History of Kähler Geometry

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Chapter 1

The Origin of Complex Geometry and Kähler Condition

1.1 The Development of 19th Century Complex Functional Theory and Riemann Surfaces

The history of Kähler geometry, though it was clearly developed in the 20th century, has its deepest conceptual origins in the innovative mathematical work of the 19th century. During this period, complex analysis evolved from a set of calculation methods into a profound and independent field. Its core objects—complex functions and the surfaces they rely on—required new ways of understanding geometry and topology. This intellectual progress provided the essential foundation: ideas like complex manifolds, holomorphic functions, and intrinsic geometry. These would later form the basis of Kähler geometry.

1.1.1 The Emergence of Complex Analysis as a Unified Field

In the early 19th century, Augustin-Louis Cauchy established a solid, rigorous basis for complex function theory. His integral theorem (around 1825, see [Cau74]) and integral formula (see [Cau31]) revealed the key, remarkable property of holomorphic (complex-differentiable) functions: their local behavior determines their global behavior. For a function to be holomorphic is a very strict condition. It means the function is infinitely differentiable, can be expanded locally as a convergent power series, and the useful residue calculus applies to it. This "rigidity" of holomorphic functions, which is very different from the flexibility of real differentiable functions, suggested there was a deep underlying geometric structure governing their domain. Bernhard Riemann, a student of Gauss, would bring this geometric perspective to the forefront.

1.1.2 Riemann's vision: Riemann surfaces

In Riemann's pioneering 1851 doctoral thesis, *Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Größe* (Foundations for a general theory of functions of a complex variable, see [Rie76]), Riemann put forward an innovative idea. To fully understand multi-valued complex functions, such as $w = \sqrt{z}$ or $w = \log(z)$, we cannot view them as mappings on the complex plane. Instead, we need a new kind of object: the Riemann surface.

Riemann's insight combined topology and geometry. He thought of a multi-valued function as being single-valued on a suitable covering surface. For $w = \sqrt{z}$, this surface has two "sheets", famously visualized as a "spiral staircase" connecting two copies of the complex plane, with a branch point at $z = 0$. The Riemann surface was not just a tool for visualization, it was the "true" domain of the function. On this surface, the function becomes holomorphic and one-to-one. This shifted the focus from the function itself to the intrinsic properties of the surface where it is defined.

1.1.3 Key Conceptual Advances from Riemann Surface Theory

The study of Riemann surfaces led to several important conceptual breakthroughs, which directly paved the way for Kähler geometry:

Early forms of complex manifolds. By definition, a Riemann surface is a one-dimensional complex manifold. It is a topological surface with a set of coordinate charts, where the transition maps between charts are biholomorphic (holomorphic with holomorphic inverse). This local model—patches of the complex plane glued together by holomorphic maps—is the basic blueprint for all higher-dimensional complex manifolds.

The Central Role of Holomorphic Functions and Forms. Riemann surfaces are the natural setting for holomorphic (and meromorphic) functions. More importantly, Riemann introduced the concept of holomorphic and meromorphic differentials (1-forms like $f(z)\mathrm{d}z$). Studying these differentials globally, including their integrals around closed loops, their singularities, etc., became a key focus. This portends the central role of holomorphic differential forms (and later, the Kähler form) in higher dimensions.

Geometry from Analysis: Conformal Structure. A Riemann surface has a canonical geometric structure derived from its complex structure: a conformal structure (an equivalence class of metrics up to a positive scaling factor). Since holomorphic maps preserve angles (they are conformal), any metric on a Riemann surface that matches its complex structure must take the form $\mathrm{d}s^2 = \lambda(z)|\mathrm{d}z|^2$, where $\lambda(z) > 0$. This close connection between complex-analytic and geometric structures is a one-dimensional precursor to the Hermitian condition in higher dimensions, where a complex structure and a Riemannian metric must be compatible.

Topology and Analysis: The Genus. Riemann linked the analytic properties of a compact Riemann surface to its topology, most notably its genus g (see [Rie57]). Intuitively, the genus is the number of "holes" in the surface, and it determines key analytic invariants:

- The space of globally defined holomorphic 1-forms has dimension g.
- The Riemann-Roch theorem (established by Riemann and later refined by Gustav Roch, see [Rie57] and [Roc65]) connects the dimension of spaces of meromorphic functions (with specified poles) to the genus. This theorem is a model for powerful index theorems that link analysis, topology and complex geometry—a theme that reached its peak with the Hirzebruch-Riemann-Roch and Atiyah-Singer theorems on complex manifolds.

Uniformization and Canonical Metrics. The highest achievement of 19th-century Riemann surface theory is the Uniformization Theorem (see [Kle83] and [Poi82]), while it is fully proven in the early 20th century by Poincaré and Koebe (see [Poi07], [Koe07c], [Koe07a] and [Koe07b]). It states that every simply connected Riemann surface is conformally equivalent to one of three model spaces: the complex plane $\mathbb{C}$, the unit disk $\mathbb{D}$, or the Riemann sphere $\hat{\mathbb{C}}$. Each of these spaces has a standard metric with constant curvature: Euclidean (curvature 0) for $\mathbb{C}$, hyperbolic (curvature -1) for $\mathbb{D}$, and spherical (curvature +1) for $\hat{\mathbb{C}}$. This creates a perfect three-way relationship between topology (the universal cover), complex structure, and canonical geometry (a metric with constant curvature in the conformal class). Finding such "optimal" or "canonical" metrics on a given complex manifold is a core problem in Kähler geometry—the Kähler-Einstein condition is the higher-dimensional equivalent of constant curvature.

1.1.4 Legacy for Kähler Geometry

By the end of the 19th century, the groundwork for Kähler geometry had been laid. The study of Riemann surfaces established a strong framework based on these key ideas:

- Complex Manifold: The idea of a space that is locally modeled on $\mathbb{C}^n$, with holomorphic transition maps between local patches.
- Holomorphic Objects: The importance of holomorphic functions, maps, and differential forms as the main objects of study.
- Geometry from Complex Structure: A complex structure naturally leads to a specific type of geometric structure (like a conformal or Hermitian structure).
- Global Topology vs. Local Analysis: The deep interaction between global topological invariants, such as genus, and the dimensions of spaces of analytic objects.
- The Canonical Metric Ideal: The search for a distinct, "best" metric (like one with constant curvature) on a given complex manifold.

Moving to higher dimensions required new tools and insights, especially the differential geometric language of connections, curvature, and characteristic classes, which were developed in the early 20th century. However, the fundamental questions and core objects of Kähler geometry were already clear in the elegant, rich theory of Riemann surfaces. Kähler geometry can be seen as the higher-dimensional continuation of this 19th-century tradition. It seeks to extend the harmonious relationship between complex analysis, topology, and differential geometry from complex curves (Riemann surfaces) to higher-dimensional complex manifolds.

1.2 The Early 20th Century: The Foundations of Hermitian Geometry

Moving from the one-dimensional world of Riemann surfaces to the higher-dimensional space of complex manifolds was a major conceptual step. While the 19th century established the basic framework of complex structures, the 20th century's task was to develop a complete differential geometry for these structures. This meant creating a calculus to measure lengths, angles, areas, and curvature on spaces that are locally like $\mathbb{C}^n$, not just $\mathbb{R}^{2n}$ with an extra condition. The link between complex analysis and Riemannian geometry is the Hermitian geometry.

1.2.1 From Riemannian to Hermitian Metrics

On a smooth manifold, a Riemannian metric is a smoothly varying, inner product g defined on the real tangent spaces. For a complex manifold M of complex dimension n, each tangent space T_pM is not just a $2n$-dimensional real vector space, but also an n-dimensional complex vector space, equipped with an almost complex structure J (where $J^2 = -I$). A Hermitian metric h is, at each point, a complex inner product (positive-definite Hermitian form) on this complex vector space. In other words, it is a map

$$h_p : T_pM \times T_pM \to \mathbb{C}$$

that is $\mathbb{C}$-linear in its first argument, $\mathbb{C}$-antilinear in its second, and satisfies two key conditions: $h(Y, X) = \overline{h(X, Y)}$ and $h(X, X) > 0$ when $X \neq 0$. Its real part, $g = \mathrm{Re}(h)$, is naturally a standard Riemannian metric on the underlying real manifold. Its imaginary part, $\omega = -\mathrm{Im}(h)$, is a real, non-degenerate 2-form.

The key compatibility condition is that the Riemannian metric g and the almost complex structure J satisfy

$$g(JX, JY) = g(X, Y)$$

for all tangent vectors X, Y. This condition means that J acts as an isometry of the metric g. A triple (M, J, g) that meets this condition is called an almost Hermitian manifold. When the almost complex structure J is integrable, i.e. it comes from a true complex manifold structure, the manifold is a Hermitian manifold.

1.2.2 The Fundamental 2-Form and the Concept of Compatibility

The imaginary part of the Hermitian form,

$$\omega(X, Y) = g(JX, Y),$$

is called the fundamental 2-form. It is a real (1,1)-form with respect to J. This enriches the geometry: a Hermitian manifold has not only a metric g, but also a compatible nondegenerate 2-form ω.

This compatible triple (J, g, ω) is the foundation. The metric g measures lengths, the complex structure J defines "multiplication by i" on tangent vectors, and the 2-form ω carries geometric information about the interaction between J and g. Their mutual relationships

$$g(X, Y) = \omega(X, JY),\ \omega(X, Y) = g(JX, Y)$$

show that they are completely interdependent. Given the complex structure J, any one of the pair (g, ω) determines the other. This deep connection between complex, Riemannian, and differential form data within a single object is what makes Hermitian (and later, Kähler) geometry uniquely rich.

1.2.3 Curvature in the Complex Realm

A key achievement of early 20th-century differential geometry was developing a workable theory of connections and curvature for Hermitian structures. The Levi-Civita connection ∇ of the underlying Riemannian metric g is the natural choice for differentiation (see [Lev17]). However, a basic question arises: does this torsion-free, metric-preserving connection also preserve the complex structure? In other words, does $\nabla J = 0$ hold? This condition requires parallel transport to commute with the action of J, it is not guaranteed by the integrability of J alone. It imposes a far stronger constraint on the compatibility between the metric and the complex structure, ultimately leading to the Kähler condition. Studying the curvature tensor R of g in this complex context led to the identification of specific components. By contracting the curvature tensor in a way that matches the complex structure, we get the Hermitian Ricci curvature form, which is a closed real (1,1)-form that would later be recognized as representing the manifold's first Chern class.

Local coordinate methods proved essential. In holomorphic coordinates $(z^1, \ldots, z^n)$, a Hermitian metric is given by

$$\mathrm{d}s^2 = h_{i\bar{j}} \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j,$$

where $\left(h_{i\bar{j}}\right)$ is a positive-definite Hermitian matrix of smooth functions. The fundamental 2-form is then

$$\omega = \frac{\sqrt{-1}}{2} h_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j.$$

In these coordinates, the details of the connection and its curvature, especially the difference between the Levi-Civita connection and the Chern connection (the unique Hermitian connection compatible with the holomorphic structure), could be calculated and analyzed more clearly.

1.3 Élie Cartan and the Global Influence of the Differential Geometry School

The local, coordinate-dependent differential calculus of the late 19th and early 20th centuries was transformed into a powerful, intrinsic, and global mathematical field mainly thanks to the genius of Élie Cartan. While he was not primarily a complex geometer, Cartan's innovative restructuring of differential geometry provided the universal language and tools without which the advanced global theory of Kähler manifolds would be impossible. His influence is present throughout modern geometry.

1.3.1 The Exterior Calculus and Differential Forms

Cartan's masterful development and promotion of exterior differential calculus was perhaps his most important contribution to geometry (see [Car99]). Replacing the complicated classical tensor calculus with the elegant, coordinate-invariant language of differential forms transformed global analysis.

Integration and Topology. The generalized Stokes theorem, $\int_M d\alpha = \int_{\partial M} \alpha$, neatly linked local differentiation and global integration, becoming the foundation of de Rham cohomology theory. This provided a natural way to discuss global topological invariants (like Betti numbers) using differential forms and became a key connection for later Hodge theory on complex manifolds.

Intrinsic Calculations. Operations like the exterior derivative d, wedge product $\wedge$, and Lie derivative became standard tools for intrinsic geometric calculations, freeing geometers from the limitations of coordinate-based expressions.

1.3.2 The Method of Moving Frames and Connections

Cartan's method of moving frames was a geometric revolution. Instead of fixing a single coordinate system, he equipped the manifold with a field of orthonormal frames that changes from point to point. The change in the frame from one point to an infinitely close point is measured by connection 1-forms ω_i^j and their curvature 2-forms Ω_i^j.

Structural Equations. Cartan's first and second structural equations are (see [Car23])

$$\begin{cases} d\theta^i = -\omega^i_j \wedge \theta^j \\ \Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \end{cases}$$

They packaged the entire local geometry (torsion and curvature) into a compact, manageable system of differential forms. This formalism was perfectly suited to complex geometry, where we naturally use unitary frames adapted to the Hermitian structure $(e_1, \ldots, e_n, Je_1, \ldots, Je_n)$.

Principal Bundles and G-Structures. The moving frame method laid the groundwork for the modern theory of principal bundles and connections. For example, a Hermitian structure is a reduction of the frame bundle (a $\mathrm{GL}(2n, \mathbb{R})$-structure) to a $\mathrm{U}(n)$-structure. Cartan's techniques for studying geometries defined by reducing the structure group became a model for understanding the intrinsic geometry of complex and Kähler manifolds.

1.3.3 Cartan's Work on Symmetric Spaces and Holonomy

In his classification of Riemannian symmetric spaces (See [Car26] and [Car27]), Cartan not only solved a major classification problem but also developed deep insights into curvature and holonomy.

Holonomy Groups. The holonomy group of a Riemannian connection, which measures the linear transformation caused by parallel transport around closed loops, became a central object. Cartan recognized that special geometric structures (like complex or Kähler structures) are reflected in a reduction of the holonomy group. As would later become clear, a Kähler manifold is characterized by its Levi-Civita connection having holonomy contained in $U(n)$ (the unitary group). Cartan's work on classifying holonomy groups laid the foundation for Berger's later classification (see [Ber55]), which lists Kähler manifolds as one of the fundamental types of special holonomy.

Symmetric Spaces as Models. Many of the irreducible Hermitian symmetric spaces that Cartan classified (such as complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric) became key examples and model spaces in Kähler geometry. They show the perfect harmony between complex structure, metric and curvature.

1.3.4 A School of Thought and a Legacy

Cartan's influence extended far beyond his own papers. Through his students and collaborators (like Shiing-Shen Chern, who would become a central figure in complex geometry), and through the power and clarity of his methods, he established a school of geometric thinking.

Global, Intrinsic Viewpoint. He promoted for studying geometry from a global, coordinate-free perspective, which is essential for dealing with the global objects of complex manifold theory.

Algebraic Tools for Differential Problems. His use of Lie algebras and group theory to solve differential geometric problems created a powerful technique. This approach is everywhere in Kähler geometry, from the algebra of differential forms to the moment map construction in symplectic geometry.

In summary, by the 1930s, the groundwork was perfectly laid for the synthesis that would define Kähler geometry. The 19th century provided the complex analytic objects (Riemann surfaces, holomorphic functions). The early 20th century provided the specific geometric framework (Hermitian metrics) for higher-dimensional complex manifolds. And Cartan's school provided the universal, powerful, intrinsic differential geometric language (forms, connections, curvature, holonomy) needed to describe their global properties and discover profound new conditions. The next step required a mind capable of uniting these three strands into a single, beautifully restrictive, and highly productive definition. That mind belonged to Erich Kähler.

