M = \mathbb{S}^2 \times \mathbb{S}^2$, $H^*(M;\mathbb{R}) = \mathbb{R}[a,b]/(a^2, b^2)$ with $|a| = |b| = 2$, so $\sigma(M) = 1 - 1 = 0$.

Exer 19.7. *Solve the problem which Russell Crowe assigns to his students in the movie "A beautiful mind" (2001):*

$$V = \{F : \mathbb{R}^3 - X \to \mathbb{R}^3 \text{ such that } \boldsymbol{\nabla} \times F = 0\}$$

$$W = \{F = \nabla g\}$$

$$\dim(V/W) =?$$

First give the general answer for any closed $X \subset \mathbb{R}^3$, and then specialize it to

(1) $X = \{x = y = z = 0\}$,

(2) $X = \{x = y = 0\}$,

(3) $X = \{x = 0\}$.

Let $\omega = F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z$, where $F = (F_x, F_y, F_z)$.
Then $\boldsymbol{\nabla} \times F = 0$ gives us that

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}, \frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}.$$

So $\mathrm{d}\omega = 0$ is closed for $F \in V$.
And for $F \in W$, $F_x = \frac{\partial g}{\partial x}, F_y = \frac{\partial g}{\partial y}, F_z = \frac{\partial g}{\partial z}$.
Therefore $\omega = \mathrm{d}g$ is exact.
Hence $\dim(V/W) = \dim H^1_{dR}(\mathbb{R}^3 - X)$.

(1) $H^1_{dR}(\mathbb{R}^3 - X) = H^1_{dR}(\mathbb{S}^2) = 0$, *i.e.* $\dim(V/W) = 0$.

(2) $H^1_{dR}(\mathbb{R}^3 - X) = H^1_{dR}(\mathbb{S}^1 \times \mathbb{R}) = \mathbb{R}$, *i.e.* $\dim(V/W) = 1$.

(3) $H^1_{dR}(\mathbb{R}^3 - X) = H^1_{dR}(\{p_1, p_2\}) = 0$, *i.e.* $\dim(V, W) = 0$.

Exer 19.8. *Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$.*

(1) Find a 6-form α on $\mathbb{R}^7 - \{0\}$ such that

$$\mathrm{d}\alpha = 0, \int_{\mathbb{S}^6} \alpha = 1$$

(2) For any smooth map $f : \mathbb{S}^{11} \to \mathbb{S}^6$, show that there exists a 5-form φ on $\mathbb{S}^{11}$ such that

$$f^*\alpha = \mathrm{d}\varphi.$$

(3) Let $H(f) = \int_{\mathbb{S}^{11}} \varphi \wedge \mathrm{d}\varphi$. Show that $H(f)$ is independent of the choice of φ satisfying $f^*\alpha = \mathrm{d}\varphi$.

(1) Let

$$\omega = \frac{1}{((x^1)^2 + \cdots + (x^7)^2)^{\frac{7}{2}}} \sum_{i=1}^{7} (-1)^{i-1} x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^7.$$

Then

$$\begin{aligned} \mathrm{d}\omega &= \frac{-7(x^1\mathrm{d}x^1 + \cdots + x^7\mathrm{d}x^7)}{((x^1)^2 + \cdots + (x^7)^2)^{\frac{9}{2}}} \sum_{i=1}^{7} (-1)^{i-1} x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^7 \\ &\quad + \frac{7\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^7}{((x^1)^2 + \cdots + (x^7)^2)^{\frac{7}{2}}} \\ &= \frac{-7\left((x^1)^2 + \cdots + (x^7)^2\right)}{((x^1)^2 + \cdots + (x^7)^2)^{\frac{9}{2}}} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^7 + \frac{7\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^7}{((x^1)^2 + \cdots + (x^7)^2)^{\frac{7}{2}}} \\ &= 0 \end{aligned}$$

And by Stokes theorem

$$\begin{aligned} \int_{\mathbb{S}^6} \omega &= \int_{\mathbb{S}^6} \sum_{i=1}^{7} (-1)^{i-1} x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^7 \\ &= \int_{\mathbb{D}^7} 7\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^7 = 7\mathrm{Vol}(\mathbb{D}^7) \end{aligned}$$

So we take $\alpha = \frac{\omega}{7\mathrm{Vol}(\mathbb{D}^7)}$.

