Riemannian Geometry
Lecture Notes

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Contents

Chapter 1

Curvature on Riemannian manifolds

1.1 Vector bundles and affine connections

Def 1.1 (tensor)**.** V, W are vector spaces, tensor $V^* \otimes W$ is a set of all linear maps $f : V \to W$.

Assume $\{e_i\}, \{w_\alpha\}$ are basis of V, W resp., then linear map f can be represented by

$$f(e_i) = a_i^\alpha w_\alpha,$$

So

$$V^* \otimes W = \mathrm{span}\{e^i \otimes w_\alpha\},$$

where $\{e^i\}$ are dual basis of $\{e_i\}$, $e^i(e_j) = \delta_i^j$.

Def 1.2. Symmetric tensor: $\mathrm{Sym}^{\otimes 2} V \subset V \otimes V$ contains all linear maps whose representation matrix is symmetric.

Skew-Symmetric tensor: $\wedge^2 V \subset V \otimes V$ contains all linear maps whose representation matrix is skew-symmetric.

Def 1.3. Let M be a C^∞ manifold, a real vector bundle of rank r over M is a C^∞ manifold $E \xrightarrow{\pi} M$, where π is a submersion, and there exist an open cover $\{U_i\}_{i \in J}$ of M, such that

(1) for each j, there is a diffeomorphism

$$\varphi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^r,$$

such that the restriction

$$\varphi_j\Big|_{\pi^{-1}(\{x\})} : \pi^{-1}(\{x\}) \to \{x\} \times \mathbb{R}^r$$

is an isomorphism.

(2) for each $i, j \in J$, the map

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r$$

is an isomorphism on each fiber and it can written as

$$\varphi_{ij}(x, v) = (x, g_{ij}(x)v),$$

where $g_{ij} \in C^\infty(U_i \cap U_j, GL(r))$ satisfy

$$\begin{cases} g_{ij} \circ g_{ji} = \mathrm{Id} & \text{on } U_i \cap U_j \\ g_{ij} \circ g_{jk} \circ g_{ki} = \mathrm{Id} & \text{on } U_i \cap U_j \cap U_k \end{cases}.$$

Exam 1.1. *(a)* $E = M \times \mathbb{R}^r$ *(trivial bundle)*

(b) TM *(transition:* $\left(\frac{\partial x^i}{\partial y^\alpha}\right)$*)*

(c) T^*M

Def 1.4 (homomorphisms between vector bundles)**.**

$$\begin{tikzcd} E_1 \arrow[r, "h"] \arrow[d] & E_2 \arrow[d] \\ M \arrow[r] & M \end{tikzcd}$$

h is a homomorphism if for each point p, $h\big|_p : E_1\big|_p \to E_2\big|_p$ is linear.
h is an endomorphism if the image is E_1, and h is isomorphism if $h\big|_p$ are isomorphism.

Def 1.5 (sections of a vector bundle)**.** $s : M \to E$ is a smooth map *s.t.*

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[dr, "\mathrm{Id}"'] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

And $s(x) \in E\big|_x$ for every point $x \in M$.

Remark 1.1. Let s be a section, consider the its local expression over a chart U of M.
Consider the local trivialization $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^r$, and let $\mathbb{R}^r = \mathrm{span}\{E_1, \cdots, E_r\}$.
Then

$$e_\alpha(x) \overset{\Delta}{=\!=} \psi^{-1}(x, E_\alpha)$$

is a local basis and

$$s(x) = s^\alpha(x) e_\alpha(x), x \in U.$$

The space of C^∞ section of $E \xrightarrow{\pi} M$ is denoted by $\Gamma(M, E)$.

Def 1.6 (Affine connection)**.** Connection is a rule of taking derivative.
An affine connection ∇ of $E \to M$ is a map

$$\nabla : \Gamma(M, TM) \times \Gamma(M, E) \to \Gamma(M, E), (X, s) \mapsto \nabla_X s$$

such that for every $s, t \in \Gamma(M, E), X, Y \in \Gamma(M, TM), f \in C^\infty(M)$, we have:

(1) Linearly:

$$\nabla_{fX+Y} s = f \nabla_X s + \nabla_Y s.$$

(2) Leibniz rule:

$$\nabla_X(fs + t) = X(f)s + f\nabla_X s + \nabla_X t.$$

Remark 1.2. Over a chart U, let $X = X^i \frac{\partial}{\partial x^i}, s = s^\alpha e_\alpha$.
We then have

$$\nabla_X s = \nabla_{X^i \frac{\partial}{\partial x^i}} s^\alpha e_\alpha = X^i \left(\frac{\partial s^\alpha}{\partial x^i} e_\alpha + s^\alpha \nabla_{\frac{\partial}{\partial x^i}} e_\alpha \right).$$

The only unknown term in the expression is

$$\nabla_{\frac{\partial}{\partial x^i}} e_\alpha = \Gamma^\beta_{i\alpha} e_\beta$$

where $\Gamma_{i\alpha}^{\beta}$ are called the Christoffel symbols.

For two different bases,

$$\nabla_{\frac{\partial}{\partial x^i}} e_\alpha = \Gamma_{i\alpha}^{\beta} e_\beta, \nabla_{\frac{\partial}{\partial \tilde{x}^i}} \tilde{e}_\alpha = \tilde{\Gamma}_{i\alpha}^{\beta} \tilde{e}_\beta$$

with

$$e_\alpha = g_\alpha^\beta \tilde{e}_\beta, \frac{\partial}{\partial x^i} = h_i^j \frac{\partial}{\partial \tilde{x}^j},$$

we have:

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^i}} e_\alpha &= h_i^j \left(\frac{\partial g_\alpha^\beta}{\partial \tilde{x}^j} \tilde{e}_\beta + g_\alpha^\beta \nabla_{\frac{\partial}{\partial \tilde{x}^j}} \tilde{e}_\beta \right) \\ &= \left(h_i^j \frac{\partial g_\alpha^\beta}{\partial \tilde{x}^j} + h_i^j g_\alpha^\gamma \tilde{\Gamma}_{j\gamma}^{\beta} \right) \tilde{e}_\beta\end{aligned}$$

So the Christoffel symbols do not transform as tensors under coordinates transformation since there is an extra term $h_i^j \frac{\partial g_\alpha^\beta}{\partial \tilde{x}^j}$.