1.4 The Entry of Erich Kähler: The 1932 Paper and the Birth of Kähler Condition

Building on the theoretical foundation laid by Hermitian geometry and Cartan's calculus, the key synthesis came in 1932 with Erich Kähler's paper *Über eine bemerkenswerte Hermitische Metrik* (On a remarkable Hermitian metric, see [Käh33]). Unlike a broad, programmatic statement, this work was a focused, technical study that accidentally identified the key condition for a new type of geometry. Kähler's goal was not to establish a new field, but to systematically study a specific kind of Hermitian structure merely.

1.4.1 The Context and Core Object

Kähler started with a real 2n-dimensional manifold equipped with a Hermitian metric

$$\mathrm{d}s^2 = \sum g_{i\bar{k}} \mathrm{d}x_i \otimes \mathrm{d}\bar{x}_k,$$

and considered the invariants under "pseudo-conformal" transformations (commuting with conjugate). His main analytical tool was the alternating quadratic differential form

$$\omega = \sum g_{i\bar{k}} \mathrm{d}x_i \wedge \mathrm{d}\bar{x}_k,$$

built directly from the metric coefficients. Using the calculus of symbolic differential forms, Kähler discussed about the derivative

$$\mathrm{d}\omega = \sum \frac{\partial g_{i\bar{k}}}{\partial x_l} x_l \wedge x_i \wedge \bar{x}_k + \sum \frac{\partial g_{i\bar{k}}}{\partial \bar{x}_l} \bar{x}_l \wedge x_i \wedge \bar{x}_k,$$

which is an example of invariant form under pseudo-conformal transformations.

1.4.2 The Discovery of the Condition and the Potential

Within this general framework, Kähler singled out a "special case" of great importance: the condition

$$\mathrm{d}\omega = 0.$$

He proved that this closure condition forces the metric to take an extremely constrained and useful form. Specifically, there must exist a real-valued potential function U such that

$$g_{i\bar{k}} = \frac{\partial^2 U}{\partial x_i \partial x_{\bar{k}}}, \ \mathrm{d}s^2 = \sum \frac{\partial^2 U}{\partial x_i \partial x_{\bar{k}}} \mathrm{d}x_i \mathrm{d}\bar{x}_k.$$

This marked the birth of the Kähler potential. Kähler immediately pointed out that metrics of this type were not just formal curiosities, they appeared naturally in the modern theory of automorphic functions at the time, for example, in the context of hyperfuchsian and hyperabelian transformations.

1.4.3 Curvature and Invariants

Based on this foundation, Kähler calculated the Riemann curvature tensor for such metrics. A key result was a simple formula for the Ricci curvature components:

$$\mathrm{R}_{i\bar{k}} = \frac{\partial^2}{\partial x_i \partial x_{\bar{k}}} \log(D(U)), \text{ where } D(U) = \det(U_{i\bar{k}}) = \det\left(\frac{\partial^2 U}{\partial x_i \partial x_{\bar{k}}}\right).$$

From this Ricci form, he constructed a second alternating differential form

$$\Omega = \mathrm{R}_{i\bar{k}} \, \mathrm{d}x_i \wedge \mathrm{d}\bar{x}_k,$$

whose differential $\mathrm{d}\Omega$ vanishes.

He then turned his attention to the exterior multiplication of ω and Ω to get invariant differential forms. A central goal was to understand the scalar curvature integral $\int R \mathrm{d}v$. By applying Stokes' theorem, he expressed this integral as a boundary term, but noted that the underlying integrand was not fully invariant under pseudoconformal transformations. For the one-dimensional complex case ($n = 1$), he showed how a correction term involving geodesic curvature restored invariance, and suggested the problem of finding a similar correction invariant for higher dimensions.

1.4.4 Link to Physics and Variational Problems

Kähler explicitly connected his geometric structures to modern physics at the time. He showed that when the Einstein field equations $R_{\alpha\beta} = \lambda g_{\alpha\beta}$ are applied to his metrics, they reduce to the nonlinear equation

$$\log D(U) - \lambda U = \psi(x) + \bar{\psi}(\bar{x}),$$

which can be simplified by choosing appropriate coordinates to the standard form

$$D(U) = e^{\lambda U}.$$

He provided examples, including the hyperfuchsian metric (with $\lambda = n+1$) and the hyperabelian metric (with $\lambda = 2$).

Furthermore, he recognized a deep variational principle at work. He observed that the determinant $D(U)$ arose as the variational derivative of Wirtinger's integral invariant $\mathfrak{C}(U)$. Therefore, solving the equation $D(U) = e^{kU}$ was equivalent to solving a specific variational problem. He also noted that the equation $D(U) = 0$ appeared in the theory of hyperabelian functions under the constraint $\mathfrak{C}(U) > 0$.

1.4.5 The Geometric Synthesis and Its Implications

The true, long-lasting significance of Kähler's 1932 paper lies exactly in identifying the closure condition $\mathrm{d}\omega = 0$ and its immediate mathematical consequences:

- The Symplectic Structure: This condition makes ω a closed, nondegenerate 2-form, giving the complex manifold a compatible symplectic structure.
- The Local Potential: It guarantees the existence of a local scalar function from which the entire metric is derived. This is a dramatic simplification of the Hermitian condition.
- The Curvature Simplification: It leads to an extremely simple and powerful expression for the Ricci curvature, in terms of the determinant of the potential's Hessian.

Kähler did not use the terms "Kähler manifold" or "Kähler metric", nor did he fully explain that this condition is equivalent to the complex structure being parallel ($\nabla J = 0$), which would be refined by later geometers using Cartan's connection theory. His contribution was discovering the essential, fruitful constraint itself. By singling out the condition $\mathrm{d}\omega = 0$ and showing its links to a potential, simplified curvature, and important problems in analysis and physics, Erich Kähler provided the seed from which the entire structure of Kähler geometry would grow. His "remarkable Hermitian metric" was, in fact, the key that would unlock the modern era of complex differential geometry.

Chapter 2

The Golden Age of Kähler Geometry

2.1 Hodge's Harmonic Integration Theory: A Perfect Structure on Kähler Manifolds

The discovery of the Kähler condition brought to light a class of manifolds where complex, Riemannian, and symplectic structures coexist harmoniously. However, the deep topological consequences of this harmony were not explored until the work of the British mathematician William Vallance Douglas Hodge. Building on Georges de Rham's foundational ideas, Hodge developed a theory that, when applied to Kähler manifolds, revealed an algebraic structure with amazing symmetry and rigidity in their cohomology rings. This structure seemed almost "too perfect to be true."

2.1.1 The Hodge Theorem: From Differential Forms to Harmonic Representatives

De Rham's theorem (around 1931, see [De 31]) had built a deep link between analysis and topology: the cohomology groups of a smooth manifold M, which are topological invariants, can be calculated using differential forms. Specifically, the k-th de Rham cohomology group $H^k_{\mathrm{dR}}(M, \mathbb{C})$ is equivalent to the vector space of closed k-forms, with exact k-forms treated as equivalent.

Hodge's great contribution, which was presented in his work of the 1930s and later in his 1941 book *The Theory and Applications of Harmonic Integrals* (see [Hod34], [Hod35], [Hod36] and [Hod41]), was to provide a standard and analytically natural way to pick a single, unique representative for each cohomology class on a Riemannian manifold. He introduced the Laplace-Beltrami operator $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$, an elliptic differential operator that acts on differential forms and generalizes the classic Laplacian. A form η that satisfies $\Delta\eta = 0$ is called a harmonic form.

The Hodge theorem states that on a compact, oriented Riemannian manifold, every de Rham cohomology class has a unique harmonic representative. Furthermore, the space of harmonic k-forms (denoted $\mathcal{H}^k(M)$) is finite-dimensional and equivalent to $H^k_{\mathrm{dR}}(M)$. This creates a direct bridge between topology and analysis.

2.1.2 Refinement Through Kähler Geometry: Decomposition by (p,q)-Type

On a general Riemannian manifold, the Hodge theorem is powerful but lacks structure. The key breakthrough happens when the manifold is a Kähler manifold. Here, the complex structure J and the compatible Kähler metric g make the Laplacian behave in a surprisingly well-behaved way with respect to the complex decomposition of forms.

On a complex manifold M of dimension n, complex differential forms can be decomposed by their type (p, q), where p is the number of holomorphic differentials $\mathrm{d}z^i$ and q is the number of anti-holomorphic differentials $\mathrm{d}\bar{z}^j$. Let $\mathcal{A}^{p,q}(M)$ stand for the space of smooth (p,q)-forms.

The central "miracle" of Kähler geometry is that on a compact Kähler manifold, the Laplacian Δ preserves the (p,q)-type of forms. In other words, if a k-form is harmonic, all its (p,q)-type components with $p+q=k$ are harmonic on their own. As a result, the harmonic forms themselves can be decomposed as

$$\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M),$$

where $\mathcal{H}^{p,q}(M)$ denotes the harmonic forms of type (p,q).

This has an immediate and remarkable effect on cohomology. Since each de Rham cohomology class is uniquely represented by a harmonic form, the de Rham cohomology groups inherit this decomposition

$$H^k_{\mathrm{dR}}(M,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M).$$

This is the Hodge decomposition of the cohomology of a compact Kähler manifold. It tells us that the complex cohomology of such a manifold is not just a vector space, but a bigraded vector space (graded by two indices p and q).

2.1.3 The Hodge Diamond: A Crystalline Symmetry

The Hodge decomposition interacts with several basic operations, resulting in a set of symmetries that can be represented by the elegant symbol of the Hodge diamond.

Complex Conjugation. Taking the complex conjugate of forms swaps the holomorphic and anti-holomorphic parts. This gives the symmetry

$$H^{p,q}(M) \cong \overline{H^{q,p}(M)}.$$

The Hodge Star and Serre Duality. The Hodge star operator $*$ associated to the Riemannian metric induces an isomorphism between forms of different degrees. On a complex manifold of dimension n, it maps (p,q)-forms to $(n-p,n-q)$-forms. On a compact Kähler manifold, this intertwines with Dolbeault cohomology, through Hodge decomposition, and induces a canonical isomorphism known as Serre duality

$$H^{p,q}(M) \cong H^{n-p,n-q}(M)^*.$$

This duality provides a canonical pairing between these cohomology groups and imposes a central symmetry on the Hodge diamond.

Consequences of Closeness $\mathrm{d}\omega = 0$. The Kähler condition $\mathrm{d}\omega = 0$ means that the Kähler form ω itself is harmonic and of type (1,1). Taking the wedge product with ω (via the Lefschetz operator L) maps harmonic (p,q)-forms to harmonic $(p+1,q+1)$-forms, creating a structural link between different parts of the cohomology. Furthermore, this induces the Hodge-Lefschetz decomposition and the hard Lefschetz theorem.

These symmetries—complex conjugation, Serre duality, and the action of the Kähler class—force the dimensions $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(M)$ to arrange into a diamond shape for a manifold of complex dimension n. For example, for a Kähler surface ($n=2$), the Hodge diamond is

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

With $h^{0,0} = h^{2,2} = 1$, $h^{2,0} = h^{0,2}$, and $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2}$, the diamond shows perfect bilateral and vertical symmetry. The central number $h^{1,1}$ is a particularly important topological invariant.

2.1.4 The Geometric Impact

When applied to Kähler manifolds, Hodge theory revealed a degree of order that went well beyond its original roots in Riemannian geometry. It showed that the Kähler condition is not just a nice differential constraint, but a source of deep algebraic structure in topology.

Topology with an Algebraic Flavor. The Hodge decomposition places strong restrictions on the possible Betti numbers

$$b_k = \sum_{p+q=k} h^{p,q}.$$

For example, all odd Betti numbers b_{2k+1} must be even, because they are sums of paired dimensions $h^{p,q}$ and $h^{q,p}$.

A Signature of Kählerity. The existence of a Hodge decomposition on cohomology (with its associated symmetries) became a powerful necessary condition for a compact complex manifold to have a Kähler metric. Many complex manifolds (such as the Hopf surface) violate these conditions (for example, $b_1 = 1$, which is odd), proving they are non-Kähler.

A Bridge to Algebraic Geometry. For projective algebraic manifolds (which are Kähler), Hodge theory created a direct link between purely topological invariants ($h^{p,q}$) and the dimensions of spaces of holomorphic objects. This comes from Dolbeault's theorem: $H^{p,q}(M) \cong H^q(M, \Omega^p)$, where Ω^p is the sheaf of holomorphic p-forms. This solidified the union of differential geometry and algebraic geometry.

In summary, Hodge's theory of harmonic integrals did not just provide a useful analytical tool. It acted like a powerful X-ray, revealing the hidden, perfectly symmetrical structure within the cohomology of a Kähler manifold. The "Hodge diamond" became the symbol of this refined structure, a crystal-clear mathematical object that proved the deep, rigid harmony enforced by the simple condition $d\omega = 0$. It established Kähler geometry as the natural field where topology, analysis, and complex algebraic geometry meet in a state of beautiful balance.

2.2 The Deep Integration of Kähler Geometry and Complex Algebraic Geometry

Hodge theory revealed the deep internal structure that the Kähler condition imposes on a manifold's topology. At the same time, another equally important development was the growing connection between Kähler geometry and complex algebraic geometry—the study of complex manifolds defined by polynomial equations. This relationship evolved from recognizing that algebraic manifolds are naturally Kähler, to the great achievement of figuring out which Kähler manifolds are actually algebraic. This combination was connected through standard metrics, characteristic classes, and a fundamental embedding theorem.

2.2.1 The Prototype: Projective Space and the Fubini-Study Metric

Complex projective space $\mathbb{CP}^n$ is the basic object of algebraic geometry. As a set, it includes all complex lines passing through the origin in $\mathbb{C}^{n+1}$. It is naturally a compact complex manifold, covered by $n + 1$ coordinate charts.

The Fubini-Study metric is the standard Kähler metric on $\mathbb{CP}^n$. It can be defined in two equivalent ways, both highlighting its natural, group-invariant property.

Quotient Construction. It comes from the standard Hermitian metric on $\mathbb{C}^{n+1}\setminus\{0\}$, by taking the quotient under the $\mathbb{C}^*$-action (multiplying by non-zero complex scalars) and restricting it to the unit sphere. More simply, in homogeneous coordinates $[Z_0 : \cdots : Z_n]$, the Kähler form is defined locally as

$$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Big(|Z_0|^2 + \cdots + |Z_n|^2\Big).$$

Invariance. Up to scaling, it is the only Kähler metric on $\mathbb{P}^n$ that is invariant with respect to the unitary group $U(n+1)$ acting projectively.

The Fubini-Study metric is very important. Its Ricci form is proportional to the Kähler form itself: $\mathrm{Ric}(\omega_{\mathrm{FS}}) = (n+1)\omega_{\mathrm{FS}}$. So it is a Kähler-Einstein metric with a positive Einstein constant. This simple, beautiful object became a model for positive curvature Kähler metrics on algebraic manifolds.

2.2.2 Algebraic Varieties as Kähler Manifolds

A smooth projective algebraic variety V is defined as the set of points in $\mathbb{CP}^N$ where a collection of homogeneous polynomials equal zero. Such a variety gets a complex manifold structure from the surrounding projective space. Importantly, it also gets a Kähler structure: the restriction of the Fubini-Study form ω_{FS} to V (by pulling back the inclusion map $V \hookrightarrow \mathbb{CP}^N$) is still a closed, positive $(1,1)$-form. This makes $\big(V, \omega_{\mathrm{FS}}\big|_V\big)$ a compact Kähler manifold.