(2) Since $H^6_{dR}(\mathbb{S}^{11}) = 0$.

So $f^*\alpha$ is exact, *i.e.* there exists 5-form φ such that $f^*\alpha = \mathrm{d}\varphi$.

(3) Let ψ be another 5-form such that $\mathrm{d}\psi = f^*\alpha$.

Since $H^5_{dR}(\mathbb{S}^{11}) = 0$ and $\mathrm{d}(\varphi - \psi) = f^*\alpha - f^*\alpha = 0$.

So $\varphi - \psi$ is exact, *i.e.* there exists 4-form η such that $\varphi - \psi = \mathrm{d}\eta$.

Hence by Stokes theorem,

$$\int_{\mathbb{S}^{11}} (\varphi \wedge \mathrm{d}\varphi - \psi \wedge \mathrm{d}\psi) = \int_{\mathbb{S}^{11}} (\varphi - \psi) \wedge f^*\alpha = \int_{\mathbb{S}^{11}} \mathrm{d}\eta \wedge f^*\alpha = \int_{\mathbb{S}^{11}} \mathrm{d}(\eta \wedge f^*\alpha) = 0.$$

Exer 19.9. *Consider a smooth map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ with $n \geqslant 2$. Let v be a volume form on $\mathbb{S}^n$ with volume* 1*.*

(1) Show that $f^*\nu$ is exact.

(2) Write $f^*\nu = \mathrm{d}\alpha$. Show that the integral

$$\int_{\mathbb{S}^{2n-1}} \alpha \wedge f^*\nu$$

is independent of the choice of α.

(3) Show that the integral above is acturally an invariant of the homotopy class of f.

(4) Show that the integral is 0 *if n is odd.*

(5) Calculate this integral if f is the Hopf map $(z_0, z_1) \to [z_0 : z_1]$.

(1) Same as the last exercise.

(2) Same as the last exercise.

(3) Let $H : \mathbb{S}^{2n-1} \times [0,1] \to \mathbb{S}^n$ be the homotopy from f to g.
Since $H^n_{dR}(\mathbb{S}^{2n-1} \times [0,1]) = 0$.
So $H^*\nu = \mathrm{d}\varphi$ is exact.
WLOG, we assume $\varphi\big|_{\mathbb{S}^{2n-1}\times\{0\}} = \alpha$ and $\varphi\big|_{\mathbb{S}^{2n-1}\times\{1\}} = \beta$.
By Stokes theorem,

$$\begin{aligned}\int_{\mathbb{S}^{2n-1}} \alpha \wedge f^*\nu - \beta \wedge g^*\nu &= \int_{\partial(\mathbb{S}^{2n-1}\times[0,1])} \varphi \wedge H^*\nu \\ &= \int_{\mathbb{S}^{2n-1}\times[0,1]} \mathrm{d}\varphi \wedge H^*\nu \\ &= \int_{\mathbb{S}^{2n-1}\times[0,1]} H^*(\nu \wedge \nu) \\ &= 0\end{aligned}$$

So the integral is an invariant of the homotopy class of f.

(4) Since n is odd, so

$$\mathrm{d}(\alpha \wedge \alpha) = \mathrm{d}\alpha \wedge \alpha + (-1)^{n-1}\alpha \wedge \mathrm{d}\alpha = 2\alpha \wedge \mathrm{d}\alpha.$$

Hence the integral is 0 by Stokes theorem.

(5) $\nu = \frac{1}{4\pi}(x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y)$.
And $f : \mathbb{S}^3 \to \mathbb{S}^2$, $\left(x^1, x^2, x^3, x^4\right) \mapsto (x, y, z)$ with

$$x = 2(x^1x^3 + x^2x^4), y = 2(x^2x^3 - x^1x^4),$$
$$z = (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2.$$

Therefore $f^*\nu = \frac{1}{\pi}(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4)$.
Let $\alpha = \frac{1}{\pi}(x^1\mathrm{d}x^2 - x^4\mathrm{d}x^3)$.
Then

$$\mathrm{d}\alpha = \frac{1}{\pi}\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4\right) = f^*\nu.$$