Prop 1.1. *If ∇_1 and ∇_2 are two affine connections over $E \to M$, $\nabla_1 - \nabla_2$ is a tensor.*

Proof.

$$(\nabla_1)_X (fs) - (\nabla_2)_X (fs) = f((\nabla_1)_X s - (\nabla_2)_X s).$$

Hence

$$\nabla_1 - \nabla_2 \in \Gamma(M, T^*M \otimes E^* \otimes E), (\nabla_1 - \nabla_2)_X s \in \Gamma(M, E).$$

□

Def 1.7 (curvature of an affine connection)**.**

$$\mathrm{R}(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

is a well-defined map from $\Gamma(M, TM) \times \Gamma(M, TM) \times \Gamma(M, E)$ to $\Gamma(M, E)$.

You may think it is strange why we prefer to define the curvature by this rather than simply $\nabla_X \nabla_Y$, this is because we want the curvature to be a tensor(this idea is very **important**) :

Prop 1.2.

$$\mathrm{R} \in \Gamma(M, \wedge^2 T^*M \otimes \mathrm{End}(E)).$$

Proof.

$$\begin{aligned}\mathrm{R}(fX, gY)hs =& \nabla_{fX} \nabla_{gY} hs - \nabla_{gY} \nabla_{fX} hs - \nabla_{[fX,gY]} hs \\ =& fgh \nabla_X \nabla_Y s + fgX(h) \nabla_Y s + fgY(h) \nabla_X s + fgX(Y(h))s \\ &+ fX(g) \nabla_Y hs - fgh \nabla_Y \nabla_X s - fgY(h) \nabla_X s - fgX(h) \nabla_Y s \\ &- fgY(X(h))s - gY(f) \nabla_X hs - fgh \nabla_{[X,Y]} s - fg[X, Y](h)s \\ &- fX(g) \nabla_Y hs + gY(f) \nabla_X hs \\ =& fgh(\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s)\end{aligned}$$

Hence R is a tensor, *i.e.* $\mathrm{R} \in \Gamma(M, \wedge^2 T^*M \otimes \mathrm{End}(E))$. □

Remark 1.3. Consider the local expression, let

$$\mathrm{R} = \mathrm{R}_{ij\alpha}^{\beta} \mathrm{d}x^i \otimes dx^j \otimes e^\alpha \otimes e_\beta, \mathrm{R}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) e_\alpha \xlongequal{\Delta} \mathrm{R}_{ij\alpha}^{\beta} e_\beta.$$

Then we have

$$\mathrm{R}_{ij\alpha}^{\beta} = \frac{\partial \Gamma_{j\alpha}^{\beta}}{\partial x^i} - \frac{\partial \Gamma_{i\alpha}^{\beta}}{\partial x^j} + \Gamma_{j\alpha}^{\mu} \Gamma_{i\mu}^{\beta} - \Gamma_{i\alpha}^{\mu} \Gamma_{j\mu}^{\beta}.$$

Def 1.8. Given $s^* \in \Gamma(M, E^*)$, we can define a connection $\nabla_X^* s^*$ of E^*, such that

$$X\langle s, s^*\rangle = \langle \nabla_X s, s^*\rangle + \langle s, \nabla_X^* s^*\rangle$$

for every $s \in \Gamma(M, E)$, which is called the Leibniz rule.

1.2 Levi-Civita connection

Def 1.9 (Riemannian metric)**.** $g \in \Gamma(M, \mathrm{Sym}^{\otimes 2} T^*M)$ and g_p is a positive definite for each $p \in M$.

Thm 1.1. *There exists a C^∞ Riemannian metrics on every C^∞ manifold.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an open cover of M and $\{\rho_\alpha\}$ be a partition of unity subordinate to it.

Then Let g_0 be the canonical metric on $\mathbb{R}^n$ and $g_\alpha = \varphi_\alpha^* g_0$.

So $g = \sum_\alpha \rho_\alpha g_\alpha$ is a Riemannian metric on M. □

Thm 1.2 (Existence of Levi-Civita Connection)**.** *Let (M, g) be a C^∞ Riemannian manifold, there exists an **unique** affine connection ∇ on $TM \to M$, such that:*

(1) ∇ is compatible with g:

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

(2) torsion free(symmetry):

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Proof. Consider three equations

$$\begin{cases} Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(X, Z) = g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \\ -Zg(X, Y) = -g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \end{cases}$$

Adding them up we can get

$$\begin{aligned} &Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ =&g(\nabla_X Y, Z) + g(Y, [X, Z]) + g(X, [Y, Z]) + g(\nabla_X Y + [X, Y], Z) \\ =&2g(\nabla_X Y, Z) + g(Y, [X, Z]) + g(X, [Y, Z]) + g([X, Y], Z) \end{aligned}$$

Hence

$$\begin{aligned} g(\nabla_X Y, Z) =&\frac{1}{2}(Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &- g(Y, [X, Z]) - g(X, [Y, Z]) - g([X, Y], Z)) \end{aligned}$$

□

Remark 1.4. Since we have

$$g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \Gamma_{ij}^l g_{kl},$$

we can obtain that

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

Def 1.10. Curvature tensor of Levi-Civita connection ∇ is

$$\mathrm{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

And we define $\mathrm{R}(X,Y,Z,W) \xlongequal{\Delta} g(\mathrm{R}(X,Y)Z,W)$.

Remark 1.5. Torsion free tells us that

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i},$$

this is very useful.
We have

$$\mathrm{R}_{ijkl} = g_{lp}\mathrm{R}^p_{ijk}, \mathrm{R}^p_{ijk} = \frac{\partial\Gamma^p_{kj}}{\partial x^i} - \frac{\partial\Gamma^p_{ki}}{\partial x^j} - \Gamma^p_{qj}\Gamma^q_{ki} + \Gamma^p_{qi}\Gamma^q_{kj}$$

$$\mathrm{R}_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{jl}}{\partial x^i\partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^j\partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j\partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i\partial x^l}\right) + g_{pq}\left(\Gamma^p_{ik}\Gamma^q_{jl} - \Gamma^p_{il}\Gamma^q_{jk}\right)$$

There are some tricks that can help you prove this equation:

$$g^{ij}g_{ik} = \delta^j_k, g^{ij}\frac{\partial g_{ik}}{\partial x^l} = -\frac{\partial g^{ij}}{\partial x^l}g_{ik}.$$

And it needs a lot of patience for you to write down the complete proof (:

Prop 1.3. *(1) Skew-symmetry:*

$$\mathrm{R}(X,Y,Z,W) = -\mathrm{R}(Y,X,Z,W) = -\mathrm{R}(X,Y,W,Z) = \mathrm{R}(Z,W,X,Y).$$

(2) The first Bianchi identity:

$$\mathrm{R}(X,Y,Z,W) + \mathrm{R}(Y,Z,X,W) + \mathrm{R}(Z,X,Y,W) = 0$$

Proof. (2)Let $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}, Z = \frac{\partial}{\partial x^k}$.

$$\begin{aligned}\mathrm{R}(X,Y)Z + \mathrm{R}(Y,Z)X &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X\\ &= \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y\\ &= \mathrm{R}(X,Z)Y\end{aligned}$$

So

$$\mathrm{R}(X,Y)Z + \mathrm{R}(Y,Z)X + \mathrm{R}(Z,X)Y = 0.$$

□

1.3 Sectional curvature and Ricci curvature

Def 1.11 (section curvature)**.**

$$\mathrm{K}(X,Y) \xlongequal{\Delta} \frac{\mathrm{R}(X,Y,Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}$$

where X, Y are linearly independent.
We say that sectional curvature $K \geqslant k$ if for any linearly independent $X, Y \in \Gamma(M, TM)$,

$$K(X,Y) \geqslant k.$$

Similarly, we can define $K \leqslant k, K = k$.