Thus, the whole field of projective algebraic geometry naturally fits within Kähler geometry. Every projective manifold has a special Kähler metric whose cohomology class $\big[\omega_{\mathrm{FS}}\big|_V\big] \in H^2(V,\mathbb{R})$ is integral—it comes from a class in $H^2(V,\mathbb{Z})$, which is the pullback of the hyperplane class from $\mathbb{CP}^N$. This integrality is a key topological feature that shows the manifold has an algebraic origin.

2.2.3 Kodaira's Embedding Theorem: From Kähler to Algebraic

The reverse question is one of the most central in complex geometry: Which compact Kähler manifolds are actually projective algebraic varieties? Not all of them are. A complex torus $\mathbb{C}^n/\Lambda$ is always Kähler, but it is projective (called an abelian variety) only if it satisfies the Riemann bilinear relations, which a non-trivial condition on the lattice Λ.

The final answer was given by Kunihiko Kodaira in his important 1954 paper *On Kähler Varieties of Restricted Type* (see [Kod54]). In the paper, Kodaira stated the theorem (which became known as the Kodaira Embedding Theorem): A compact complex manifold M is a projective algebraic variety (this is equivalent to saying that it can be holomorphically embedded into some $\mathbb{CP}^N$ thanks to the Chow's theorem) if and only if it has a Kähler metric where the Kähler form ω corresponds to an integral cohomology class. In other words, $[\omega] \in H^2(M,\mathbb{Z}) \subset H^2(M,\mathbb{R})$.

Such a Kähler metric is called a Hodge metric since it first appeared in Hodge's paper in 1951 (see [Hod51]). The condition $[\omega] \in H^2(M,\mathbb{Z})$ means there exists a holomorphic line bundle $L \to M$ whose first Chern class is $c_1(L) = [\omega]$. Kodaira's proof is a masterpiece applying a blend of complex geometry and algebraic methods, focusing on the study of sections of positive line bundles.

2.2.4 The Role of the First Chern Class and Positivity

Kodaira's theorem can be rephrased using a fundamental topological invariant: the first Chern class of the manifold, $c_1(M) = c_1(T^{1,0}M)$, where $T^{1,0}M$ is the holomorphic tangent bundle.

For a projective manifold M with a Hodge metric ω, the associated line bundle L is positive. A key observation is that for large enough integers m, the bundle $L^m \otimes K_M^{-1}$, where K_M is the canonical bundle, becomes positive. This means the class $m[\omega] + c_1(M)$ can be represented by a positive $(1,1)$-form. In particular, this implies that the first Chern class $c_1(M)$ cannot be too negative compared to a positive class.

The most notable special case occurs when the Kähler class itself is proportional to the first Chern class. If M has a Kähler metric ω such that $[\omega] = \lambda c_1(M)$ for some $\lambda > 0$, then $c_1(M)$ is a positive class. By Kodaira embedding theorem, M is projective. This condition—the positivity of the first Chern class—defines the important class of Kähler manifolds called Fano manifolds.

2.2.5 Synthesis and Legacy

The combination of Kähler geometry and algebraic geometry by the mid-20th century was deep. The Fubini-Study metric provided a universal model of positive curvature. The principle of inheritance ensured that the large and rich field of algebraic geometry can be seem as a subfield of Kähler geometry. Most importantly, Kodaira Embedding Theorem provided a classification principle: within compact Kähler manifolds, the algebraic ones are exactly those with a Hodge metric—a condition that can be detected in topology through the integrality or positivity of cohomology classes. This established a powerful relationship:

- From Algebra to Geometry: Start with an algebraic variety, give it the inherited Fubini-Study metric, and use analytic methods (Hodge theory, PDE) to study its properties.
- From Geometry to Algebra: Start with a compact Kähler manifold, check if its Kähler class is integral (or if $c_1(M)$ is positive), and if so, conclude it is algebraic and can be studied with algebraic geometry tools.

This interaction became the driving force for many future developments. It made Kähler geometry the necessary analytic framework for algebraic geometry, while also raising the challenging problem of understanding the exact differences between general Kähler manifolds and projective ones—a distinction that would later be deeply explored in the context of the Kähler-Einstein problem and stability theory. Kodaira's theorem did not just end a chapter; it opened the door to the modern era where the existence of special metrics is closely linked to global algebraic and topological stability.

2.3 The Early Quest for Canonical Metrics: The Einstein Condition

The close connection between Kähler geometry and algebraic geometry—strengthened by Kodaira's theorem—gave a strong motivation to find "best" or "standard" metrics on these manifolds. In Riemannian geometry, the most natural candidate for such a special metric is an Einstein metric: one where the Ricci curvature tensor is proportional to the metric tensor itself. On a Kähler manifold, this search becomes particularly elegant and constrained. It leads directly to key topological obstruction and sets the stage for one of the field's defining conjectures.

2.3.1 The Kähler-Einstein Condition

On a general Riemannian manifold (M, g), the Einstein condition is written as:

$$\mathrm{Ric}(g) = \lambda g,$$

where λ is a real constant called the Einstein constant. The Ricci tensor $\mathrm{Ric}(g)$ is a symmetric 2-tensor derived by contracting the full Riemann curvature tensor.

A big simplification in Kähler geometry comes from the compatibility of all its structures. The Ricci tensor of a Kähler metric is not only symmetric but also J-invariant, meaning $\mathrm{Ric}(JX, JY) = \mathrm{Ric}(X, Y)$. This invariance ensures it is compatible with the complex structure. As a result, the Ricci tensor can be identified with a real, closed (1,1)-form known as the Ricci form, denoted by ρ or $\mathrm{Ric}(\omega)$.

Locally, in holomorphic coordinates, we can write the Ricci form as:

$$\rho = \mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det\big(g_{i\bar{j}}\big).$$

Importantly, this formula shows that ρ is a globally defined closed (1,1)-form. Its cohomology class does not depend on the specific Kähler metric within a given Kähler class.

Given this, the Einstein condition for a Kähler metric g (with associated Kähler form ω) translates neatly into a condition on forms:

$$\mathrm{Ric}(\omega) = \lambda\omega.$$

This is the Kähler-Einstein condition. The proportionality constant λ must be real, and the equation means the Ricci form is proportional to the Kähler form itself.

2.3.2 The First Chern Class as a Topological Obstruction

A key result in complex geometry is identifying a fundamental topological invariant from curvature. The first Chern class of a complex manifold M, denoted $c_1(M)$, is the first Chern class of its holomorphic tangent bundle $T^{1,0}M$. A remarkable fact directly following from the local formula for ρ is that

$$[\rho] = 2\pi\, c_1(M) \in H^2(M, \mathbb{R}).$$

In other words, the cohomology class of the Ricci form divided by 2π is exactly the first Chern class. This is a purely topological invariant, determined only by the complex structure of M.

The Kähler-Einstein equation $\mathrm{Ric}(\omega) = \lambda\omega$ therefore implies a cohomological constraint:

$$2\pi\, c_1(M) = [\mathrm{Ric}(\omega)] = \lambda[\omega].$$

Thus, for a Kähler-Einstein metric to exist, the first Chern class $c_1(M)$ must be a real multiple of some Kähler class $[\omega]$. This single equation splits the entire problem into three distinct, exhaustive cases based on the sign of the topological invariant $c_1(M)$:

- The Chern Zero Case ($c_1(M) = 0$): Topologically, this means $\lambda[\omega] = 0$. Because $[\omega] \neq 0$ (it is a Kähler class), λ must be 0. So The required metric must satisfy $\mathrm{Ric}(\omega) = 0$, such a metric is called a Ricci-flat Kähler metric. The existence question becomes: On a compact Kähler manifold with $c_1(M) = 0$, does there exist a Kähler metric with zero Ricci form in any given Kähler class?

- The Chern Negative Case ($c_1(M) < 0$): The equation $2\pi\, c_1(M) = \lambda[\omega]$ forces the Einstein constant λ to be negative. So the required metric satisfies $\mathrm{Ric}(\omega) = \lambda\omega$ with $\lambda < 0$. Since $[\omega]$ must be in the negative direction of $c_1(M)$, the Kähler class $[\omega]$ is not arbitrary. The existence question is: On a compact Kähler manifold with $c_1(M) < 0$, does there exist a Kähler-Einstein metric with a negative constant in the specific class $[\omega] = -\frac{2\pi\, c_1(M)}{|\lambda|}$?

- The Chern Positive Case ($c_1(M) > 0$): The equation forces the Einstein constant λ to be positive. The required metric satisfies $\mathrm{Ric}(\omega) = \lambda\omega$ with $\lambda > 0$. Again, the Kähler class is not free—it must be proportional to $c_1(M)$. Manifolds with $c_1(M) > 0$ are exactly Fano manifolds (by Kodaira's theorem, they are projective). The question is: On a Fano manifold, does there exist a Kähler-Einstein metric with a positive constant in the class proportional to $c_1(M)$?

2.3.3 Early Obstruction and the Emerging Landscape

By the mid-1950s, the three-way classification based on the sign of $c_1(M)$ had provided a clear framework for the Kähler-Einstein problem. However, it was clear right away that this necessary condition was far from enough. Without a general theory, the search for canonical metrics remained a set of difficult, unsolved questions in each case.

The field was made up of isolated examples and unclear boundaries. No unifying principle explained when and why such special metrics should exist. It was exactly at this moment, when questions were clear but answers were scarce, that Eugenio Calabi put forward a visionary conjecture. This conjecture promised to bring order to the entire field. His proposal did not start with the Einstein equation. Instead, it came from a deeper, more basic question about the very nature of Ricci curvature on Kähler manifolds.

2.4 The Foresight of Calabi: The Calabi Conjecture

Against the background of the three-way classification based on the first Chern class, the Italian-American mathematician Eugenio Calabi put forward a deep and unifying conjecture in the 1950s (see [Cal54] and [Cal57]). His goal was not to solve the Kähler-Einstein problem directly, but to answer a more basic question about the relationship between a Kähler metric and its Ricci form.

2.4.1 The Precise Statement of the Conjecture

Calabi's key idea focused on the cohomological nature of Ricci curvature in Kähler geometry. He identified the core problem: how to realize a given Ricci form within a fixed Kähler class. It stated as follows:

Let (M, ω) be a compact Kähler manifold. Let Ω be any closed real $(1,1)$-form that belongs to the cohomology class $2\pi\, c_1(M)$. Then, there exists a unique Kähler metric ω' in the Kähler cohomology class $[\omega]$ whose Ricci form $\mathrm{Ric}(\omega')$ is exactly the given form Ω.

In simple terms, we can choose a specific closed $(1,1)$-form, which lies in the cohomology class determined by the manifold's first Chern class, as the Ricci curvature. The conjecture claims that there is exactly one Kähler metric in a given class that achieves this specific Ricci curvature target.

2.4.2 Geometric and Topological Significance

The depth of the conjecture comes from how it redefines the problem of finding canonical metrics:

A Problem of Realization. It is important to tell this apart from the Kähler-Einstein problem. The Calabi conjecture claims that there exists a metric with a fixed, given Ricci form Ω, while the Kähler-Einstein condition ($\mathrm{Ric}(\omega) = \lambda\omega$) requires the Ricci form to be proportional to the unknown metric form itself. This is a very specific condition on Ω. The Calabi conjecture does not guarantee that Kähler-Einstein metrics exist. Instead, it asks whether any representative of $2\pi\, c_1(M)$ can be the Ricci form of some metric in $[\omega]$. These two problems are closely related but logically different.

The Primacy of the Cohomology Class. The conjecture shows a basic constraint and a possible freedom. The possible Ricci curvatures are not arbitrary, their cohomology class is fixed by topology as $2\pi\, c_1(M)$. However, Calabi boldly proposed that within this topological constraint, there is full freedom at the form level: every representative of this class can be the Ricci form of some Kähler metric. This transformed Ricci curvature from a local, analytical object derived from a metric into a global, topological piece of data that we can choose freely.

A Uniqueness Principle. The claim of uniqueness was just as important as the claim of existence. It implied a strong rigidity: a Kähler metric in a fixed cohomology class is uniquely determined by its Ricci curvature form. This established a clear idea of a "canonical" relationship between metrics and their Ricci forms within a class.

A Foundational Framework. Calabi's conjecture laid the foundation for finding "best" metrics. If it were true, we could in principle choose a particularly desirable Ricci form (for example, one that is harmonic, or one proportional to a given form) and be certain that a unique metric producing that curvature exists. This did not completely solve the Einstein problem, but it created the mathematical framework needed to tackle that problem.

2.4.3 The Mathematical Challenge

Calabi turned the conjecture into a difficult problem in geometric analysis. He set out the necessary and sufficient conditions for proving it. This established a clear mathematical target, but it did not include the explicit PDE (partial differential equation) formulation that would later be key to solving the conjecture. The main challenge was to use global nonlinear analysis on complex manifolds to show that the topological condition $[\Omega] = 2\pi c_1(M)$ is both necessary and sufficient for the desired metric to exist.

For more than two decades, the Calabi Conjecture remained a central unsolved problem in complex differential geometry. Its statement was a reflection of Calabi's deep geometric intuition—he saw a basic link between curvature and topology that was far from obvious. It served as a guiding hypothesis, promising that if the topology allows it, we can shape the curvature accordingly. The conjecture set a clear research direction, shifting the focus from finding obstacles to building solutions through a deep combination of geometry and analysis. Proving it would require a new level of technical skill and would redefine the field.

Chapter 3

The Proof of Calabi Conjecture and Its Impact

3.1 The Complex Monge-Ampère Equation

The Calabi Conjecture put forward a broad geometric vision. To turn this vision into a manageable mathematical problem, we needed to express it using the clear language of partial differential equations (PDEs). This process of turning the conjecture into an equation showed that its core is a specific, fully nonlinear elliptic PDE called the complex Monge-Ampère equation (see [Cal57]).

3.1.1 From Geometric Prescription to Analytic Equation

We start with a compact Kähler manifold (M, ω_0), where ω_0 is the initial Kähler form. Let Ω be a given closed real (1,1)-form that belongs to the cohomology class $2\pi\, c_1(M)$. Our goal is to find a new Kähler form ω_φ in the same cohomology class $[\omega_0]$ such that

$$\mathrm{Ric}(\omega_\varphi) = \Omega.$$

Since ω_φ is in the same class as ω_0, we can write it as

$$\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where φ is a smooth real-valued function on M, known as the Kähler potential. The condition $\omega_\varphi > 0$ (meaning it remains a positive-definite (1,1)-form, and thus a valid metric) is called the positivity condition. This condition makes the problem a geometric PDE rather than just an analytic one.