Hence by Stokes theorem,

$$\begin{aligned}\int_{\mathbb{S}^3} \alpha \wedge f^*\nu &= \frac{1}{\pi^2}\int_{\mathbb{S}^3} (x^1\mathrm{d}x^2 - x^4\mathrm{d}x^3) \wedge (\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4) \\ &= \frac{1}{\pi^2}\int_{\mathbb{S}^3} (x^1\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 - x^4\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3) \\ &= \frac{1}{\pi^2}\int_{\mathbb{D}^4} 2\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ &= \frac{2}{\pi^2} \cdot \frac{\pi^2}{2} = 1\end{aligned}$$

Exer 19.10. *Let $P_1 = (a_1, b_1), \ldots, P_n = (a_n, b_n)$ be n distinct points in $\mathbb{R}^2$. Define a 1-form ω on $\mathbb{R}^2 - \{P_1, \ldots, P_n\}$ by*

$$\omega = \sum_{i=1}^{n} \frac{(x - a_i)\mathrm{d}y - (y - b_i)\mathrm{d}x}{(x - a_i)^2 + (y - b_i)^2}$$

Let C be a simple closed curve containing $P_1, \ldots, P_n$ inside and rotate in the positive direction. Compute the line integral

$$\int_C \omega.$$

Let γ_i be a small circle around P_i rotate in the positive direction, r_i is the radius and D_i is the small disk with boundary γ_i, then

$$\int_C \omega = \sum_{i=1}^{n} \int_{\gamma_i} \omega.$$

Let ω_i be the i-th term of ω.

Then by Stokes theorem, for $j \neq i$

$$\int_{\gamma_i} \omega_j = \int_{D_i} \mathrm{d}\left(\frac{(x-a_j)\mathrm{d}y - (y-b_j)\mathrm{d}x}{(x-a_j)^2 + (y-b_j)^2}\right) = \int_{D_i} 0 = 0.$$

And

$$\int_{\gamma_i} \omega_i = \int_0^{2\pi} \frac{r_i^2 \cos\theta \mathrm{d}\sin\theta - r_i^2 \sin\theta \mathrm{d}\cos\theta}{r_i^2\cos^2\theta + r_i^2\sin^2\theta} = \int_0^{2\pi} \mathrm{d}\theta = 2\pi.$$

Hence

$$\int_C \omega = \sum_{i=1}^{n} \int_{\gamma_i} \omega_i = 2\pi n.$$

Exer 19.11. *Let $\phi : \mathbb{S}^2 \to \mathbb{T}^2$ be a smooth map, show that for any top de Rham cohomology class $[\nu] \in H^2(\mathbb{T}^2; \mathbb{R})$, we have $\phi^*[\nu] = 0$.*

Since $H^*(\mathbb{S}^2;\mathbb{R}) = \mathbb{R}[a]/(a^2)$ with $|a| = 2$ and $H^*(\mathbb{T}^2;\mathbb{R}) = \Lambda_{\mathbb{R}}[b,c]/(b^2,c^2)$ with $|b| = |c| = 1$.
So $[\nu]$ corresponds to $b \smile c$.
And since $\phi^* b = \phi^* c = 0 \in H^1(\mathbb{S}^2;\mathbb{R})$.
Hence $\phi^*[\nu] = \phi^* b \smile \phi^* c = 0$.

Exer 19.12. *(1) Define the degree $\deg(f)$ of a C^∞ map $f : \mathbb{S}^2 \to \mathbb{S}^2$ and prove that $\deg(f)$ as you present it is well-defined and independent of any choices you need to make in your definition.*

(2) Prove in detail that for each integer k (possibly negative), there is a C^∞ map $f : \mathbb{S}^2 \to \mathbb{S}^2$ of degree k.

(1) Let $[\mathbb{S}^2] \in H_2(\mathbb{S}^2, \mathbb{Z})$ be the fundamental class of $\mathbb{S}^2$.

Then the integer k such that $f_*[\mathbb{S}^2] = k[\mathbb{S}^2]$ is the degree of f.

(2) Using spherical coordinate (θ, φ) with $(x,y,z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

Let $f : \mathbb{S}^2 \to \mathbb{S}^2, (\theta,\varphi) \mapsto (\theta, k\varphi)$.

Then $f_*[\mathbb{S}^2] = k[\mathbb{S}^2]$, *i.e.* $\deg(f) = k$.