Remark 1.6.

$$\mathrm{R}_0(X,Y,Z,W) = \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle$$

also satisfy the skew-symmetry and first Bianchi identity properties, and

$$\mathrm{R}_0(X,Y,Y,X) = |X|^2|Y|^2 - |\langle X,Y\rangle|^2.$$

Lemma 1.1. *If* $\mathrm{span}\{X,Y\} = \mathrm{span}\{Z,W\} = \Sigma$ *is a plane, then*

$$\mathcal{K}(X,Y) = \mathcal{K}(Z,W) = \mathcal{K}(\Sigma).$$

Proof. Let $Z = aX + bY, W = cX + dY$.

$$\begin{aligned}\mathrm{R}(Z,W,W,Z) &= \mathrm{R}(aX+bY, cX+dY, cX+dY, aX+bY)\\ &= (ad-bc)^2\mathrm{R}(X,Y,Y,X)\end{aligned}$$

So

$$|z|^2|w|^2 - |\langle z,w\rangle|^2 = (ad-bc)^2\left(|x|^2|y|^2 - |\langle x,y\rangle|^2\right)$$

□

Thm 1.3. *The sectional curvature* $\mathcal{K}$ *determines the full curvature tensor.*

Proof. Given $X,Y,Z,W \in \Gamma(M,TM)$, define

$$F:\mathbb{R}^2 \to C^\infty(M), (s,t) \mapsto \mathrm{R}(x+sz, y+tw, y+tw, x+sz)$$

Then

$$\begin{aligned}&\left.\frac{\partial^2 F}{\partial s\partial t}\right|_{s=t=0}\\ =&\left.\frac{\partial^2}{\partial s\partial t}\left(\mathrm{R}(sZ,tW,Y,X)+\mathrm{R}(sZ,Y,tW,X)+\mathrm{R}(X,tW,Y,sZ)+\mathrm{R}(X,Y,tW,sZ)\right)\right|_{s=t=0}\\ -&\left.\frac{\partial^2}{\partial s\partial t}\left(\mathrm{R}(sW,tZ,Y,X)+\mathrm{R}(sW,Y,tZ,X)+\mathrm{R}(X,tZ,Y,sW)+\mathrm{R}(X,Y,tZ,sW)\right)\right|_{s=t=0}\\ =&\mathrm{R}(Z,W,Y,X)+\mathrm{R}(Z,Y,W,X)+\mathrm{R}(X,W,Y,Z)+\mathrm{R}(X,Y,W,Z)\\ -&\mathrm{R}(W,Z,Y,X)-\mathrm{R}(W,Y,Z,X)-\mathrm{R}(X,Z,Y,W)-\mathrm{R}(X,Y,Z,W)\\ =&-\mathrm{R}(X,Y,Z,W)+\mathrm{R}(Y,Z,X,W)+\mathrm{R}(Y,Z,X,W)-\mathrm{R}(X,Y,Z,W)\\ -&\mathrm{R}(X,Y,Z,W)+\mathrm{R}(Z,X,Y,W)+\mathrm{R}(Z,X,Y,W)-\mathrm{R}(X,Y,Z,W)\\ =&-6\mathrm{R}(X,Y,Z,W)\end{aligned}$$

□

Prop 1.4. *Let* (M,g) *be a Riemannian manifold and* $p\in M$*, TFAE:*

(1) For a plane $\Sigma\subset T_pM$*,* $\mathcal{K}_p(\Sigma)$ *is independent of* Σ

(2) $\exists$*some constant* k_p*, such that*

$$\mathrm{R}_{ijkl} = k_p(g_{il}g_{kj} - g_{ik}g_{jl})$$

(3) $\exists$*some constant* k_p*, such that*

$$\frac{\mathrm{R}(X,Y,Y,X)}{|X|^2|Y|^2 - |\langle X,Y\rangle|^2} = k_p$$

for every $X,Y\in T_pM$ *and* $\mathrm{span}\{X,Y\}$ *is a plane.*

Proof. (2) $\Rightarrow$ (3):

$$\mathrm{R}(X,Y,Y,X) = \mathrm{R}_{ijkl}x^i x^l y^j y^k$$
$$|X|^2|Y|^2 - |\langle X,Y\rangle|^2 = (g_{il}g_{kj} - g_{ik}g_{jl})x^i x^l y^j y^k$$

(3) $\Rightarrow$ (2): Consider

$$\begin{cases} F(s,t) = \mathrm{R}(X+sZ, Y+tW, Y+tW, X+sZ) \\ F_0(s,t) = \mathrm{R}(X+sZ, Y+tW, Y+tW, X+sZ) \end{cases}$$

Then

$$\begin{cases} \frac{\partial^2 F}{\partial s \partial t}\Big|_{s=t=0} = -6\mathrm{R}(X,Y,Z,W) \\ \frac{\partial^2 F_0}{\partial s \partial t} = -6\mathrm{R}(X,Y,Z,W) \end{cases}$$

Hence $\mathrm{R}(X,Y,Z,W) = k_p \mathrm{R}_0(X,Y,Z,W)$. □

Def 1.12 (Ricci curvature)**.**

$$\mathrm{Ric} = \mathrm{R}_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$$

where

$$\mathrm{R}_{ij} = g^{kl}\mathrm{R}_{iklj} = g^{kl}\mathrm{R}_{kijl} = \mathrm{R}^k_{kij}.$$

We say that Ricci curvature $\mathrm{Ric}(g) \geqslant C$ if for any $X \in \Gamma(M, TM)$,

$$\mathrm{Ric}(X,X) \geqslant C.$$

Similarly, we can define $\mathrm{Ric}(g) \leqslant C$.