The key step is related to the Ricci form. Remember its local expression in holomorphic coordinates for any Kähler metric ω

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det\big(g_{i\bar{j}}\big),$$

where $g_{i\bar{j}}$ are the components of the metric tensor corresponding to ω. Applying this to our unknown metric ω_φ and the given form Ω, the equation $\mathrm{Ric}(\omega_\varphi) = \Omega$ becomes

$$-\sqrt{-1}\partial\bar{\partial}\log\det\left((g_0)_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right) = \Omega.$$

Next, we use the cohomological condition. Since both $\mathrm{Ric}(\omega_0)$ and Ω belong to the same class $2\pi\, c_1(M)$, their difference is an exact form. So, there exists a smooth real function F on M (unique up to an additive constant) such that

$$\Omega - \mathrm{Ric}(\omega_0) = \sqrt{-1}\partial\bar{\partial}F.$$

Substituting this into the previous equation gives

$$-\sqrt{-1}\partial\bar{\partial}\log\det\big((g_0)_{i\bar{j}}+\varphi_{i\bar{j}}\big)=\mathrm{Ric}(\omega_0)+\sqrt{-1}\partial\bar{\partial}F.$$

We know that $\mathrm{Ric}(\omega_0)=-\sqrt{-1}\partial\bar{\partial}\log\det\big((g_0)_{i\bar{j}}\big)$, so the equation simplifies greatly. We can factor out the $\sqrt{-1}\partial\bar{\partial}$ operator, leading to:

$$\log\left(\frac{\det\big((g_0)_{i\bar{j}}+\varphi_{i\bar{j}}\big)}{\det\big((g_0)_{i\bar{j}}\big)}\right)=-F+C,$$

where C is a constant. The constant is found by integrating both sides over M. A standard normalization (requiring the integral of e^F with respect to the volume form of ω_0 to equal the total volume) lets us absorb the constant into F. This results in the standard form:

$$\det\big((g_0)_{i\bar{j}}+\varphi_{i\bar{j}}\big)=e^F\det\big((g_0)_{i\bar{j}}\big),\ \omega_0+\sqrt{-1}\partial\bar{\partial}\varphi>0.$$

This is the complex Monge-Ampère equation linked to the Calabi Conjecture.

3.1.2 The Nature of the Equation

The equation has several key features that make it both difficult and deeply geometric:

Fully Nonlinear. The unknown function φ appears nonlinearly inside the determinant of its second derivatives. This is not a quasilinear or semilinear equation. The nonlinearity is in the highest-order terms, so standard PDE methods don't work.

Elliptic. For the equation to be elliptic, the linearization of the operator must be a positive-definite operator. This ellipticity condition is exactly the same as the positivity condition $\omega_\varphi=\omega_0+i\partial\bar{\partial}\varphi>0$ since the linearized operator is essentially the Laplacian Δ_{ω_φ}. So the equation is only elliptic for admissible functions (those where ω_φ is a positive (1,1)-form). This closely connects analysis to geometry: we must work within the convex cone of Kähler potentials.

Determinantal Structure. The left-hand side is the determinant of the complex Hessian matrix $(\varphi_{i\bar{j}})$ with the background metric added. In one complex dimension, it reduces to the Poisson equation $\Delta\varphi=f$. In higher dimensions, its structure is much more complex. But it inherits special algebraic properties from being a determinant, which can be used to derive a priori estimates.

Global Setting on a Manifold. Unlike many classic PDEs studied on domains in Euclidean space, this equation is defined inherently on a compact complex manifold with no boundary. This means we can't use boundary conditions to control solutions. Instead, all control must come from the global geometry and topology of M, the function F, and the nonlinear structure of the equation itself.

In short, Calabi's visionary geometric conjecture was simplified to a precise, inherent analytic problem: Prove that there exists a smooth, admissible function φ that satisfies the complex Monge-Ampère equation on a compact Kähler manifold. The fate of the conjecture now depended on analysts' ability to solve this nonlinear elliptic equation in a global geometric setting—a task that required groundbreaking new methods.

3.2 Yau's Proof Strategy: Continuity Method and *A Priori* Estimates

Shing-Tung Yau's 1978 work supplied the decisive analytic framework (see [Yau78]). He organized the proof around the continuity method and, crucially, established a hierarchy of *a priori* estimates that are uniform along the continuity path.

3.2.1 The continuity method: openness and closedness

In his paper, Yau began by fixing a compact Kähler manifold (M, ω_0) and a smooth function F (normalized so that the volume constraint held). He reduced the problem to solving (the complex Monge-Ampère equation)

$$\det\big(g_{i\bar{j}} + \varphi_{i\bar{j}}\big) = e^F \det\big(g_{i\bar{j}}\big),\ g_{i\bar{j}} + \varphi_{i\bar{j}} > 0,$$

with a normalization such as $\int_M \varphi\, \omega_0^n = 0$ to remove the additive ambiguity.

He then introduced a 1-parameter family of equations interpolating between a trivial solvable case and the desired one:

$$\det\big(g_{i\bar{j}} + (\varphi_t)_{i\bar{j}}\big) = e^{tF+c_t} \det\big(g_{i\bar{j}}\big),\ g_{i\bar{j}} + (\varphi_t)_{i\bar{j}} > 0,$$

where $c_t \in \mathbb{R}$ was chosen so that the integrals on both sides were equal, and defined

$$S = \{t \in [0,1] \mid \text{the } t\text{-equation admits an admissible solution } \varphi_t \in C^{2,\alpha}(M)\}.$$

His argument then consisted of two standard steps:

- Openness: he proved that solvability at t_0 implied solvability for all t near t_0.
- Closedness: he proved that if $t_j \to t_\infty$ with $t_j \in S$, then a subsequence of φ_{t_j} converged (in a strong enough topology) to a solution at t_∞.

The openness followed by an implicit function type argument: the linearized operator at a solution was elliptic and invertible on mean-zero functions, so local solvability reduced to standard elliptic theory once the background metric remained Kähler along the path.

The heart of the paper was the closedness step, which required uniform *a priori* estimates for solutions that did not deteriorate as t approached the endpoint.

3.2.2 The main *a priori* estimates in Yau's argument

The proof was organized around a sequence of *a priori* estimates whose role was to make the continuity method compact. The key steps could be summarized as follows:

Uniform C^0 and gradient estimates. Yau first established a global bound for $\sup_M |\varphi|$ in terms of the given data (such as bounds on F). This C^0 control prevented the potential from drifting along the continuity path and served as the starting point for compactness. Building on it, he then derived a bound for $\sup_M |\nabla\varphi|$, again in terms of the given data, by applying the maximum principle to a carefully chosen auxiliary quantity involving $|\nabla\varphi|^2$ and φ.

The key second-order estimate: uniform equivalence of metrics. Yau established a two-sided bound on the eigenvalues of ω_φ with respect to ω_0, often phrased as controlling a trace quantity like $\operatorname{tr}_{\omega_0}(\omega_\varphi)$. This was the step where the determinant equation was converted into differential inequalities for the metric coefficients: he worked in holomorphic normal coordinates, computed Laplacians with respect to ω_φ, and applied the maximum principle to an auxiliary function such as $\log \operatorname{tr}_{\omega_0}(\omega_\varphi) - A\varphi$ (for a suitable constant A). Once this step was proved, the Monge-Ampère equation became uniformly elliptic along the whole continuity path.

Higher-order estimates and Schauder bootstrapping. With uniform ellipticity secured, Yau derived higher derivative bounds. He next proved a uniform bound on third derivatives of the potential, that is, on quantities such as $\varphi_{i\bar{j}k}$ in terms of the given metric and bounds on derivatives of f. And then Schauder theory upgraded the control to $C^{2,\alpha}$.

With these estimates established, Yau obtained closedness by a compactness argument. Along any sequence $t_j \to t_\infty$ with solutions φ_{t_j}, uniform $C^{2,\alpha}$ bounds yielded precompactness by the Arzelà-Ascoli theorem. Thus a subsequence converged to a limit solving the equation at t_∞.

Combining openness and closedness, Yau concluded that $S = [0, 1]$, and hence the target equation admitted an admissible solution $\varphi \in C^{2,\alpha}(M)$ at $t = 1$. Standard elliptic regularity (bootstrapping) then upgrades this solution to a smooth one. Moreover, the uniqueness (already anticipated by Calabi) was obtained by comparing two solutions in the same Kähler class and using an integration by parts argument to show their difference was constant.

3.3 Corollaries: Kähler-Einstein Metrics in the Non-Positive Cases

Yau's solution of the complex Monge-Ampère equation does not only provide an abstract existence theorem for a nonlinear PDE. It immediately yields canonical metrics in the two cases where the first Chern class is non-positive.

3.3.1 The $c_1(M) = 0$ case and Ricci-flat Kähler metrics

Assume $c_1(M) = 0$. Then the cohomology class of the Ricci form vanishes: $[\mathrm{Ric}(\omega)] = 2\pi\, c_1(M) = 0$. In particular, given any background Kähler form ω_0, we may prescribe $\Omega = 0$ in Calabi's formulation and seek a metric in $[\omega_0]$ with

$$\mathrm{Ric}(\omega_\varphi) = 0.$$

Yau's theorem produces such a Kähler metric uniquely in each Kähler class. This is the analytic birth of what are now called *Calabi-Yau metrics*: Ricci-flat Kähler metrics on compact Kähler manifolds with vanishing first Chern class.

From the viewpoint of holonomy, Ricci-flatness is only the first visible consequence. The Kähler condition forces the Levi-Civita holonomy to lie in $U(n)$, and the vanishing of $c_1(M)$ corresponds to the existence of a nowhere-vanishing holomorphic volume form (at least after passing to a finite cover in many geometric situations). In favorable cases, the holonomy further reduces to $SU(n)$, producing manifolds with special holonomy that became central examples in differential geometry and (later) in mathematical physics.

3.3.2 The $c_1(M) < 0$ case and negative Kähler-Einstein metrics

The negative first Chern class case ($c_1(M) < 0$) was the first setting in which the general existence of Kähler-Einstein metrics was established: Aubin proved it in 1976, and Yau gave an independent proof as part of his 1978 work (see [Aub76] and [Yau78]).

Cohomological normalization. Here the Kähler class is no longer a free parameter: the Einstein condition forces $[\omega] = -2\pi\, c_1(M)$. After rescaling, we may normalize the Einstein constant to $\lambda = -1$ and look for $\mathrm{Ric}(\omega) = -\omega$.

Analytic method. In the negative case, we look for an admissible potential φ such that

$$\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0,\ \mathrm{Ric}(\omega_\varphi) = -\omega_\varphi.$$

Equivalently, in local holomorphic coordinates this becomes a Monge-Ampère equation of the form

$$\det\big(g_{i\bar{j}} + \varphi_{i\bar{j}}\big) = e^{F+\varphi}\det\big(g_{i\bar{j}}\big),\ g_{i\bar{j}} + \varphi_{i\bar{j}} > 0,$$

where F is a smooth function determined by the background data. The proof then follows the same overall pattern by the continuity method together with uniform *a priori* estimates: openness by linear elliptic theory, and closedness by uniform C^0, second-order, and higher-regularity estimates.

Conclusion. Every compact Kähler manifold with $c_1(M) < 0$ admits a unique Kähler-Einstein metric of negative Ricci curvature in the canonical class $-2\pi c_1(M)$. Together with the $c_1(M) = 0$ corollary of Yau's solution of the Calabi conjecture, this completes the non-positive side of the Kähler-Einstein existence theory and leaves the positive (Fano) case as the subtle one.

3.4 Impact on Geometry and the Remaining Fano Case

The solution of the Calabi conjecture was far more than the resolution of a single existence problem. It changed the working style of complex differential geometry by showing that a global, fully nonlinear PDE on a compact manifold can be tamed by geometric ideas using *a priori* estimates. And it provided a systematic way to produce canonical metrics directly from cohomological data.

At a conceptual level, the theorem also sharpened a guiding philosophy: on a Kähler manifold, curvature is not merely a local invariant extracted from a chosen metric, but a global object that can be prescribed within a fixed cohomology class. This viewpoint would later be repeated, in a much subtler form, in the search for Kähler-Einstein and constant scalar curvature Kähler metrics.

3.4.1 Special holonomy and a new supply of geometric examples

As noted in the previous subsection, the $c_1(M) = 0$ corollary yields a canonical Ricci-flat Kähler metric in each Kähler class. What is historically striking is how this existence result immediately fed into the emerging theory of special holonomy: it produced a broad class of compact metrics whose holonomy is expected to reduce (typically to $SU(n)$ after mild assumptions), thereby providing a systematic source of "new" Riemannian geometries beyond the classical symmetric spaces.

Just as importantly, the theorem turned the existence of special holonomy manifolds from a rare curiosity into a systematic construction. It supplied a large and flexible family of compact examples that could be studied by deformation, degeneration, and surgery, and it gave geometers a reliable source of test cases for broader conjectures in Riemannian geometry and topology.

3.4.2 A new analytic toolkit for complex geometry

Yau's proof made the complex Monge-Ampère equation a standard tool. The strategy—set up a continuity path, prove uniform estimates, and conclude existence by compactness—became a template for many later problems in Kähler geometry. The emphasis on uniform C^0 control, second-order trace estimates, and bootstrapping also clarified which quantities are geometrically natural and robust under deformation.

In practice, the ability to solve a global, nonlinear elliptic equation on a compact complex manifold meant that geometers could translate algebro-geometric or topological input into metric information. In particular, we can start from a chosen cohomology class, build a canonical representative with prescribed Ricci curvature, and then feed the resulting metric into curvature identities, Bochner formulas, and maximum principle arguments, etc.

3.4.3 Applications to classification and algebraic geometry

Once the existence of canonical metrics became available, it could be used as a structural hypothesis in classification problems. A celebrated example is Yau's use of Kähler-Einstein metrics in his proof of the Frankel conjecture: the existence of a positive Kähler-Einstein metric provides strong curvature control, which in turn forces strong restrictions on the complex and topological structure of the manifold.

More broadly, Calabi-Yau metrics became central objects in complex and algebraic geometry, not only as geometric structures on a fixed manifold but also as metrics varying in families. They revealed deep links between analytic limits of metrics and algebraic degenerations of complex structures, a theme that would later become essential in the study of moduli.

3.4.4 The unresolved positive (Fano) case and the coming change of viewpoint

The solution of the Calabi conjecture also made clear that the non-positive cases are, in a sense, the "stable" ones. When $c_1(M) \leqslant 0$, the relevant Monge-Ampère equations admit estimates that lead to existence and uniqueness, and the outcome is a canonical metric determined by cohomological data.

When $c_1(M) > 0$, the situation changes dramatically. The Kähler-Einstein equation forces the metric class to be $[\omega] = 2\pi\, c_1(M)$, and new obstructions arise that have no analogue in the non-positive cases. In particular, the obstruction is no longer purely analytic; it is closely tied to holomorphic vector fields, degenerations of the complex structure, and the presence of additional symmetries.

This is where the historical narrative naturally turns. The positive (Fano) case would force geometers to look beyond existence theory for a single PDE and to search for a conceptual criterion governing when a canonical metric should exist. That search ultimately led to the modern paradigm linking canonical Kähler metrics to algebro-geometric stability.

Chapter 4

The Paradigm Shift: From Analysis to Stability

4.1 Obstructions on Fano Manifolds: Matsushima and Futaki

The non-positive cases of the Kähler-Einstein problem fit neatly into the framework of Yau's solution of the Calabi conjecture: once the cohomological sign is favorable, the complex Monge-Ampère equation admits the estimates needed for existence and uniqueness. The Fano case, by contrast, immediately revealed a new phenomenon. Even when the natural cohomological condition $[\omega] = 2\pi\, c_1(M)$ holds, the search for a positive Kähler-Einstein metric runs into obstructions coming from the complex automorphism group.

In hindsight, this is the point at which the Kähler-Einstein problem stopped being "just" a nonlinear PDE. The presence of nontrivial holomorphic vector fields means that the geometry is constrained by symmetry, and the right invariant to measure this constraint turns out to be algebraic in nature.

4.1.1 Why the Fano case is different

On a compact Kähler manifold, a Kähler-Einstein metric with $\lambda > 0$ must lie in the class $2\pi\, c_1(M)$. Thus, for a Fano manifold M we are naturally led to solve

$$\mathrm{Ric}(\omega) = \omega, \qquad [\omega] = 2\pi\, c_1(M).$$

At the level of the Monge-Ampère equation, the sign changes in the exponent are already a warning: the analytic estimates used in the non-positive cases do not directly carry over. But the deeper issue is that the Einstein equation in the positive case interacts strongly with the group $\mathrm{Aut}(M)$ of holomorphic automorphisms.