Exer 19.13. *Let Ω be the 2-form on $\mathbb{R}^3 - \{0\}$ defined by*

$$\Omega = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y)$$

(1) Prove that Ω is closed.

(2) Let $f : \mathbb{R}^3 - \{0\} \to \mathbb{S}^2$ be the map $(x,y,z) \mapsto \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}(x,y,z)$. Prove that Ω is the pullback along f of a form on $\mathbb{S}^2$.

(3) Prove that Ω is not exact.

(1) Similar to exercise 19.8(1).

(2) Let $\omega = x\mathrm{d}y\wedge\mathrm{d}z + y\mathrm{d}z\wedge\mathrm{d}x + z\mathrm{d}x\wedge\mathrm{d}y$ and denote $r = (x^2+y^2+z^2)^{\frac{1}{2}}$.
Since

$$f^*(\mathrm{d}x) = \frac{r^2-x^2}{r^3}\mathrm{d}x - \frac{xy}{r^3}\mathrm{d}y - \frac{xz}{r^3}\mathrm{d}z,$$

$$f^*(\mathrm{d}y) = -\frac{yx}{r^3}\mathrm{d}x + \frac{r^2-y^2}{r^3}\mathrm{d}y - \frac{yz}{r^3}\mathrm{d}z,$$

$$f^*(\mathrm{d}z) = -\frac{zx}{r^3}\mathrm{d}x - \frac{zy}{r^3}\mathrm{d}y + \frac{r^2-z^2}{r^3}\mathrm{d}z.$$

So

$$\begin{aligned} f^*\omega &= \left(\frac{zxr^2}{r^6}\frac{x}{r} + \frac{yzr^2}{r^6}\frac{y}{r} + \frac{r^4-x^2r^2-y^2r^2}{r^6}\frac{z}{r}\right)\mathrm{d}x\wedge\mathrm{d}y \\ &+ \left(\frac{xyr^2}{r^6}\frac{y}{r} + \frac{zxr^2}{r^6}\frac{z}{r} + \frac{r^4-y^2r^2-z^2r^2}{r^6}\frac{x}{r}\right)\mathrm{d}y\wedge\mathrm{d}z \\ &+ \left(\frac{yzr^2}{r^6}\frac{z}{r} + \frac{xyr^2}{r^6}\frac{x}{r} + \frac{r^4-z^2r^2-x^2r^2}{r^6}\frac{y}{r}\right)\mathrm{d}z\wedge\mathrm{d}x \\ &= \frac{z}{r^3}\mathrm{d}x\wedge\mathrm{d}y + \frac{x}{r^3}\mathrm{d}y\wedge\mathrm{d}z + \frac{y}{r^3}\mathrm{d}z\wedge\mathrm{d}x \end{aligned}$$

(3) By Stokes theorem,

$$\int_{\mathbb{S}^2}\Omega = \int_{\mathbb{S}^2}\omega = \int_{\mathbb{D}^3} 3\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = 4\pi.$$

So Ω is not exact.

Exer 19.14. *Suppose $f: M\to N$ is a smooth map between smooth manifolds, and is smoothly homotopic to a locally constant map. Prove $f^*\omega$ is exact for any closed differential k-form ω on N (with $k>0$).*

Since f is smoothly homotopic to a locally constant map.
So $f^*: H^k_{dR}(N)\to H^k_{dR}(M)$ is trivial.
Hence $f^*[\omega]=0$, *i.e.* $f^*\omega$ is exact.

Exer 19.15. *Let $\omega\in\Omega^n_c(\mathbb{R}^n)$ be a compactly supported n-form. Show that $\omega = \mathrm{d}\eta$ for some compactly supported $(n-1)$-form $\eta\in\Omega^{n-1}_c(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n}\omega = 0$.*

If $\omega = \mathrm{d}\eta$, by Stokes theorem, $\int_{\mathbb{R}^n}\omega = 0$.
Conversely, by Poincaré duality, $[\omega] = 0\in H^n_c(\mathbb{R}^n)$.
So $\omega = \mathrm{d}\eta$ for some $\eta\in\Omega^{n-1}_c(\mathbb{R}^n)$.