Def 1.13 (scalar curvature)**.**

$$s = g^{ij}\mathrm{R}_{ij} = \mathrm{tr}_g \,\mathrm{Ric}.$$

Def 1.14 (curvature operator)**.**

$$\mathcal{R} : \Gamma(M, \wedge^2 TM) \to (M, \wedge^2 TM),$$

satisfying

$$g(\mathcal{R}(X \wedge Y), Z \wedge W) = \mathrm{R}(X,Y,W,Z).$$

Exam 1.2.

$$f : S^n(K) \hookrightarrow (\mathbb{R}^{n+1}, g_{can}), g_K = f^* g_{can}.$$

$$f(x^1, \cdots, x^n) = \left(x^1, \cdots, x^n, \sqrt{K^2 - \sum_{i=1}^n (x^i)^2}\right), g_{ij} = \delta_{ij} + \frac{x^i x^j}{\sqrt{K^2 - \sum (x^i)^2}}.$$

Then

$$\mathrm{R}_{ijkl} = \frac{1}{K^2}(g_{il}g_{kj} - g_{ik}g_{jl}), \mathrm{R}_{ij} = \frac{n-1}{K^2}g_{ij}, \mathrm{S} = \frac{n(n-1)}{K^2}$$

Prop 1.5. *Let (M,g) be a Riemannian manifold, prove that the scalar curvature $s(p)$ at $p \in M$ is given by*

$$s(p) = \frac{n}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \mathrm{Ric}_p(x,x)\mathrm{d}S.$$

Proof. Assume $(x^1, \cdots, x^n)$ is an orthogonal coordinate of T_pM.

$$\begin{aligned} \frac{n}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \mathrm{Ric}_p(x,x)\mathrm{d}S &= \frac{n}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \mathrm{R}_{ij}x^i x^j \mathrm{d}S \\ &= \frac{n}{\omega_{n-1}} \mathrm{R}_{ij} \frac{\delta^{ij}\omega_{n-1}}{n} \\ &= \mathrm{tr}\,(\mathrm{Ric}_p) = s(p) \end{aligned}$$

□

Prop 1.6. *Let (M, g) be a Riemannian n-manifold and $p \in M$. For each unit vector $v \in T_pM$, prove that*

$$\mathrm{Ric}_p(v, v) = \frac{n-1}{\mathrm{vol}(\mathbb{S}^{n-2})} \int_{w \in S_v^\perp} \sec(v, w) \mathrm{d}V_{\hat{g}}$$

where $S_v^\perp$ denotes the set of unit vectors in T_pM that are orthogonal to v and $\hat{g}$ denotes the Riemannian metric on $S_v^\perp$ induced from the flat metric g_p on T_pM.

Proof. Let $\{e_1, \cdots, e_{n-1}\}$ be an orthogonal basis of $S_v^\perp$, $e_n = v$.
Then $\{e_1, \cdots, e_n\}$ is an orthogonal basis of T_pM, so

$$\mathrm{Ric}_p(v, v) = \delta^{ij} R_{innj} = \sum_{i=1}^{n-1} \sec(e_i, v).$$

So we can obtain

$$\begin{aligned}
\frac{n-1}{\mathrm{vol}(\mathbb{S}^{n-2})} \int_{\mathbb{S}_v^\perp} \sec(v, w) \mathrm{d}V_{\hat{g}} =& \frac{n-1}{\mathrm{vol}(\mathbb{S}^{n-2})} \int_{\mathbb{S}_v^\perp} \mathrm{R}_{innj} w^i w^j \mathrm{d}V_{\hat{g}} \\
=& \sum_{i=1}^{n-1} \frac{n-1}{\mathrm{vol}(\mathbb{S}^{n-2})} \mathrm{R}_{inni} \frac{\mathrm{vol}(\mathbb{S}^{n-2})}{n-1} \\
=& \mathrm{Ric}_p(v, v)
\end{aligned}$$

□

Prop 1.7. *Let (M, g) be a Riemannian n-manifold with sectional curvature $K \geqslant k$, then*

$$\mathrm{Ric}(g) \geqslant (n-1)k.$$

Proof. Follows by proposition 1.6. □

Remark 1.7. In general, we can not get any information about sectional curvature from Ricci curvature, but in 3-dimensional case, we have the following theorem:

Thm 1.4. *Let (M, g) be a connected Einstein manifold,* i.e. $\mathrm{Ric}(g) = cg$ *for some* $c \in \mathbb{R}$, *suppose* $\dim M = 3$, *then (M, g) has constant sectional curvature $\frac{c}{2}$.*

Proof. Consider the orthogonal frame $\{e_1, e_2, e_3\}$ such that $(g_{ij}) = I$, then

$$\mathrm{R}_{11} = -\mathrm{R}_{1212} - \mathrm{R}_{1313}, \mathrm{R}_{22} = -\mathrm{R}_{2323} - \mathrm{R}_{2121}, \mathrm{R}_{33} = -\mathrm{R}_{3131} - \mathrm{R}_{3232},$$
$$\mathrm{R}_{12} = -\mathrm{R}_{3132}, \mathrm{R}_{23} = -\mathrm{R}_{1213}, \mathrm{R}_{31} = -\mathrm{R}_{2321}.$$

So we deduce

$$R_{1212} = R_{1313} = R_{2323} = R_{11} - \frac{R_{11} + R_{22} + R_{33}}{2} = -\frac{1}{2}c, i.e. K \equiv \frac{1}{2}c.$$

□

Remark 1.8. This theorem allow us to research 3-dimensional manifold more easily. The structure of most of manifolds with dimension bigger than 3 are unclear, e.g. whether there is a Ricci-flat metric on $\mathbb{S}^4$ or not. We will discuss these question in the last chapter.

Here are some big theorem and conjugate in Riemannian geometry:

Thm 1.5 (Hamilton 1982)**.** *Let (M, g) be a simply connected compact* 3*-dim Riemannian manifold, if Ric* > 0, *then M is diffeomorphic to $\mathbb{S}^3$.*

Thm 1.6 (Böhm-Wilking 2008)**.** *Let (M, g) be a simply connected compact Riemannian manifold for $n \geqslant 4$, and curvature operator$> 0 \Rightarrow M$ is diffeomorphic to $\mathbb{S}^n$.*

Conj 1.1. $\dim_{\mathbb{R}} M = 4$, *sectional curvature* > 0 *and Einstein* $\mathrm{R}_{ij} = kg_{ij}$, *then it is $\mathbb{S}^4$ or $\mathbb{CP}^2$.*

Chapter 2

Basic concepts in Riemannian geometry

2.1 Pullback vector bundles & connections

Def 2.1. For a smooth map $M \xrightarrow{f} N$ and a vector bundle $E \xrightarrow{\pi} N$, f^*E is called the pullback vector bundle, defined as

$$f^*E = \{(p, v) \in M \times E | \pi(v) = f(p)\}.$$

So the following diagram commutes.

$$(f^*E)_p = \tilde{\pi}^{-1}(\{p\}) \cong E_{f(p)} = \pi^{-1}(\{f(p)\}).$$

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "\tilde{\pi}"] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Remark 2.1. If $\{e_A\} \to E$ is a local basis, then define

$$(f^*e_A)(p) = e_A(f(p)),$$

so $\{f^*e_A\}$ is a local basis of $f^*(E)$.