When $\mathrm{Aut}(M)$ is nontrivial, it acts on the space of Kähler metrics in the class $2\pi\, c_1(M)$, and we cannot expect an elliptic equation to have an isolated solution without taking quotient by this symmetry. Historically, this manifested first as a necessary Lie-theoretic condition due to Matsushima, and then as a more delicate numerical obstruction discovered by Futaki.

4.1.2 Matsushima's reductivity obstruction

In 1957, Matsushima proved a fundamental restriction on the automorphism group of a Kähler-Einstein Fano manifold: if M admits a Kähler-Einstein metric with $\lambda > 0$, then the Lie algebra $\mathfrak{aut}(M)$ of holomorphic vector fields is reductive (i.e. it splits as $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}$ with $\mathfrak{g}_{\mathrm{ss}}$ is semisimple). Equivalently, the identity component of $\mathrm{Aut}(M)$ is a complex reductive algebraic group. This result was the first clear indication that the Fano existence problem has built-in algebraic constraints (see [Mat57]).

Geometrically, reductivity can be viewed as a compatibility condition between the complex symmetries and the Riemannian structure: the Kähler-Einstein metric forces the automorphism group to behave as if it came from an isometry group after complexification. In particular, the existence problem cannot be studied solely by local PDE methods; we must track how global holomorphic symmetries affect both solvability and uniqueness.

4.1.3 Futaki's invariant: a numerical obstruction

Matsushima's theorem is powerful but coarse: reductivity is a structural condition on a Lie algebra. In 1983, Futaki discovered a finer obstruction, now called the Futaki invariant, which assigns to each holomorphic vector field a complex number, and whose nonvanishing rules out the existence of a positive Kähler-Einstein metric (see [Fut83]).

One convenient way to present Futaki's construction is as follows. Fix a Kähler form $\omega \in 2\pi\, c_1(M)$. Since $[\mathrm{Ric}(\omega) - \omega] = 0$, there exists a (real-valued) Ricci potential f_ω satisfying

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial} f_\omega, \quad \int_M e^{f_\omega}\omega^n = \int_M \omega^n.$$

For a holomorphic vector field $X \in \mathfrak{aut}(M)$, Futaki defined

$$\mathrm{Fut}(X) = \int_M X(f_\omega)\, \omega^n.$$

A nontrivial fact is that $\mathrm{Fut}(X)$ is independent of the choice of $\omega \in 2\pi c_1(M)$: it depends only on the complex structure of M and the vector field X. In particular, it is a character on the Lie algebra of holomorphic vector fields.

This invariance has two immediate conceptual consequences.

- It shows that obstructions to Kähler-Einstein metrics can be detected without knowing the metric itself, and we can compute them from algebro-geometric data.
- It clarifies why the Fano case cannot be reduced to a purely analytic existence theorem: even if we could solve the equation for a given background form, the answer is constrained by a deformation-invariant, algebraically defined quantity.

Moreover, when a Kähler-Einstein metric ω_{KE} exists, we have $\mathrm{Ric}(\omega_{\mathrm{KE}}) = \omega_{\mathrm{KE}}$, so the corresponding Ricci potential is constant. Hence $X(f_{\omega_{\mathrm{KE}}}) = 0$ and the Futaki invariant must vanish. Thus, $\mathrm{Fut} \neq 0$ is a definitive obstruction.

4.1.4 From obstructions to a new guiding principle

Taken together, Matsushima's theorem and Futaki's invariant changed the interpretation of the problem. They suggested that the right criterion for existence should be expressed not as a piecemeal collection of analytic estimates, but as a global, deformation-stable condition on the underlying polarized variety. This shift in viewpoint is the seed of the modern stability paradigm: the expectation that canonical Kähler metrics exist exactly when the algebraic geometry of $(M, -K_M)$ satisfies an appropriate notion of stability.

4.2 Yau's Programmatic Conjecture: From PDE to Stability

In the 1980s and 1990s, Yau proposed a further, far-reaching guiding principle: rather than accumulating separate obstructions, one should expect a single, invariant algebro-geometric notion of stability to govern the existence of canonical Kähler metrics (see e.g. [Yau93], [Tia87], [Tia90]).

At a heuristic level, the proposal can be summarized as follows.

Canonical metrics as "best" representatives. The Kähler-Einstein equation (and its generalizations, such as constant scalar curvature Kähler metrics) should be understood as selecting the most symmetric, most balanced metric within a fixed cohomology class. This viewpoint is consistent with the non-positive cases, where solvability and uniqueness follow from the sign of $c_1(M)$.

Existence should be equivalent to stability. In the positive case, where the equation is constrained by automorphisms and degeneration phenomena, the correct existence criterion should be a stability condition phrased purely in terms of the polarized variety. In particular, it should be stable under deformation of complex structures and compatible with taking quotients by symmetries.

Failure of existence should be explained by degeneration. If a Fano manifold fails to admit a Kähler-Einstein metric, the reason should not be an analytic accident. Instead, it should manifest as an algebro-geometric instability: the variety should admit a degeneration to a "more unstable" limit in which a numerical invariant detects the failure.

This conjectural equivalence was not merely philosophical. It served as a bridge between two sets of objects that had been developing in parallel:

- on the analytic side, the study of nonlinear elliptic equations and variational functionals on the (infinite-dimensional) space of Kähler potentials;
- on the algebraic side, Geometric Invariant Theory (GIT) and the idea that the existence of "balanced" representatives is controlled by stability with respect to group actions.

In this way, Yau's program reframed the historical narrative: the central question became not only how to solve a PDE, but when the underlying complex manifold should be expected to support a canonical metric at all.

The next step was to make the word "stability" precise in the Kähler-Einstein context, by defining an algebraic notion that can be tested on all possible degenerations and produces a computable numerical invariant reducing to Futaki's character in special cases. This leads directly to Tian's introduction of K-stability and the Donaldson-Futaki invariant.

4.3 K-Stability: Its Birth and the Evolution of the Concept

With the guiding principle in place, the historical narrative now turns from "what obstructs existence" to "how to encode obstruction by degeneration". The key idea is to treat a Fano manifold not as an isolated object but as a polarized variety that may admit meaningful one-parameter limits. The stability paradigm predicts that such limits should decisively control whether a Kähler-Einstein metric can exist.

4.3.1 Tian's foundational work: the emergence of K-stability

To turn Yau's stability philosophy into a testable criterion, Tian provided the first systematic framework in the 1990s: the idea that one should probe a polarized manifold by one-parameter degenerations, and read (in)stability off the behavior of the polarization along the degeneration (see [Tia97] and [Tia00]).

In Tian's original viewpoint, he fixed a polarization (M, L) (for the Fano Kähler-Einstein problem we take $L = -K_M$) and then studied special $\mathbb{C}^*$-equivariant one-parameter degenerations of (M, L), frequently constructed after embedding M by a large multiple of L. The guiding principle is that such degenerations should be regarded as "tests" for whether (M, L) can support a canonical Kähler metric.

A Futaki-type numerical invariant for degenerations. A decisive feature of Tian's picture is that each degeneration should carry a numerical "weight" measuring how the polarization behaves along the limit. Historically, Tian's early formulations of stability for the Fano Kähler-Einstein problem were phrased using (often embedded) degenerations and a generalized Futaki-type invariant associated to such degenerations, extending Futaki's character from holomorphic vector fields to algebro-geometric limits.

The definition of K-(poly/semi)stability (Tian's core idea). With such a Futaki-type invariant assigned to each nontrivial degeneration, Tian declared (M, L) to be K-semistable if all weights are nonnegative, and K-stable if they are strictly positive. The variant most compatible with the Kähler-Einstein problem is what is now called K-polystability (Tian referred to it as weakly K-stability in his terminology): one allows equality only for the "trivial" (product) degenerations generated by holomorphic automorphisms. While this later terminology was standardized only after the test-configuration formalism became widespread, the underlying point—that symmetry forces a nontrivial equality case and must be quotiented out—is already present in Tian's early framework.

Geometric meaning. In Tian's paradigm, the role of K-stability is to rule out destabilizing one-parameter degenerations: a negative Futaki-type weight is the invariant numerical signature that the polarization becomes "more unbalanced" along a degeneration and hence obstructs the existence of a canonical metric.

With this guiding idea in place, we next turn to Donaldson's work, which reformulated and standardized the algebro-geometric testing objects and invariants in a way that made the analogy with Geometric Invariant Theory explicit.

4.3.2 Donaldson's key advance: test configurations and the Donaldson-Futaki invariant

If Tian's picture provided the guiding philosophy—probe a polarized manifold by one-parameter degenerations and read stability from a Futaki-type weight—Donaldson's contribution in the early 2000s was to supply a precise, flexible, and functorial formalism in which that philosophy could be carried out without choosing auxiliary embeddings. (see [Don02] and [Don05b])

From embedded degenerations to test configurations. Donaldson introduced the notion of a test configuration for a polarized manifold (M, L). Fixing an integer $r > 0$, A test configuration of exponent r consists of

- a normal variety $\mathcal{X}$ equipped with an algebraic $\mathbb{C}^*$-action;
- a $\mathbb{C}^*$-equivariant flat proper morphism $\pi : \mathcal{X} \to \mathbb{C}$ (where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication);
- a $\mathbb{C}^*$-linearized, π-relatively ample line bundle $\mathcal{L} \to \mathcal{X}$,

such that for every $t \neq 0$ there is a $\mathbb{C}^*$-equivariant isomorphism between the fiber $(\mathcal{X}_t, \mathcal{L}_t)$ and (M, L^r). The central fiber $(\mathcal{X}_0, \mathcal{L}_0)$ is the algebro-geometric "limit" produced by the degeneration. This construction packages, in a canonical way, what geometric invariant theory tests via one-parameter subgroups. And it does so for arbitrary polarized varieties, rather than for points in a fixed projective space.

The Donaldson-Futaki invariant. Given a test configuration, Donaldson defined a numerical invariant by extracting the leading asymptotics of the induced $\mathbb{C}^*$-representation on the spaces of sections. Let

$$N_k = \dim H^0(\mathcal{X}_0, \mathcal{L}_0^k),\ w_k = \text{total weight of the } \mathbb{C}^*\text{-action on } H^0(\mathcal{X}_0, \mathcal{L}_0^k).$$

For $k \gg 1$ these admit polynomial expansions

$$N_k = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),\ w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

The Donaldson-Futaki invariant is then defined by the combination

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{b_0 a_1 - b_1 a_0}{a_0}.$$

This expression is designed so that, for product test configurations induced by holomorphic vector fields, DF reduces to (a normalization of) Futaki's original character, thereby extending Futaki's obstruction from infinitesimal symmetries to genuine algebraic degenerations.

Standardizing K-stability and clarifying the equality case. With test configurations and DF in place, Donaldson proposed the now-standard definition: (M, L) is K-semistable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geqslant 0$ for all nontrivial test configurations, and K-polystable if equality occurs only for product configurations (the ones coming from $\mathrm{Aut}(M, L)$). This "polystable" equality case is not an afterthought: it is forced by the presence of continuous automorphisms, mirroring the passage from stability to polystability in classical GIT.

The GIT and moment-map analogy made precise. Donaldson's framework made the historical analogy with GIT far more than a slogan. On the one hand, test configurations are the algebro-geometric analogue of one-parameter subgroups used in the Hilbert-Mumford criterion. On the other hand, the Donaldson-Futaki invariant plays the role of the numerical weight that detects (in)stability along such degenerations. This viewpoint is also compatible with the symplectic "moment map" picture of scalar curvature, where a canonical metric is expected to arise as a zero of an infinite-dimensional moment map. The test configuration formalism provides the finite-dimensional shadows of this picture.

Taken together, these innovations made K-stability into a workable, robust algebro-geometric condition: it can be tested on a large, canonical class of degenerations (test configurations) and measured by a computable numerical invariant (DF).

In parallel, the same period also witnessed a major analytic development: rather than focusing on degenerations, analysts and geometers began to organize the existence problem through variational principles on the (infinite-dimensional) space of Kähler metrics.

4.4 Analytic developments: energy functionals and variational structure

A key shift in the analytic viewpoint was to treat canonical metrics as minimizers (or critical points) of natural energy functionals on a fixed Kähler class. In this approach, the obstruction to existence is not only detected by symmetry or degeneration, but also by the large-scale behavior (properness) of these functionals.

Throughout this section, we let (M, ω) be a compact Kähler manifold of complex dimension n, and work in the fixed Kähler class $[\omega]$. The space of Kähler potentials is

$$\mathcal{H} = \{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\},$$

and we also denote the volume by $V = \int_M \omega^n$. Moreover, there is naturally an infinite-dimensional Riemannian metric on $\mathcal{H}$. Indeed, for each $\varphi \in \mathcal{H}$ we identify the tangent space $T_\varphi\mathcal{H} \cong C^\infty(M, \mathbb{R})$, and we endow $\mathcal{H}$ with the Mabuchi L^2 metric

$$\langle u, v\rangle_\varphi = \frac{1}{V}\int_M u\, v\, \omega_\varphi^n, \qquad u, v \in T_\varphi\mathcal{H}.$$

This Riemannian viewpoint makes it meaningful to consider (weak) geodesics and geodesic rays φ_t in $\mathcal{H}$, and to study the asymptotic slopes of energy functionals along such paths.

4.4.1 Monge-Ampère energy and the Aubin I/J-functionals

Among the most basic functionals on $\mathcal{H}$ is the Monge-Ampère energy (also called the Aubin-Mabuchi energy; see [Aub84] and [Mab86]), defined by

$$\mathcal{AM}(\varphi) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_M \varphi\, \omega_\varphi^j \wedge \omega^{n-j}.$$

Its first variation has a particularly simple form: along any smooth path $\varphi_t \in \mathcal{H}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{AM}(\varphi_t) = \frac{1}{V}\int_M \dot{\varphi}_t\, \omega_{\varphi_t}^n.$$

Thus $\mathcal{AM}$ is (up to normalization) a primitive of the Monge-Ampère measure ω_φ^n on $\mathcal{H}$.

In practice, we rarely use $\mathcal{AM}$ alone. Aubin introduced several closely related "energy-type" functionals that can be viewed as different normalizations of the same Monge-Ampère data, and that provide a coercive scale for variational problems.

The I-functional. For $\varphi \in \mathcal{H}$ one convenient normalization is

$$I(\varphi) = \frac{1}{V}\int_M \varphi\,(\omega^n - \omega_\varphi^n).$$

It is insensitive to adding constants to φ (since $\int_M(\omega^n - \omega_\varphi^n) = 0$), and after a standard normalization of the potential (e.g. $\int_M \varphi\, \omega^n = 0$ or $\sup_M \varphi = 0$) we have $I(\varphi) \geqslant 0$.