Exer 19.16. *Let M be a $2n$-dimensional smooth manifold. A symplectic form on M is a smooth closed 2-form $\omega\in\Omega^2(M)$ so that $\omega\wedge\cdots\wedge\omega\in\Omega^{2n}(M)$ is a volume form (that is, nowhere vanishing). Determine all pairs of positive integers (k,l) so that $\mathbb{S}^k\times\mathbb{S}^l$ has a symplectic form.*

Suppose $\omega^m = \mathrm{d}\eta$ is exact for some $m>0$.
Then by Stokes theorem,

$$\int_M\omega^n = \int_M(\mathrm{d}\eta)\wedge\omega^{n-m} = \int_M \mathrm{d}\Big(\eta\wedge\omega^{n-k}\Big) = 0.$$

Contradiction!

So for symplectic form ω, $[\omega^m] \neq 0$ for any $1 \leqslant m \leqslant n$.
And since $H^{2m}_{dR}(\mathbb{S}^k \times \mathbb{S}^l) = 0$ for $m \neq 0, k, l$ and $k + l$.
Hence we must have $k = l = 2$.
On the other hand, for $M = \mathbb{S}^2 \times \mathbb{S}^2$, $H^*(M) = \mathbb{R}[a, b]/(a^2, b^2)$ with $|a| = |b| = 2$.
Take ω to be the representative form of $a + b$.
Then $[\omega \wedge \omega] = 2ab \neq 0$ is a volume form, *i.e.* ω is symplectic.

Exer 19.17. *Let M be a closed $2n$-dimensional manifold and let ω be a closed 2-form on M which is non-degenerate,* i.e. *for any $p \in M$, the map $T_pM \to T_p^*M, X \mapsto i_X\omega(p)$, is an isomorphism. Show that the de Rham cohomology groups $H^{2k}_{dR}(M) \neq 0$ for $0 \leqslant k \leqslant n$.*

By exercise 18.5, ω is a symplectic form.
And by the last exercise, ω^k is not exact for any k.
So $H^{2k}_{dR}(M) \neq 0$ for $0 \leqslant k \leqslant n$.

Exer 19.18. *Let M be a closed n-dimensional manifold. Let ω be a closed k-form on M, $1 \leqslant k \leqslant n$. Prove that for any $p \in M$ there is another closed k-form τ which vanishes on a neighborhood of p and such that $[\omega] = [\tau] \in H^k_{dR}(M)$.*

Let $U \ni p$ be a chart and ρ be a bump function with $\mathrm{Supp}(\rho) \subset U$ and $\rho(V) \equiv 1$ for $V \subset U$.
Since $H^k_{dR}(U) = H^k(dR)(\mathbb{R}^n) = 0$.
So $\omega\big|_U = \mathrm{d}\eta$ for some $(k-1)$-form η in U.
Take $\tau = \omega - \mathrm{d}(\rho\eta)$.
Then in V, $\tau = \omega - \mathrm{d}\eta = 0$ and $[\omega] = [\tau] \in H^k_{dR}(M)$.

Exer 19.19. *Consider the map $\mathrm{d}f : \Omega^i(M) \to \Omega^{i+1}(M)$ given by $\omega \mapsto \mathrm{d}\omega + \mathrm{d}f \wedge \omega$, where M is a smooth manifold, $\Omega^i(M)$ is the set of smooth i-forms on M, and f is a smooth function on M.*

(1) Show that $\mathrm{d}f$ is a cochain map, i.e. $\mathrm{d}f \circ \mathrm{d}f = 0$.

(2) Let $H^i_f(M)$ be the i-th cohomology group of the cochain complex $(\Omega^i(M), \mathrm{d}f)$. Show that $H^0_f(M) \cong \mathbb{R}$ when M is the real line $\mathbb{R}$.

(1)
$$\begin{aligned}\mathrm{d}f \circ \mathrm{d}f(\omega) &= \mathrm{d}(\mathrm{d}\omega + \mathrm{d}f \wedge \omega) + \mathrm{d}f \wedge (\mathrm{d}\omega + \mathrm{d}f \wedge \omega)\\ &= \mathrm{dd}\omega - \mathrm{d}f \wedge \mathrm{d}\omega + \mathrm{d}f \wedge \mathrm{d}\omega + \mathrm{d}f \wedge \mathrm{d}f \wedge \omega\\ &= 0\end{aligned}$$

(2) Consider $g \in \Omega^0(M)$, *i.e.* g is a smooth function.
Then $\mathrm{d}f(g) = \mathrm{d}g + \mathrm{d}f \wedge g = (g' + gf')\mathrm{d}x$.
So for $g \in \ker \mathrm{d}f$, $(\log g)' = \frac{g'}{g} = -f'$, *i.e.* $g = Ce^{-f}$ for some constant $C \in \mathbb{R}$.
Hence $H^0_f(M) \cong \{Ce^{-f}\} \cong \mathbb{R}$.