Def 2.2. The pullback connection $\hat{\nabla}$ on $\hat{E} = f^*E$ is a map

$$\Gamma(M, TM) \times \Gamma(M, \hat{E}) \to \Gamma(M, \hat{E}),$$

satisfying the following commutative diagram:

$$\begin{tikzcd} (f^*E, \hat{\nabla}) \arrow[r] \arrow[d, "\hat{\pi}"] & (E, \nabla) \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Remark 2.2. If we have

$$\nabla_{\frac{\partial}{\partial y^\alpha}} e_A = \Gamma^B_{\alpha A} e_B,$$

then

$$\hat{\nabla}_{\frac{\partial}{\partial x^i}} \hat{e}_A = \hat{\Gamma}^B_{iA} \hat{e}_A = \Gamma^B_{\alpha A} \frac{\partial f^\alpha}{\partial x^i} \hat{e}_A.$$

Def 2.3. define the pullback metric

$$\hat{g}(\hat{e}_A, \hat{e}_B)(p) = g(e_A, e_B)(f(p)), \hat{g} = f^*g.$$

It satisfy the commutative diagram below:

$$\begin{tikzcd} (f^*E, \hat{g}) \arrow[r] \arrow[d, "\hat{\pi}"] & (E, g) \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Remark 2.3. For $g = g_{AB}e^A \otimes e^B$, we have

$$\begin{aligned} f^*g &= f^*(g_{AB}) f^*e^A \otimes f^*e_B \\ &= g_{AB}(f) f^*e^A \otimes f^*e_B \end{aligned}$$

Prop 2.1. *suppose ∇ is a metric compatible affine connection on (E, g),* i.e.

$$Xg(s,t) = g(\nabla_X s, t) + g(s, \nabla_X t).$$

Then $\hat{\nabla}$ is a compatible with $(\hat{E}, \hat{g})$, so the diagram below commutes.

$$\begin{tikzcd} (f^*E, \hat{\nabla}, \hat{g}) \arrow[r] \arrow[d, "\tilde{\pi}"] & (E, \nabla, g) \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Def 2.4. For

$$\mathrm{R}^E = \mathrm{R}^B_{\alpha\beta A} \mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta \otimes e^A \otimes e_B,$$

we define the pullback curvature:

$$\mathrm{R}^{\hat{E}} = \hat{\mathrm{R}}^B_{ijA} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \hat{e}^A \otimes \hat{e}_B,$$

where

$$\hat{\mathrm{R}}^B_{ijA}(p) = \mathrm{R}^B_{\alpha\beta A}(f(p)) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}.$$

2.2 Parallel transport

Prop 2.2. *If $\gamma : [a,b] \to (M, g)$ is a C^∞ curve, $v \in T_{\gamma(a)}M$, then there exists a unique vector field $V \in \Gamma([a,b], \gamma^*TM)$ such that*

$$\begin{cases} \hat{\nabla} V \equiv 0 \\ V(a) = v \end{cases}$$

Remark 2.4. $\hat{\nabla} V$ is actually $\hat{\nabla}_{\frac{\partial}{\partial t}} V$ since $[a,b]$ is a 1-dimensional manifold.

Proof. Choosing basis $\{e_i(t)\}$ in $T_{\gamma(t)}M$ for $t \in [a,b]$.

For $V \in \Gamma([a,b], \gamma^*TM)$, let

$$V = V^i(t) e_i(t)$$

and we write

$$\hat{\nabla} e_i(t) = \omega_i^j(t) e_j(t).$$

So

$$\left(\frac{\mathrm{d}V^i}{\mathrm{d}t} + \omega_j^i v^j \right) e_i = \hat{\nabla} V = 0.$$

Hence V is unique by the uniqueness of solution of ODE. $\square$

Def 2.5. For $\gamma : [a,b] \to (M,g)$, define

$$P_{t_0,t,\gamma} : T_{\gamma(t_0)}M \to T_{\gamma(t)}M, v \mapsto V(t)$$

called the parallel transport along γ.

Thm 2.1. $\gamma : I \to (M,g)$, *then*

(1)

$$P_{s,t,\gamma} : T_{\gamma(s)}M \to T_{\gamma(t)}M$$

is linear

(2)

$$P_{t_2,t_3,\gamma} \circ P_{t_1,t_2,\gamma} = P_{t_1,t_3,\gamma}, P_{t_1,t_1,\gamma} = \mathrm{Id}$$

(3) $P_{s,t,\gamma}$ *is a linear isometry*

(4)

$$F : I \times \gamma^*TM \to \gamma^*TM, F(t,(s,v)) = (t, P_{s,t,\gamma}v)$$

(5) For any $V \in \Gamma([a,b], \gamma^*TM), t, t_0 \in I$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(P_{t,t_0,\gamma}(V(t))) = P_{t,t_0,\gamma}(\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V(t))$$

Proof. (1) Let V_1, V_2 be the unique parallel vector field along γ such that $V_1(s) = v_0, V_2(s) = w_0$.

Then

$$\widehat{\nabla}(cV_1 + V_2) \equiv 0, (cV_1 + V_2)(s) = cv_0 + w_0.$$

So

$$P_{s,t,\gamma}(cv_0 + w_0) = (cV_1 + V_2)(t) = cP_{s,t,\gamma}v_0 + P_{s,t,\gamma}w_0.$$

(2) Let V be the unique parallel vector field along γ such that $V_1(t_1) = v_0$.

Then

$$P_{t_1,t_2,\gamma}(v_0) = V(t_2), P_{t_2,t_3,\gamma}(V(t_2)) = V(t_3).$$

(3) Let V_1, V_2 be the unique parallel vector field along γ such that $V_1(s) = v_0, V_2(s) = w_0$.

Then

$$\left\langle \widehat{\nabla}V_1, \widehat{\nabla}V_2 \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle V_1, V_2\rangle = 0,$$

i.e. $\langle V_1, V_2\rangle$ is constant.