The J-functional. Another basic scale is the (Aubin) J-functional

$$J(\varphi) = \frac{1}{V}\sum_{j=0}^{n-1}\frac{j+1}{n+1}\int_M \sqrt{-1}\,\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^j \wedge \omega_\varphi^{n-1-j}.$$

It is also invariant under $\varphi \mapsto \varphi + c$, and $J(\varphi) \geqslant 0$ for normalized potentials. Equivalently, up to an additive constant depending only on the chosen normalization, we can write J using the Aubin-Mabuchi energy as

$$J(\varphi) = \frac{1}{V}\int_M \varphi\omega^n - \mathcal{AM}(\varphi).$$

Comparability and basic relations. A main reason I and J are used throughout Kähler geometry is that they are quantitatively comparable and behave like an "energy norm" on $\mathcal{H}$. In particular, after fixing a normalization of φ, we have inequalities of the form

$$0 \leqslant J(\varphi) \leqslant I(\varphi) \leqslant (n+1)J(\varphi).$$

Moreover, $I - J$ admits an explicit positive expression (equivalently, $I - J \geqslant 0$), and both I and J control the size of φ in various weak topologies (for instance, they are comparable to the d_1-distance on the space of finite-energy Kähler potentials).

Role in coercivity/properness. Many refined energy functionals (Mabuchi's K-energy, Ding functional, etc.) are studied via their growth along sequences where $J(\varphi) \to \infty$: we say such a functional is coercive/proper if it dominates J up to an additive constant. This turns the existence problem for canonical metrics into a variational question about lower bounds.

4.4.2 Mabuchi's introduction of the K-energy

Fix a compact Kähler manifold (M, ω) and a Kähler class $[\omega]$, and consider the corresponding space $\mathcal{H}$ of potentials. Within the fixed space $\mathcal{H}$, the scalar curvature $S(\omega_\varphi)$ varies with φ, while its average

$$\bar{S} = \frac{1}{V} \int_M S(\omega_\varphi) \omega_\varphi^n$$

is independent of φ (it is a topological number determined by $[\omega]$).

In the 1980s, Mabuchi introduced a remarkable functional, now called the (Mabuchi) K-energy or Mabuchi energy, whose first variation is precisely the scalar curvature defect $S - \bar{S}$ (see [Mab86] and [Mab87]). Concretely, for a smooth path $\varphi_t \in \mathcal{H}$ ($t \in [0, 1]$) connecting 0 to φ, Mabuchi defined a 1-form on $\mathcal{H}$ by

$$\alpha_{\varphi_t}(\dot{\varphi}_t) = \frac{1}{V} \int_M \dot{\varphi}_t (\bar{S} - S(\omega_{\varphi_t})) \omega_{\varphi_t}^n,$$

and showed that this 1-form is closed. As a consequence, the functional

$$\mathcal{M}(\varphi) = \int_0^1 \alpha_{\varphi_t}(\dot{\varphi}_t) \, \mathrm{d}t$$

is well-defined (independent of the chosen path) up to an additive constant.

Critical points and cscK metrics. The key geometric meaning is immediate from the variational formula: a potential φ is a critical point of $\mathcal{M}$ if and only if $S(\omega_\varphi) \equiv \bar{S}$, *i.e.* ω_φ has constant scalar curvature (cscK). In particular, Kähler-Einstein metrics are special critical points in the cases where the Einstein equation forces constant scalar curvature.

The Chen-Tian formula. A useful way to rewrite $\mathcal{M}$ is to decompose it into a relative entropy term plus Monge-Ampère-type energies (often referred to as the Chen-Tian formula, see [Che00]). Define the entropy

$$\mathrm{Ent}(\omega^n, \omega_\varphi^n) = \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n,$$

and for any closed real $(1, 1)$-form η define the twisted Aubin-Mabuchi energy

$$\mathcal{AM}_\eta(\varphi) = \frac{1}{nV} \sum_{j=0}^{n-1} \int_M \varphi \eta \wedge \omega_\varphi^j \wedge \omega^{n-1-j}.$$

Then we have an identity of the form

$$\mathcal{M}(\varphi) = \mathrm{Ent}(\omega^n, \omega_\varphi^n) + \bar{S} \, \mathcal{AM}(\varphi) - n \, \mathcal{AM}_{\mathrm{Ric}(\omega)}(\varphi) + C,$$

where C is a constant depending only on the choice of reference form ω (and the normalization convention).

Connection to Futaki's invariant. One basic reason that $\mathcal{M}$ is closely tied to algebro-geometric stability is that its variation along holomorphic automorphisms recovers Futaki's character. Concretely, let X be a holomorphic vector field and consider the 1-parameter family obtained by pulling back the metric:

$$\omega_{\varphi_s} = \exp(sX)^*\omega.$$

Then the derivative of Mabuchi energy at $s = 0$ equals the Futaki invariant of X:

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{M}(\varphi_s) = \mathrm{Fut}(X).$$

So in some sense, we can say that $\mathcal{M}$ is a non-linear version of the Futaki invariant.

4.4.3 Tian's key contributions: properness and the variational bridge

At the level of formal calculus, Mabuchi's construction already suggests that canonical metrics should be found by minimizing $\mathcal{M}$ on the space of Kähler potentials. Tian's decisive contribution was to make this slogan into a concrete analytic strategy for the positive first Chern class case, and at the same time to connect it to the emerging stability picture.

Properness/coercivity as the right global condition. The functional $\mathcal{M}$ is invariant under adding constants to φ, and in the presence of nontrivial automorphisms it is also natural to consider it modulo the $\mathrm{Aut}(M)$-action. Tian emphasized that the existence problem should be controlled not by local critical point equations alone, but by the global growth of $\mathcal{M}$ along rays in $\mathcal{H}$ (see [Tia94]). Concretely, he asked for a coercive lower bound of the form

$$\mathcal{M}(\varphi) \geqslant \delta\, J(\varphi) - C, \;\; \varphi \in \mathcal{H},$$

for some constants $\delta > 0$ and $C \in \mathbb{R}$ (after fixing a normalization of φ), or more invariantly, that $\mathcal{M}(\varphi) \to +\infty$ whenever $J(\varphi) \to +\infty$. This "properness" condition is the variational analogue of stability: it rules out escaping directions in which the energy drops without bound, and it is exactly what we need to turn minimizing sequences into convergent ones.

From Moser-Trudinger type inequalities to existence. In the Fano case, Tian established a systematic route to such coercive estimates by combining the Monge-Ampère energy scale (I, J) with complex-analytic input measuring singularities. A key tool is his introduction and use of the α-invariant, which controls integrability properties of plurisubharmonic potentials and leads to Moser-Trudinger type inequalities. In particular, Tian proved that sufficiently strong lower bounds on this invariant imply the existence of a Kähler-Einstein metric (see [Tia87], [Tia90] and [Tia97]). From the modern viewpoint, these inequalities can be read as concrete instances where the appropriate energy functional becomes proper, hence admitting a minimizer.

A conceptual bridge to stability. Tian's work did not treat the Mabuchi energy merely as one functional among many: it elevated $\mathcal{M}$ (and its close relatives) to the central object where analysis, geometry, and algebra meet. On the one hand, the first variation of $\mathcal{M}$ detects the scalar curvature defect, so minimizers should be cscK (KE) metrics. On the other hand, its behavior under automorphisms and degenerations reflects Futaki-type obstructions. This dual nature is precisely what later allowed the Yau-Tian-Donaldson picture to be formulated as a single equivalence: algebro-geometric stability on one side, and properness of an energy functional on the other.

For the KE problem on Fano manifolds, an especially useful feature is that we can replace $\mathcal{M}$ by closely related functionals that are more directly adapted to the Monge-Ampère equation. Historically, this is where the Ding variational approach enters the story.

4.4.4 The introduction of the Ding functional

The Mabuchi energy $\mathcal{M}$ is tailored to the scalar curvature equation, and it is the natural functional for the general cscK problem. In the Fano Kähler-Einstein setting, however, Ding introduced a closely related functional whose definition is more directly aligned with the complex Monge-Ampère equation (see [Din88]). This functional, which now commonly called the Ding functional, quickly became a fundamental tool in the variational approach to the Fano case.

Definition of the Ding functional. Suppose $\omega \in 2\pi\, c_1(M)$, f_ω is the normalized Ricci potential and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for $\varphi \in \mathcal{H}$, Ding's functional can be written (up to an additive constant depending on the reference choice) as

$$\mathcal{D}(\varphi) = -\mathcal{AM}(\varphi) - \log\left(\frac{1}{V}\int_M e^{f_\omega - \varphi}\omega^n\right).$$

One key point is that, unlike $\mathcal{M}$, the second term uses the fixed background volume form ω^n rather than the moving volume ω_φ^n. This makes $\mathcal{D}$ particularly natural for the Monge-Ampère equation coming from the KE problem.

Critical points: Kähler-Einstein metrics. A standard variational calculation shows that critical points of $\mathcal{D}$ are exactly Kähler-Einstein metrics in the class $2\pi\, c_1(M)$ (modulo the expected action of $\mathrm{Aut}(M)$). In this sense, $\mathcal{D}$ provides a "more directly Monge-Ampère" variational formulation of the Fano KE equation.

Relation to Futaki-type obstructions and stability. Ding's approach is not disconnected from the stability story. For example, Ding and Tian developed the generalized Futaki invariant and related inequalities (see [DT92]), clarifying again that the obstruction to existence is encoded in deformation-invariant quantities and that variational functionals have built-in sensitivity to degeneration.

From the modern perspective, the Ding functional is one of the cleanest bridges between analysis and algebra in the Fano case: its properness/coercivity is expected to be equivalent to (uniform) K-stability, and it is often technically more convenient than $\mathcal{M}$ when working with weak geodesics and pluripotential theory.

4.4.5 The central conjecture: K-stability and coercivity of the Mabuchi energy

By the end of the 1990s, two parallel stories had taken shape. On the one hand, the algebraic side supplied a numerical stability test—K-stability, defined via test configurations and the Donaldson-Futaki invariant. On the other hand, the analytic side supplied variational functionals whose critical points are canonical metrics—most notably Mabuchi's energy $\mathcal{M}$. The natural expectation, already emphasized by Tian in the 1990s, is that these are in fact two descriptions of the same obstruction phenomenon (see [Tia94]).

The "core conjecture" behind the modern variational approach can be summarized as follows: the K-(poly)stability of the polarized variety (M, L) is expected to be equivalent to the coercivity/properness of the Mabuchi energy on the corresponding space of Kähler metrics. More precisely, the semistable/polystable distinction is reflected analytically by whether $\mathcal{M}$ is bounded below (proper, resp.) after quotienting by the automorphism group.

At a heuristic level, the equivalence is explained by comparing "directions to infinity" in the space of metrics with algebraic degenerations. Along a suitably chosen geodesic ray in the space of Kähler potentials, the asymptotic slope of $\mathcal{M}$ should match a purely algebro-geometric slope invariant: the Donaldson-Futaki invariant of the associated degeneration. In this picture, destabilizing test configurations are exactly those rays along which $\mathcal{M}$ fails to grow, while coercivity rules out such escaping directions and forces the existence of a minimizer, hence a canonical metric.

This conjectural bridge is the analytic core of the Yau-Tian-Donaldson paradigm, and it also motivated much of the subsequent synthesis of PDE methods, pluripotential theory, and algebraic geometry that led to complete proofs in the Kähler-Einstein Fano case.

Chapter 5

The Synthesis of Theories and Proofs

5.1 Donaldson's programmatic contributions: quantization and finite-dimensional approximation

By the late 1990s, the Yau-Tian picture had clarified what the sought-for criterion should "look" like: an algebro-geometric stability condition, tested against degenerations, should control the existence of canonical Kähler metrics. The gap was methodological. Geometers still needed a robust mechanism to pass between the infinite-dimensional geometry of the space of Kähler metrics and the finite-dimensional algebra of projective geometry.

Donaldson's work in the early 2000s supplied precisely such a mechanism. Complementing (and logically independent from) the test-configuration formalism described in the previous chapter, he proposed a systematic "quantization" philosophy: approximate a Kähler metric by Bergman/Fubini-Study metrics coming from embeddings by high tensor powers of an ample line bundle, and study canonical metrics through the asymptotic behavior of these finite-dimensional approximations (see [Don01] and [Don05b]).

5.1.1 The Hitchin-Kobayashi analogy and the moment-map viewpoint

A guiding theme in Donaldson's approach is an analogy with gauge theory, in particular the Hitchin-Kobayashi correspondence for holomorphic vector bundles. There, a differential-geometric equation (the Hermitian-Einstein condition) is equivalent to an algebro-geometric stability notion (Mumford-Takemoto slope stability).

In the scalar curvature setting, the relevant "object" is no longer a vector bundle but the complex structure together with a polarization (M, L); the "equation" is the constant scalar curvature Kähler (cscK) equation (or its Fano specialization, the Kähler-Einstein equation). The philosophical expectation is the same: a canonical metric should be the differential-geometric representative of an algebro-geometric stability property.

A technical bridge for making this analogy precise is the symplectic "moment map" picture of scalar curvature (developed by Fujiki and Donaldson, see [Fuj92] and [Don99]). Roughly speaking, we consider the (infinite-dimensional) space of compatible almost complex structures with a fixed symplectic form, on which the Hamiltonian symplectomorphism group acts; in this framework the scalar curvature plays the role of a moment map. In the finite-dimensional GIT world, moment maps and stability are linked by the Kempf-Ness philosophy. Donaldson's program can be read as importing this finite-dimensional template into Kähler geometry by constructing finite-dimensional approximations where both the group action and a moment map are concretely accessible.

5.1.2 Quantization by Bergman metrics and balanced embeddings

Let (M, L) be a polarized manifold, and fix a Hermitian metric h on L whose curvature form is a Kähler metric $\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$. For each large integer k, the space $H^0(M, L^k)$ of holomorphic sections gives an embedding into projective space (when L is ample and k is sufficiently large), and any choice of inner product on $H^0(M, L^k)$ produces a Fubini-Study metric on L^k, hence a Kähler metric in $2\pi\, c_1(L)$.

Donaldson's key observation is that there are preferred "balanced" choices of inner products and embeddings. In one common formulation, a metric is balanced at level k if the associated Bergman function is constant, i.e.

$$\rho_k(\omega)(x) = \sum_{i=1}^{N_k} |s_i(x)|^2_{h^k} \equiv C,$$

where $\{s_i\}$ is an L^2-orthonormal basis of $H^0(M, L^k)$, $N_k = \dim H^0(M, L^k)$ and C is a constant. This is a finite-dimensional, purely linear-algebraic condition. It characterizes fixed points of a natural iteration built from the Hilbert map and the Fubini-Study map (often denoted Hilb and FS), where Hilb sends a Kähler metric to the induced L^2 inner product on $H^0(M, L^k)$, and FS sends such an inner product back to the associated Fubini-Study metric. It also appears as a zero of a finite-dimensional moment map for the unitary group action on the space of projective embeddings.

The strength of the quantization philosophy comes from the Tian-Catlin-Zelditch asymptotic expansion of ρ_k as $k \to \infty$, which shows that the coefficients of the expansion encode curvature invariants of ω and in particular the scalar curvature (see [Zel98]). As a consequence, balanced metrics are asymptotic solutions to the cscK equation: if we can find balanced metrics for arbitrarily large k, their limit (when it exists) is expected to solve the cscK equation.

This paradigm has two complementary payoffs:

- It replaces the nonlinear PDE by a sequence of finite-dimensional fixed-point problems.
- It makes stability enter through classical GIT: balanced embeddings are precisely the GIT-polystable points for the relevant projective action.

In particular, Donaldson proved that the existence of a cscK metric implies asymptotic Chow stability under suitable hypotheses, providing a concrete "metric $\Rightarrow$ stability" direction in the projective setting (see [Don01]).

5.1.3 Lower bounds, test configurations, and the Calabi functional

While balanced metrics realize one route to canonical metrics by approximation, Donaldson's work also clarified how to extract "numerical obstructions" from degenerations in a way that mirrors the variational viewpoint.