Exer 19.20. *Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus and let C be the curve which is the image of the line $\{2x - 5y = 0\} \subset \mathbb{R}^2$ under the projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.*

(1) Write a differential form on $\mathbb{T}^2$ which represents the Poincaré dual to C.

(2) Is there a differential form which represents the Poincaré dual to C and is zero on a neighborhodd of the point $(0, 0) \in \mathbb{T}^2$?

(1) Let $H_1(\mathbb{T}^2) = \mathbb{Z}[a, b]$ with a represent $\pi(\{x = 0\})$ and b represent $\pi(\{y = 0\})$.

Then a corresponds to $\mathrm{d}x$ and b corresponds to $-\mathrm{d}y$.

And since $S = \pi(\{2x - 5y = 0\})$ is represented by $2a + 5b$.

So it corresponds to $2\mathrm{d}x - 5\mathrm{d}y$.

(2) Let $\rho : B_{0.2}(0, 0) \to \mathbb{R}$ be a bump function that maps the boundary to 0 and $B_{0.1}(0, 0)$ to 1.

Then we can extend ρ to $\mathbb{R}^2/\mathbb{Z}^2$ by zero.

Take $\omega = 2\mathrm{d}x - 5\mathrm{d}y - \mathrm{d}(\rho(2x - 5y))$.

So $[\omega] = [2\mathrm{d}x - 5\mathrm{d}y]$ and $\omega = 2\mathrm{d}x - 5\mathrm{d}y - \mathrm{d}(2x - 5y) = 0$ in $B_{0.1}(0, 0)$.

Exer 19.21. *Let M, N be closed oriented n-manifolds with N connected. Show that if $f : M \to N$ has nonzero degree, then $f^* : H^*_{dR}(N; \mathbb{R}) \to H^*_{dR}(M; \mathbb{R})$ is injective.*

Let Ω_M and Ω_N be the generator of $H^n_{dR}(M; \mathbb{R})$ and $H^n_{dR}(N; \mathbb{R})$ resp.
Since $f^* : H^n_{dR}(N; \mathbb{R}) \to H^n_{dR}(M; \mathbb{R}), \Omega_N \mapsto \deg(f)\Omega_M$ is injective.
For $\omega \in H^k_{dR}(N; \mathbb{R})$, let $\tau \in H^{n-k}_{dR}(N; \mathbb{R})$ be its Poincaré dual.
Then $\deg(f)\Omega_M = f^*(\Omega_N) = f^*(\omega) \wedge f^*(\tau)$.
So $f^*(\omega) \neq 0$, *i.e.* $f^* : H^k_{dR}(N; \mathbb{R}) \to H^k_{dR}(M; \mathbb{R})$ is injective.

Exer 19.22. *Let $\rho : \tilde{M} \to M$ be a 2-fold covering map between smooth manifolds. Show that the map $\rho^* : H^*_{dR}(M) \to H^*_{dR}(\tilde{M})$ induced by ρ on the de Rham cohomology is injective.*

Suppose $\rho^*(\omega) = \mathrm{d}\eta$ is exact.
Let $f : \tilde{M} \to \tilde{M}$ be the nontrivial deck transformation.
Then $\mathrm{d}(f^*(\eta)) = f^*(\mathrm{d}\eta) = (\rho \circ f)^*(\omega) = \rho^*(\omega)$.
So $\rho^*(\omega) = \mathrm{d}\frac{1}{2}(\eta + f^*(\eta))$.
And since $\frac{1}{2}(\eta + f^*(\eta))$ is invariant under f, *i.e.* it is $\rho^*(\zeta)$ for some form ζ on M.
Hence $\omega = \mathrm{d}\zeta$ is exact, *i.e.* ρ^* is injective.