Therefore

$$d(P_{s,t,\gamma}(v_0), P_{s,t,\gamma}(w_0)) = d(v_0, w_0).$$

(4) Since $P_{s,t,\gamma} : (\gamma^*T)_sM \to (\gamma^*T)_tM$ is a linear isometry.

So $F : I \times \gamma^*TM \to \gamma^*TM$ is smooth, because γ^*TM is vector bundle over I.

(5) Let $V(t) = V^i(t)e_i(t)$, where $\{e_i(t)\}$ is a basis of parallel vector fields along the curve.

Then

$$P_{t,s,\gamma}\left(\widehat{\nabla}V(t)\right) = P_{t,s,\gamma}\left(\frac{\mathrm{d}V^i(t)}{\mathrm{d}t}e_i(t)\right) = \frac{\mathrm{d}V^i(t)}{\mathrm{d}t}e_i(s).$$

$$\widehat{\nabla}(P_{t,s,\gamma}(V(t))) = \widehat{\nabla}(V^i(t)e_i(s)) = \frac{\mathrm{d}V^i(t)}{\mathrm{d}t}e_i(s)$$

$\square$

2.3 Hessian of smooth functions

Def 2.6. By the definition 1.8, for $\omega \in \Gamma(M, T^*M)$, we define

$$(\nabla_Y \omega)(X) \xlongequal{\Delta} Y(\omega(X)) - \omega(\nabla_Y X).$$

Remark 2.5. Assume

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j = \tilde{\Gamma}_{il}^j \mathrm{d}x^l.$$

Then we have the relation

$$\tilde{\Gamma}_{il}^j = 0 - \mathrm{d}x^j \left(\Gamma_{il}^t \frac{\partial}{\partial x^t} \right) = -\Gamma_{il}^j, i.e. \nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j = -\Gamma_{il}^j \mathrm{d}x^l.$$

Def 2.7. For a smooth function $f : M \to \mathbb{R}$, the Hessian of f is

$$\text{Hess }(f) = \nabla^2 f = \nabla \mathrm{d}f \in \Gamma(M, T^*M \otimes T^*M).$$

Lemma 2.1.

$$\text{Hess }(f) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right) \mathrm{d}x^i \otimes \mathrm{d}x^j$$

$$\text{Hess }(f)(X, Y) = g(\nabla_X \nabla f, Y) = X(Y(f)) - (\nabla_X Y)(f)$$

So Hess $(f) \in \Gamma(M, \mathrm{Sym}^{\otimes 2} T^*M)$.

Proof.

$$\begin{aligned} \text{Hess }(f)(X, Y) &= X(\mathrm{d}f(Y)) - \mathrm{d}f(\nabla_X Y) \\ &= X(Y(f)) - (\nabla_X Y)(f) \\ &= Xg(\nabla f, Y) - g(\nabla f, \nabla_X Y) \\ &= g(\nabla_X \nabla f, Y) \end{aligned}$$

So

$$\begin{aligned} \text{Hess }(f)(aX, bY) &= aX(bY(f)) - (\nabla_{aX} bY)(f) \\ &= abX(Y(f)) + aX(b)Y(f) - ab(\nabla_X Y)(f) - aX(b)Y(f) \\ &= ab\text{Hess }(f)(X, Y) \\ &= ab(Y(X(f)) + [X, Y]f - (\nabla_Y X)(f) - [X, Y]f) \\ &= ab\text{Hess }(f)(Y, X) \end{aligned}$$

Therefore Hess is a symmetric tensor.
And

$$\begin{aligned} \text{Hess }(f)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) &= \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f - \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)(f) \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \end{aligned}$$

Another method:

$$\begin{aligned} \text{Hess }(f) =& \nabla \left(\frac{\partial f}{\partial x^j} \mathrm{d}x^j \right) \\ =& \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \mathrm{d}x^j - \frac{\partial f}{\partial x^j} \Gamma_{ik}^j \mathrm{d}x^k \right) \otimes \mathrm{d}x^i \\ =& \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial x^k} \Gamma_{ij}^k \right) \mathrm{d}x^i \otimes \mathrm{d}x^j \end{aligned}$$

□

Def 2.8. Laplace of function is defined as

$$\Delta_g f = \mathrm{tr}_g \,\mathrm{Hess}\,(f) = g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\right).$$

Prop 2.3.

$$\Delta_g f = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det(g)}\frac{\partial f}{\partial x^j}\right)$$

Proof.

$$\begin{aligned}
\Delta_g f =& g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\right)\\
=& g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{1}{2}g^{ij}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)\frac{\partial f}{\partial x^k}\\
=& g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{1}{2}g^{ij}g^{kl}\frac{\partial g_{ij}}{\partial x^l}\frac{\partial f}{\partial x^k} - \frac{1}{2}g^{ij}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i}\right)\frac{\partial f}{\partial x^k}\\
=& g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{1}{2\det g}g^{kl}\frac{\partial \det g}{\partial x^l}\frac{\partial f}{\partial x^k} + g_{il}g^{kl}\frac{\partial g^{ij}}{\partial x^j}\frac{\partial f}{\partial x^k}\\
=& g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{1}{\sqrt{\det g}}g^{kl}\frac{\partial \sqrt{\det g}}{\partial x^l}\frac{\partial f}{\partial x^k} + \frac{\partial g^{kj}}{\partial x^j}\frac{\partial f}{\partial x^k}\\
=& \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det g}\frac{\partial f}{\partial x^j}\right)
\end{aligned}$$

□

Thm 2.2. *Let $f : M \to \mathbb{R}$ be a C^∞ function on a Riemannian manifold (M, g).*

(1) If $p \in M$ is a local maximum or local minimum, then $(\nabla f)(p) = 0$

(2) If $p \in M$ is a local maximum, then $(\mathrm{Hess}\ (f))(p) \leqslant 0, (\Delta f)(p) \leqslant 0$

(3) If $p \in M$ is a local minimum, then $(\mathrm{Hess}\ (f))(p) \geqslant 0, (\Delta f)(p) \geqslant 0$

Proof. Consider a local chart $(U, \varphi, x^i), x^i = r^i \circ \varphi, \psi = \varphi^{-1}$.
So for $F = f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$,

$$\frac{\partial f}{\partial x^i} = \frac{\partial F}{\partial r^i} = 0.$$

And

$$\frac{\partial^2 F}{\partial r^i \partial r^j} = \frac{\partial}{\partial r^i}\left(\frac{\partial F}{\partial r^j}\right) = \frac{\partial}{\partial r^i}\left(\frac{\partial f}{\partial x^k}\frac{\partial \psi^k}{\partial r^j}\right) = \frac{\partial^2 f}{\partial x^k \partial x^l}\frac{\partial \psi^k}{\partial r^i}\frac{\partial \psi^l}{\partial r^j} + \frac{\partial f}{\partial x^k}\frac{\partial^2 \psi^k}{\partial r^i \partial r^j}$$