A fundamental quantity for cscK geometry is the Calabi functional

$$\mathcal{C}(\omega) = \int_M (S(\omega) - \bar{S})^2\, \omega^n,$$

whose minimizers are extremal metrics (and in particular cscK metrics when the minimum is 0). Donaldson showed that, for polarized manifolds, test configurations produce explicit lower bounds for $\mathcal{C}$ in terms of algebraic data, thereby turning degenerations into quantitative estimates on how far a metric can be from being canonical (see [Don05b]).

Conceptually, this fits the same template as in gauge theory: stability should control the existence of a solution, while instability produces an explicit "energy gap". In the Kähler setting, the relevant gap is measured either by the asymptotic slope of an energy functional (Mabuchi, Ding) along geodesic rays, or by algebraic weights along test configurations. Donaldson's contribution was to give a systematic finite-dimensional framework in which these slopes and weights can be compared and computed.

5.1.4 How Donaldson's program fits the historical arc

Last chapter emphasized the conceptual shift from solving a single PDE to understanding stability via degenerations and numerical invariants. Donaldson's quantization program adds a complementary bridge: it explains why this shift is not merely philosophical but can be implemented by concrete approximations that turn an infinite-dimensional geometric problem into a sequence of finite-dimensional algebraic ones.

With this finite-dimensional approximation machinery in place, the stage is set for the decisive breakthroughs of the 2010s, where the stability criterion (in the refined form of K-stability) and analytic techniques (continuity methods, pluripotential theory, and weak geodesics) were finally combined to prove the Yau-Tian-Donaldson conjecture for Fano Kähler-Einstein metrics.

5.2 The milestone proof of the Yau-Tian-Donaldson conjecture

The Yau-Tian-Donaldson (YTD) conjecture asserts an equivalence between a metric existence statement and an algebro-geometric stability statement: Let M be a Fano manifold, then M admits a Kähler-Einstein metric in the class $2\pi\, c_1(M)$ if and only if the polarized pair $(M, -K_M)$ is K-polystable.

Historically, we should read this conjecture as two logically independent directions.

- Necessity: a Kähler-Einstein metric forces K-(poly)stability.
- Sufficiency: K-(poly)stability forces the existence of a Kähler-Einstein metric.

The first direction is conceptually closer to classical "moment map $\Rightarrow$ stability" principles, while the second is deeper, because it must turn an a priori algebraic inequality into a global nonlinear PDE solution.

5.2.1 Necessity: Tian's 1997 result (KE $\Rightarrow$ K-stability)

The first decisive milestone predates the eventual 2010s synthesis: in 1997 Tian established, in a framework close to his original definition of K-stability, that the existence of a Kähler-Einstein metric on a Fano manifold implies what Tian called weak K-stability (in modern language, a form of K-polystability, see [Tia97]).

The philosophy of Tian's argument can be summarized as an early instance of the "metric $\Rightarrow$ stability" paradigm.

Step 1: test by special degenerations. Tian started from a Fano manifold M admitting a Kähler-Einstein metric ω_{KE} with $\mathrm{Ric}(\omega_{\mathrm{KE}}) = \omega_{\mathrm{KE}}$. He then fixed a *special degeneration* $\pi : W \to \Delta$ of M with a lifted $\mathbb{C}^*$-action, and its infinitesimal generator induced a holomorphic vector field v_W on the central fiber W_0. In Tian's terminology, weak K-stability amounted to showing

$$\mathrm{Re}\, f_{W_0}(v_W) \geq 0,$$

where $f_{W_0}(v_W)$ is the generalized Futaki invariant in the sense of Ding-Tian (see [DT92]), with equality allowed only for the trivial (product) degeneration.

Step 2: embed and build a family of reference metrics. Tian chose $m \gg 1$ so that K_M^{-m} is very ample, embedded M into $\mathbb{CP}^N$, and realized the degeneration $W \subset \mathbb{CP}^N \times \Delta$ via a one-parameter subgroup $\sigma(t) \subset SL(N+1, \mathbb{C})$. Pulling back the Fubini-Study form produced a family of Kähler forms on the smooth fibers $W_t \cong M$,

$$\omega_t := \frac{1}{m}\,\sigma(t)^*\,\omega_{\mathrm{FS}}\Big|_{W_t} \in 2\pi\, c_1(M).$$

Step 3: compare with ω_{KE} and force the potential to diverge. After composing with suitable automorphisms $\tau(t) \in \mathrm{Aut}(M)$ (to account for holomorphic vector fields), Tian wrote

$$\tau(t)^*\omega_t = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar\partial\phi_t,$$

with ϕ_t normalized (in particular, orthogonal to the first eigenspace corresponding to holomorphic vector fields). A key lemma in the paper showed that if the degeneration is nontrivial then ϕ_t cannot remain uniformly bounded; rather,

$$\|\phi_t\|_{C^0} \to \infty \ (t \to 0).$$

The point is that a uniform bound would force the projective embeddings to converge to an isomorphism $M \cong W_0$, hence would make the degeneration trivial.

Step 4: use coercivity of the K-energy and compute its slope. Tian then applied a coercivity estimate for Mabuchi's K-energy based at ω_{KE}: along normalized potentials escaping to infinity we have $\nu_{\omega_{\mathrm{KE}}}(\phi) \to +\infty$. Together with $\|\phi_t\|_{C^0} \to \infty$, this implies

$$\nu_{\omega_{\mathrm{KE}}}(\phi_t) \longrightarrow +\infty \ (t \to 0).$$

On the other hand, reparametrizing $t = e^{-s}$ and writing $\sigma(e^{-s})^*\omega_{e^{-s}} = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar\partial\psi_s$, Tian differentiated $\nu_{\omega_{\mathrm{KE}}}(\psi_s)$ in s and related the derivative to an integral expression involving the Hamiltonian of the generating vector field; the limit as $t \to 0$ was identified with $\mathrm{Re}\, f_{W_0}(v_W)$.

Step 5: conclude $\mathrm{Re}\, f_{W_0}(v_W) \geq 0$ and characterize equality. Finally, Tian argued by contradiction: if $\mathrm{Re}\, f_{W_0}(v_W) < 0$, then for t sufficiently small the derivative of $\nu_{\omega_{\mathrm{KE}}}$ along the degeneration became uniformly negative, forcing the K-energy to tend to $-\infty$, contradicting the previous divergence to $+\infty$. Hence $\mathrm{Re}\, f_{W_0}(v_W) \geq 0$. Moreover, if $\mathrm{Re}\, f_{W_0}(v_W) = 0$, the same slope control showed the K-energy stayed bounded, which is incompatible with the potential-divergence lemma unless the degeneration is trivial.

While later developments (notably the test-configuration formalism and the precise Donaldson-Futaki invariant) clarified and streamlined the algebro-geometric side, Tian's 1997 work already captured the core mechanism: a Kähler-Einstein metric provides global analytic control, and this control rules out destabilizing degenerations.

5.2.2 Difficulties of the direct continuity method in the sufficiency direction

After Yau's solution of the Calabi conjecture, a natural idea for the Fano case is to try to solve the Kähler-Einstein equation by a direct continuity path of Monge-Ampère type. The prototype is Aubin's continuity method

$$\frac{\det\big(g_{i\bar j} + \varphi_{i\bar j}\big)}{\det\big(g_{i\bar j}\big)} = e^{-t\varphi + F}, \qquad t \in [0,1],$$

where F is determined by the reference metric $\omega \in 2\pi c_1(M)$, and where $t = 1$ corresponds to the Kähler-Einstein equation.

The openness part is standard, and the hard issue is closedness: given solutions for $t_i \to t_\infty$, we need uniform *a priori* estimates to prevent degeneration.

In the non-positive cases, Yau's estimates close the path. In the Fano case, however, several new phenomena appear, and they are precisely the phenomena predicted by the stability paradigm.

- Loss of uniform estimates: The sign change in the exponent makes it possible for the potential to drift and for analytic quantities to concentrate. The uniform C^0 bounds that are automatic in the $c_1 \leqslant 0$ cases are no longer available without extra input.
- Metric degeneration and singular limits: Even though we can still get lower Ricci bounds along much of the path, the natural compactness statement is only Gromov-Hausdorff precompactness. So the limit can be a singular metric space, and understanding its singular set in a way compatible with complex/algebraic geometry is subtle.
- Where stability should enter is unclear: A bare Gromov-Hausdorff limit does not automatically produce an algebro-geometric degeneration (test configuration) on which the Donaldson-Futaki invariant can be evaluated. This missing bridge is exactly what prevented early approaches (including Tian's early program) from completing the argument.

These difficulties motivated a change of viewpoint: rather than pushing a purely smooth continuity path to the endpoint, we should allow controlled singularities (cone angles) so that compactness becomes robust and the limit can be identified with an algebraic degeneration.

5.2.3 Cone-angle continuity and the Chen-Donaldson-Sun strategy

The Chen-Donaldson-Sun breakthrough was to implement precisely the missing bridge hinted above: start from a continuity method with good compactness properties, upgrade the analytic degeneration to an algebraic test configuration via a partial C^0 estimate, and then use K-polystability to rule out any nontrivial degeneration (see [CDS14], [CDS15a], [CDS15b] and [CDS15c]).

Step 1: replace Aubin's path by a cone-angle continuity method. Rather than directly pushing Aubin's path all the way to $t = 1$, Chen-Donaldson-Sun introduced a family of Kähler-Einstein cone metrics. Fix a smooth divisor $D \in |-\lambda K_M|$ for some large integer λ (so that D is an anticanonical-type divisor). For $\beta \in (0, 1]$, they sought a Kähler metric $\omega_\beta \in 2\pi\, c_1(M)$ with cone angle $2\pi\beta$ along D satisfying the current equation

$$\mathcal{R}ic(\omega_\beta) = \mu_\beta\, \omega_\beta + 2\pi(1-\beta)[D], \;\; \mu_\beta = 1 - (1-\beta)\lambda.$$

When $\beta = 1$ this becomes the smooth Kähler-Einstein equation $\mathcal{R}ic(\omega) = \omega$.

So they considered the set

$$E = \{\beta \in (0, 1] \mid \text{a Kähler-Einstein cone metric of angle } 2\pi\beta \text{ exists}\}.$$

And the overall goal is to prove $1 \in E$.

Step 2: show E is nonempty and open. Nonemptiness holds for sufficiently small cone angle by analytic existence theory for conical Monge-Ampère equations (see [JMR16]). Openness is proved by an implicit function theorem argument in suitable weighted Hölder spaces, once we have an appropriate Fredholm theory for the linearized operator around a cone metric (see [Don12] and [LS14]).

Step 3: closedness reduces to understanding the degeneration as $\beta \to \beta_\infty$. Suppose $\beta_i \in E$ increases to a limit β_∞. The main issue is: must $\beta_\infty \in E$? If not, then the sequence (M, ω_{β_i}) degenerates.

A crucial point is that the cone KE equation provides uniform Ricci lower bounds (in a weak sense compatible with cone singularities). This allows us to apply Gromov-Hausdorff compactness. After passing to a subsequence, we get a metric space limit

$$(M, \omega_{\beta_i}) \longrightarrow (M_\infty, d_\infty).$$

Cheeger-Colding theory (and its Kähler refinements) give a detailed description of the regular and singular sets of M_∞.

Step 4: partial C^0 estimate and algebraicity of the limit. The bridge from metric convergence to algebraic geometry is the partial C^0 estimate: roughly, Chen-Donaldson-Sun showed that for some fixed $k \gg 1$, the Bergman kernel associated to $H^0(M, -kK_M)$ (computed using ω_{β_i}) is uniformly bounded below.

This has two decisive consequences.

- It produces (uniform) projective embeddings of (M, ω_{β_i}) by sections of $-kK_M$, so the Gromov-Hausdorff limit is identified with an algebro-geometric limit in projective space.
- The limit space M_∞ is shown to be a normal Q-Fano variety (with mild singularities), and the limiting metric is a weak Kähler-Einstein (possibly conical) metric on the regular locus.

In particular, if $\beta_\infty \notin E$, then M_∞ is not isomorphic to M as a polarized variety.

Step 5: produce a test configuration with central fiber M_∞. Using the projective embeddings coming from the partial C^0 estimate, they showed that M_∞ arises as the central fiber of a genuine $\mathbb{C}^*$-equivariant degeneration of M (a test configuration for $(M, -K_M)$). In other words, the analytic degeneration coming from the metric limit can be upgraded to an algebraic degeneration in Donaldson's sense.

Step 6: KE metric with cone angle and vanishing (log) DF invariant. Along the cone-angle continuity method the natural algebro-geometric object is the log pair $(M, (1-\beta)D)$, and the corresponding numerical invariant for a degeneration is the log Donaldson-Futaki invariant. It can be viewed as the usual Donaldson-Futaki weight of a test configuration, corrected by an additional boundary contribution coming from the divisor D with coefficient $(1 - \beta)$. When $\beta = 1$ the boundary term disappears and one recovers the ordinary Donaldson-Futaki invariant (see [Don12] and [Li11]).

In the Chen-Donaldson-Sun setting, the Gromov-Hausdorff limit M_∞ carries a weak Kähler-Einstein structure with cone angle $2\pi\beta_\infty$. This analytic input forces the log Donaldson-Futaki invariant of the induced degeneration to vanish

$$\log \mathrm{DF}_{\beta_\infty}(M_\infty, D_\infty) = 0.$$

Intuitively, this is the "metric $\Rightarrow$ slope $= 0$" principle: the existence of a KE metric with cone angle makes the relevant energy functional minimizing, and the corresponding asymptotic slope along the ray coming from a test configuration is exactly the log Futaki weight.

Moreover, since $\log \mathrm{DF}_\beta(M_\infty, D_\infty)$ is a linear function in β, we obtain that

$$\mathrm{DF} = \log \mathrm{DF}_1(M_\infty, D_\infty) \leqslant 0.$$

Step 7: invoke K-polystability to force the degeneration to be trivial. If $(M, -K_M)$ is K-polystable, then $\mathrm{DF} \leqslant 0$ can occur only for *product* test configurations. But the construction above produces a nontrivial degeneration whenever $M_\infty \ncong M$. This contradiction shows that the bad alternative $\beta_\infty \notin E$ cannot happen.

Therefore E is closed, and since it is also nonempty and open, we conclude $E = (0, 1]$ and in particular $1 \in E$. This yields a smooth Kähler-Einstein metric on M.

Historical meaning. From a distance, the Chen-Donaldson-Sun proof is precisely the promised synthesis in the YTD paradigm: analytic compactness produces a degeneration, partial C^0 turns it into algebraic geometry, and K-stability rules out exactly the degenerations that would prevent closing the continuity method.

5.3 Generalizations and current frontiers

The proof of the Yau-Tian-Donaldson (YTD) conjecture for Fano Kähler-Einstein metrics is now best viewed as a flexible toolkit rather than a single rigid argument: different approaches emphasize different compactness mechanisms, different bridges from analysis to algebra, and different ways of formulating "stability" quantitatively.

5.3.1 Other approaches to proving YTD

Besides the cone-angle continuity method of Chen-Donaldson-Sun, several alternative routes have emerged which either reprove the same implication "(uniform) K-stability $\Rightarrow$ KE" or provide logically equivalent analytic formulations that can be closed by different techniques.

The "smooth" continuity method and partial C^0. We can attempt to run a continuity path entirely within the smooth category (without introducing cone singularities) and still keep enough compactness to extract an algebro-geometric degeneration. A key point is that the algebro-geometric identification of the metric limit again hinges on a partial C^0 estimate along the path. This strategy was carried out in the "classical" continuity-method setting by Datar and Székelyhidi (see [Szé16] and [DS16]).

Kähler-Ricci flow as a dynamical approach. Instead of solving a stationary Monge-Ampère equation, Chen-Sun-Wang evolved metrics by the normalized Kähler-Ricci flow and studied long-time behavior. In the Fano setting, the flow provides a canonical dynamical process whose limits (when they exist) should be Kähler-Einstein. The interaction with K-stability enters through the analysis of possible singularity formation and the algebro-geometric meaning of the limiting objects, leading to a flow-based proof of the YTD correspondence (see [CSW18]).

Variational and pluripotential-theoretic methods. A different approach is to recast the KE problem as a variational one on the space of (finite-energy) Kähler potentials, where stability is reflected by the coercivity/properness of suitable energy functionals (notably the Ding functional in the Fano case, see [BBJ21]). This perspective and its precise YTD-type equivalences will be discussed in detail in the third subsection.

Quantization-only proofs and probabilistic viewpoints. There are also approaches that put the "quantization philosophy" at the center: We can prove YTD conjecture by comparing algebro-geometric stability thresholds (such as the delta-invariant) with finite-dimensional Bergman-type approximations and then passes to the limit. Zhang gave a particularly short quantization proof along these lines (see [Zha23]). In a different direction, Berman developed a thermodynamic viewpoint via random point processes, where canonical metrics arise as macroscopic limits of Gibbs measures and stability is encoded by uniform Gibbs stability. This gives yet another conceptual route to YTD-type correspondence (see [Ber22] and[Ber24]).

5.3.2 The cscK version of the Yau-Tian-Donaldson conjecture

While the Fano Kähler-Einstein case has a privileged PDE (a single Monge-Ampère equation), the general Calabi program asks for canonical metrics in an arbitrary Kähler class. In this setting, a natural problem is to solve the constant scalar curvature Kähler (cscK) equation

$$S(\omega) = \bar{S},$$

where $S(\omega)$ is the scalar curvature and $\bar{S}$ is its topological average in the fixed class.

Donaldson's conjecture (polarized manifolds). Let (X, L) be a polarized manifold and consider Kähler metrics in the class $2\pi c_1(L)$. Donaldson proposed that the existence of a cscK metric should be equivalent to an algebro-geometric stability condition for (X, L), generalizing the Fano YTD picture (see [Don02]).

Automorphisms and "relative" stability. A key new phenomenon is that cscK metrics may coexist with nontrivial automorphisms, and the correct statement must be made modulo the action of $\mathrm{Aut}(X, L)$. Analytically this means we should study the Mabuchi energy on the quotient of the space of Kähler potentials by automorphisms (or works with extremal metrics in Calabi's sense), while algebro-geometrically this leads to various "relative" versions of K-stability.

Early evidence: toric surfaces and Abreu's equation. In the toric setting, the cscK equation becomes Abreu's fourth order PDE on the moment polytope. Donaldson developed a continuity method and delicate a priori estimates to prove existence results for extremal/cscK metrics on toric surfaces, providing some of the first substantial confirmations of the conjectural picture (see [Don05a], [Don08], [Don09]).

Necessity: cscK $\Rightarrow$ (relative) K-stability. Compared with the KE case, establishing necessity required new ideas because we no longer have Ricci curvature control. A sequence of works proved increasingly general necessity results, starting from Stoppa's theorem that cscK implies K-stability when $\mathrm{Aut}(X, L)$ is discrete (see [Sto09]), and its extension to some other settings (see [SS11] and [Szé15]). Moreover, a modern formulation of the cscK YTD correspondence replaces bare K-polystability by uniform (or relative uniform) K-stability and identifies it with coercivity/properness of the Mabuchi K-energy. A decisive step was the regularity theory for weak minimizers of the Mabuchi functional and its consequences for properness and stability, developed by Berman-Darvas-Lu (see [BDL20]).

Sufficiency: recent breakthroughs. On the sufficiency direction, namely stability/properness $\Rightarrow$ existence of (weighted) cscK metrics, two independent 2025 preprints announced proofs that complete the cscK version of the YTD picture in broad generality: one by Boucksom-Jonsson (see [BJ25]) and another by Darvas-Zhang (see [DZ25]).

Historical meaning. From the historical viewpoint, the cscK YTD story clarifies that the Fano KE case is not an isolated miracle: it is one prominent instance of a much broader principle linking the existence of canonical metrics (as minimizers of energy functionals) with algebro-geometric stability of polarized manifolds.

5.3.3 The singular Fano ($\mathbb{Q}$-Fano) version of the YTD correspondence

Once the smooth Fano case was understood, the next natural step was to extend the YTD picture to *singular* Fano objects—normal $\mathbb{Q}$-Fano varieties with klt (Kawamata log terminal) singularities. From the modern birational viewpoint, this is not a technical afterthought: singular Fano varieties occur naturally as degenerations of smooth Fano manifolds (for instance as K-polystable limits), and they are the expected building blocks in moduli theory.

Weak KE metrics on klt Fano varieties. On a klt $\mathbb{Q}$-Fano variety X, we cannot expect a smooth Kähler-Einstein metric globally. The correct analytic object is a weak (or current-valued) Kähler-Einstein metric: a positive closed current $\omega \in 2\pi c_1(X)$ with locally bounded potentials on the regular locus and satisfying the KE equation in the sense of currents. Pluripotential theory provides a natural variational setting for such metrics via finite-energy potentials and the Ding/Mabuchi energy functionals.

The variational approach and uniform K-stability (finite automorphism group). A major advance was the work of Berman-Boucksom-Jonsson, who established a precise YTD-type equivalence in the singular setting under the assumption that $\mathrm{Aut}(X)$ is finite: for a klt $\mathbb{Q}$-Fano variety, the existence of a weak Kähler-Einstein metric is equivalent to *uniform* K-stability (and equivalently to coercivity/properness of the Ding functional in the finite-energy space), thereby extending the metric $\leftrightarrow$ stability paradigm beyond the smooth category (see [BBJ21]).

Allowing automorphisms: G-uniform and reduced uniform stability. When $\mathrm{Aut}(X)$ is positive-dimensional, coercivity of the relevant energies cannot hold on the full space because directions coming from holomorphic vector fields create "flat" directions. We must instead formulate stability and properness modulo automorphisms. This led to refined notions such as G-uniform K-stability and reduced uniform K-stability, which measure coercivity only after projecting away the automorphism directions. Building on this idea, Li proved that for general klt $\mathbb{Q}$-Fano varieties the existence of a weak Kähler-Einstein metric is equivalent to an appropriate (reduced) uniform stability condition (see [LTW22] and [Li22]).

Back to polystability: equivalence with K-polystability. For moduli applications, the most geometric notion is still K-polystability rather than uniform variants. A further recent step is that reduced uniform K-stability can be related back to ordinary K-polystability: Liu-Xu-Zhuang proved that reduced uniform K-stability is equivalent to K-polystability, thereby completing the bridge between the analytic variational criterion and the classical algebro-geometric formulation (see [LXZ22]).

Historical meaning. Viewed in the long arc of the subject, the singular YTD correspondence shows that the "canonical metric $\Leftrightarrow$ stability" paradigm is robust under degeneration: the analytic notion of KE metric naturally weakens to a current on the regular locus, while the stability notion naturally strengthens to a coercive form modulo automorphisms. This robustness is one of the reasons why the modern theory is inseparable from moduli and the minimal model program.

5.3.4 The moduli problem: K-moduli and the minimal model program

Once stability has been identified as the correct algebro-geometric avatar of canonical metrics, it is natural to ask a more global question: can we organize all K-polystable Fano objects into a well-behaved moduli space? Historically, this "moduli problem" is where the YTD paradigm fully merges with the modern birational viewpoint: stability is not only a criterion for solving a PDE on a fixed manifold, but also the correct notion that makes families of Fano varieties compactifiable.

Why moduli is subtle. Unlike curves, Fano manifolds do not admit a straightforward stable reduction theory, and naive parameter spaces are typically non-Hausdorff under degenerations. From the metric side, sequences of Kähler-Einstein Fanos may converge (in the Gromov-Hausdorff sense) to singular spaces. From the algebraic side, we expect such limits to be klt $\mathbb{Q}$-Fano varieties. The singular YTD correspondence discussed above is precisely what legitimizes these singular objects as boundary points.

The K-moduli philosophy. The guiding principle can be summarized as follows: K-polystable $\mathbb{Q}$-Fano varieties should form an algebraic moduli space whose points correspond to canonical metrics on the smooth locus, and whose boundary records metric and algebro-geometric degenerations. In practice, we work with a functor of $\mathbb{Q}$-Gorenstein families of klt Fanos and imposes K-(poly)stability fiberwise. The resulting moduli space (often called the K-moduli space) is expected to be separated and proper after allowing singular fibers, providing the correct replacement for a naive moduli of smooth Fanos (See [Xu21]).

Compactness and the role of degeneration. A key conceptual output of the KE theory is that compactness is obtained only after admitting singular limits. Analytically, Gromov-Hausdorff compactness produces a metric limit. Algebraically, we want this limit to arise from a degeneration of polarized varieties. This is why the partial C^0 estimate and the algebro-geometric interpretation of metric limits, originally used as a tool in existence proofs, also become structural inputs in moduli theory: they explain why the boundary of moduli should consist of klt $\mathbb{Q}$-Fano varieties rather than more exotic metric spaces (see [DS19], [LWX19] and [Oda15]).

Projectivity and CM line bundles. Beyond existence and properness, we would like the moduli space to be projective. The bridge here is provided by the Chow-Mumford line bundle (or its refinements), whose positivity is closely tied to uniform stability and coercivity of energy functionals. From the YTD viewpoint, the same convexity/properness properties that force the existence of a canonical metric also supply positivity needed to embed the moduli space into projective space, much as in classical GIT.

Connections with the minimal model program. The appearance of klt singularities and $\mathbb{Q}$-Fano limits is not accidental: it reflects the minimal model program's principle that the natural compactifications of moduli problems should be built from mildly singular varieties (see [Kol13]). In this sense, the moduli problem for KE Fanos is a meeting point of three themes: analytic compactness of canonical metrics, algebro-geometric stability, and birational classification. The modern picture suggests that canonical metrics do not merely solve an equation on a fixed variety, they also single out the correct building blocks that behave well in families.

Chapter 6

Unfinished Roads and Cross-Disciplinary Echoes

The preceding chapters traced how Kähler geometry evolved from a compatibility condition in Hermitian geometry into a central arena where analysis, topology, and algebraic geometry meet. In this final chapter we briefly summarize the current landscape, highlight some persistent open problems, and point to a few places where ideas from Kähler geometry resonate beyond their original home.

6.1 Where the field stands today

Kähler geometry now plays a dual role in modern mathematics.

- As a structural framework: it supplies a rigid yet flexible setting in which complex geometry admits canonical differential-geometric tools (Hodge theory, curvature identities, Bochner techniques, and the $\partial\bar{\partial}$-lemma, etc.).
- As a unifying principle: the "canonical metric $\Leftrightarrow$ stability" paradigm has turned analytic existence questions into algebro-geometric inequalities and moduli-theoretic structure, while conversely stability has acquired analytic meaning via convexity and coercivity of energy functionals along geodesics.

From a historical perspective, a notable shift has occurred: the central objects are no longer only individual metrics or manifolds, but families and degenerations. Canonical metrics are now expected to behave well under limits, and the correct category in which to take limits often forces us to admit singular spaces (klt $\mathbb{Q}$-Fanos, varieties with mild singularities, or metric measure limits). This is precisely why the most active interface is with moduli theory and the minimal model program.

6.2 Some open problems and unfinished directions

The YTD story for Fano KE metrics and the rapidly developing picture for cscK metrics do not close the book. Instead, they reorganize the subject around some persistent challenges.

Effectivity and computability (stability and energies). Can we turn qualitative YTD-type equivalences into practical tests? Concretely, can we compute or estimate stability thresholds (e.g. δ-invariants, log canonical thresholds) in broad classes, and then derive explicit (or computable) coercivity/properness constants for the Ding and Mabuchi functionals from such algebro-geometric data (especially in the presence of automorphisms, where we work modulo $\mathrm{Aut}(X, L)$)? See e.g. [RTZ21], [Zha23], [Den24], [DR24], [Abb+25] and [Zhe25].

Degenerations, singularities, and canonical limits. How do canonical metrics behave under degeneration, and what is the correct notion of "canonical" limiting object? This includes: metric regularity of weak KE/cscK solutions on singular spaces; interaction with birational operations (flips, contractions); and dynamical/variational questions such as convergence rates of the Kähler-Ricci flow and the extraction of canonical algebraic degenerations in unstable cases. See e.g. [HL24], [His24], [Bed25a], [Bed25b], [Che25], [HS25] and [PT25].

Global geometry of K-moduli. Even when the existence/stability picture is understood pointwise, the global organization of solutions in families is subtle. Can we describe boundary strata of K-moduli explicitly (and relate them to metric phenomena such as collapsing or topology change on the smooth locus), and can we make the positivity/projectivity of K-moduli effective via CM line bundles and quantitative stability? See e.g. [XZ20], [Asc+23], [Hat23], [BL24], [Hat24], [Tam24], [BES25] and [LZ25]

6.3 Cross-disciplinary echoes

While our focus has been the internal development of Kähler geometry, many of its ideas and techniques have had a lasting impact well beyond the field itself.

Algebraic geometry and birational classification. The metric-stability dictionary has strengthened the bridge between differential geometry and the minimal model program: it suggests that canonical metrics not only solve PDEs but also select the correct objects for moduli and degeneration. In particular, admitting klt singularities is now seen as a feature rather than a defect, aligning analytic compactness with birational compactifications. See e.g. [Kol13], [DS19],[Oda15], [LWX19] and [Xu21].

Geometric analysis and nonlinear PDE. Kähler geometry continues to serve as a testing ground for nonlinear PDE methods: a priori estimates for complex Monge-Ampère equations, weak geodesics in spaces of potentials, and variational methods in pluripotential theory have all been exported to other geometric problems. Conversely, ideas from metric geometry and optimal transport increasingly feed back into the understanding of spaces of metrics. See e.g. [Din88], [Tia00], [JMR16], [CSW18] and [BBJ21].

Mathematical physics and mirror phenomena. Ricci-flat Kähler metrics on Calabi-Yau manifolds, originally emerging from the Calabi conjecture, have become indispensable in mathematical physics. Even without entering physical details, one can view this as another instance of a recurring theme: canonical metrics provide "optimal" geometric structures on spaces whose algebraic/topological data alone is insufficient to capture their finer geometry. See e.g. [Can+91], [Wit93], [Kon94], [SYZ96] and [Hor+03].

A closing thought. From the 19th-century birth of complex analysis and Riemann surfaces to the 20th-century formalization of complex manifolds and Hermitian/Kähler metrics, the narrative of Kähler geometry is one of increasing structural unity. We have seen how Hodge theory and curvature identities turned the Kähler condition into a powerful bridge between topology, analysis, and algebraic geometry, and how Calabi's vision was realized by Yau's solution of the Calabi conjecture and the central role of complex Monge-Ampère equations. In the modern era, canonical metrics (Kähler-Einstein, cscK, and related flows) are inseparable from stability, degeneration, and moduli: existence questions now come packaged with compactness, singular limits, and algebro-geometric compactifications. Ultimately, Kähler geometry teaches a broader lesson: when analytic and algebraic notions of structure are pushed to their natural limits, they do not merely coexist—they illuminate one another, revealing a single geometric idea through multiple, mutually reinforcing languages.

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