Since $\frac{\partial f}{\partial x^k} = 0$ at the extreme point.
So Hess $F = \mathrm{d}\psi \cdot \mathrm{Hess}\ f \cdot (\mathrm{d}\psi)^T = \mathrm{d}\psi \circ \mathrm{Hess}\ f \circ (\mathrm{d}\psi)^{-1}$.
Hence Hess $(f) \geqslant 0 \Leftrightarrow$ Hess $(F) \geqslant 0$ and Hess $(f) \leqslant 0 \Leftrightarrow$ Hess $(F) \leqslant 0$. □

2.4 The second fundamental form

Def 2.9. For $f : (M, g) \to (N, h)$ and given $X, Y \in \Gamma(M, TM)$, we define

$$B(X, Y) = \hat{\nabla}_X f_* Y - f_*(\nabla^g_X Y)$$

is the second fundamental form.

Exam 2.1. *If $(N,h)=(\mathbb{R},g_{can})$, then $B(X,Y)$ is the Hessian.*

Remark 2.6. For $X=X^i\frac{\partial}{\partial x^i}, Y=Y^j\frac{\partial}{\partial x^j}$, we can obtain that

$$B(X,Y)=\left(\frac{\partial^2 f^\alpha}{\partial x^i\partial x^j}+\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}-\Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k}\right)X^iY^j\frac{\partial}{\partial y^\alpha}$$

Prop 2.4. $B(X,Y)=B(Y,X),\text{i.e.}B\in\Gamma(M,\mathrm{Sym}^{\otimes 2}T^*M\otimes f^*TN)$

Coro 2.1.

$$\hat{\nabla}_X f_*Y-\hat{\nabla}_Y f_*X=f_*(\nabla_XY)-f_*(\nabla_YX)=f_*([X,Y]).$$

Remark 2.7. Define

$$\tilde{\nabla}=\nabla^{*g}\otimes\hat{\nabla},$$

which is the connection on $T^*M\otimes f^*TN$
Then $B=\tilde{\nabla}\mathrm{d}f$, where

$$\mathrm{d}f\in\Gamma(M,T^*M\otimes f^*TN)$$

is the induced map of f.

Lemma 2.2. *Consider an immersion $f:M\to(\bar{M},\bar{g})$, there is an induced metric g on TM, $g=f^*\bar{g}$.*
Then there are two system over M:

$$\begin{tikzcd}
(f^*T\bar{M},\hat{\nabla},\hat{g}) \arrow[r] \arrow[d] & (T\bar{M},\bar{\nabla},\bar{g}) \arrow[d] \\
M \arrow[r, "f"] & \bar{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
(TM,\nabla,g) \arrow[r] \arrow[d] & (T\bar{M},\bar{\nabla},\bar{g}) \arrow[d] \\
M \arrow[r, "f"] & \bar{M}
\end{tikzcd}$$

Show that $f_*(TM)$ is a subbundle of $f^*T\bar{M}$.
Moreover, there exists a subbundle $T^\perp M$, s.t.

$$f^*T\bar{M}=f_*(TM)\oplus T^\perp M.$$

Prop 2.5. *f is an immersion, then*

$$B\in\Gamma(M,\mathrm{Sym}^{\otimes 2}T^*M\otimes T^\perp M),\text{i.e.}\hat{g}(B(X,Y),f_*(Z))=0$$

Proof.

$$\begin{aligned}\hat{g}(B(X,Y),f_*Z)=&\hat{g}(\hat{\nabla}_Xf_*Y-f_*(\nabla_XY),f_*(Z))\\=&\hat{g}(\hat{\nabla}_Xf_*Y,f_*(Z))-\hat{g}(f_*(\nabla_XY),f_*Z)\\=&\hat{g}(\hat{\nabla}_Xf_*Y,f_*Z)-g(\nabla_XY,Z)\end{aligned}$$

Take $X=\frac{\partial}{\partial x^i},Y=\frac{\partial}{\partial x^j},Z=\frac{\partial}{\partial x^k}$.
Then

$$\begin{aligned}\hat{g}(B(X,Y),f_*Z)=&\hat{g}\left(\hat{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial f^\alpha}{\partial x^j}\frac{\partial}{\partial y^\alpha},\frac{\partial f^\beta}{\partial x^k}\frac{\partial}{\partial y^\beta}\right)-\Gamma^l_{ij}g_{kl}\\=&\bar{g}_{\alpha\delta}\frac{\partial f^\delta}{\partial x^k}\left(\frac{\partial^2 f^\alpha}{\partial x^i\partial x^j}+\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\Gamma^\alpha_{\beta\gamma}(f)-\frac{\partial f^\alpha}{\partial x^l}\Gamma^l_{ij}\right)\end{aligned}$$

By rank theorem, $\hat{F}$ is the representation of f such that

$$\hat{F}(x^1,\cdots,x^m)=(x^1,\cdots,x^m,0,\cdots,0).$$

So

$$\hat{g}(B(X,Y),f_*Z)=\bar{\Gamma}^\alpha_{\beta\gamma}\bar{g}_{\alpha\delta}\delta^\beta_i\delta^\gamma_j\delta^\delta_k-g_{kl}\Gamma^l_{ij}=0$$

□

Remark 2.8. We can also compute this without using rank theorem, but the equation is a little bit long. It is a very good exercise (:

Def 2.10. Given $\eta \in \Gamma(M, T^{\perp}M)$, define

$$B_\eta(X,Y) = \hat{g}(B(X,Y),\eta) \in \Gamma(M, \mathrm{Sym}^{\otimes 2}T^*M),$$

called the second fundamental form along η.

Def 2.11 (Weingarten map)**.** $W_\eta : \Gamma(M,TM) \to \Gamma(M,TM)$ such that

$$g(W_\eta(X),Y) = B_\eta(X,Y), i.e.(B_\eta)_{ij} = (W_\eta)_i^k g_{kj}.$$

Thm 2.3 (Gauss)**.** *For $X,Y,Z,W \in T(M,TM)$,*

$$R(X,Y,Z,W) - \hat{R}(X,Y,f_*Z,f_*W) = \hat{g}(B(Y,Z),B(X,W)) - \hat{g}(B(X,Z),B(Y,W))$$

In particular,

$$R(X,Y,Y,X) - \hat{R}(X,Y,f_*Y,f_*X) = \hat{g}(B(Y,Y),B(X,X)) - \hat{g}(B(X,Y),B(X,Y)).$$

Proof.

$$\begin{aligned}
&\hat{g}\left(\hat{\nabla}_X\hat{\nabla}_Y f_*Z, f_*W\right)\\
=&\hat{g}(\hat{\nabla}_X(B(Y,Z)+f_*(\nabla_Y Z)), f_*W)\\
=&X(\hat{g}(B(Y,Z),f_*W)) - \hat{g}(B(Y,Z),\hat{\nabla}_X f_*W) + X(\hat{g}(f_*(\nabla_Y Z),f_*W)) - \hat{g}(f_*\nabla_Y Z, \hat{\nabla}_X f_*W)\\
=&-\hat{g}(B(Y,Z),B(X,W)) + Xg(\nabla_Y Z,W) - g(\nabla_Y Z,\nabla_X W)\\
=&g(\nabla_X\nabla_Y Z,W) - \hat{g}(B(Y,Z),B(X,W))
\end{aligned}$$

And

$$\begin{aligned}
\hat{g}(\hat{\nabla}_{[X,Y]}f_*Z,f_*W) =&\hat{g}(f_*(\nabla_{[X,Y]}Z),f_*W)\\
=&g(\nabla_{[X,Y]}Z,W)
\end{aligned}$$

So

$$\begin{aligned}
&R(X,Y,Z,W) - \hat{R}(X,Y,f_*Z,f_*W)\\
=&R(X,Y,Z,W) - \hat{g}\left(\hat{\nabla}_X\hat{\nabla}_Y f_*Z,f_*W\right) + \hat{g}\left(\hat{\nabla}_Y\hat{\nabla}_X f_*Z,f_*W\right) + \hat{g}(\hat{\nabla}_{[X,Y]}f_*Z,f_*W)\\
=&\hat{g}(B(Y,Z),B(X,W)) - \hat{g}(B(X,Z),B(Y,W))
\end{aligned}$$

□

Coro 2.2 (Gauss' Theorema Egregium)**.** *sectional curvature equals to Gauss curvature.*

Chapter 3

Completeness and the Hopf-Rinow theorem

3.1 Geodesics and exponential maps

Def 3.1. Consider a C^∞ curve $\gamma : [a,b] \to (M,g)$, define

$$\gamma_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \xlongequal{\Delta} \gamma'(t) \in \Gamma([a,b], \gamma^*TM), \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t) \xlongequal{\Delta} \gamma''(t).$$

Remark 3.1.

$$\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\left(\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\partial}{\partial x^i}\right) = \left(\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + \Gamma^k_{ij}\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\right)\frac{\partial}{\partial x^k}, i.e. B = \hat{\nabla}\gamma'.$$

Def 3.2. $\gamma : [a,b] \to (M,g)$ is called a geodesic if

$$\hat{\nabla}\gamma' = 0, i.e. \frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + \Gamma^k_{ij}(\gamma)\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t} = 0.$$

Prop 3.1. *If γ is a geodesic, then $|\gamma'|$ is a constant.*

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\gamma', \gamma'\rangle_{\hat{g}} = 2\left\langle\hat{\nabla}_{\frac{d}{dt}}\gamma', \gamma'\right\rangle_{\hat{g}} = 0$$

□

Thm 3.1. *Let (M,g) be a Riemannian manifold. For any $p \in M, v \in T_pM$ and $t_0 \in \mathbb{R}$, there exist an open interval $I \subset \mathbb{R}$ with $t \in I$, and a geodesic $\gamma : I \to M$, such that*

$$\gamma(t_0) = p, \gamma'(t_0) = v.$$

Proof. Let (U, φ, x^i) be a local chart around $p \in M$ and $T^k = \frac{\mathrm{d}\gamma^k}{\mathrm{d}t}$.

So we have the ODE

$$\frac{\mathrm{d}T^k}{\mathrm{d}t} + \Gamma^k_{ij}T^iT^j = 0, T^k(t_0) = v^k.$$

Hence using the theory of ODE, we can prove the existence an uniquene of solutiaon. □

Def 3.3. A geodesic $\gamma : I \to (M,g)$ is said to be maximal, if it can not be extended to a geodesic on a larger interval.

Remark 3.2. On a Riemann manifold (M, g), given $p \in M, v \in T_pM$.

The maximal interval is

$$I_{p,v} = \bigcup \left\{ \begin{array}{c} I \subset \mathbb{R} | 0 \in I \text{ and there exist a geodesic} \\ \gamma : I \to (M, g), \gamma(0) = p, \gamma'(0) = v \end{array} \right\}.$$

We denote $\gamma_v(t) : I_{p,v} \to (M, g)$ is the maximal geodesic.

Prop 3.2. $\gamma_{cv}(t) = \gamma_v(ct)$.

Proof. Let $F(t) = \gamma_v(ct)$.

$$F'' = \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} F_* \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) = \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} c\gamma_v'(ct) = c^2 \gamma_v(ct) = 0, \, F_* \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) \bigg|_{t=0} = cv.$$

Hence $F(t) = \gamma_{cv}(t)$. □

Def 3.4. $\forall p \in M$, define the set

$$\mathscr{E}_p = \{ v \in T_pM | 1 \in I_{p,v} \}.$$

Def 3.5 (Exponential map)**.**

$$\exp_p : \mathscr{E}_p \to M, v \mapsto \gamma_v(1).$$

And for

$$\mathscr{E} = \bigsqcup_{p \in M} \mathscr{E}_p \subset \bigsqcup_{p \in M} T_pM = TM,$$

we can define $\exp : \mathscr{E} \to M$.

Thm 3.2. *(1) $\mathscr{E} \subset TM$ is open and* $\exp : \mathscr{E} \to M$ *is C^∞*

(2) If $p \in M$ and $v \in \mathscr{E}_p \subset T_pM$, then

$$I_{p,v} = \{ t \in \mathbb{R} | tv \in \mathscr{E}_p \}$$

(3) Each set $\mathscr{E}_p \subset T_pM$ is star-shaped w.r.t. $0 \in T_pM$

(4) For each $p \in M$,

$$\mathrm{d}\left(\exp_p\right)_0 : T_0(T_pM) \cong T_pM \to T_pM$$

is identity.

Proof. (1) Consider a local vector field on TM around (p, v):

$$G = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}$$

The integral curve of G passing through (p, v) satisfying that

$$\frac{\mathrm{d}\phi^i}{\mathrm{d}t} = \phi^{n+i}, \frac{\mathrm{d}\phi^{n+i}}{\mathrm{d}t} = -\Gamma^i_{jk} \phi^{n+j} \phi^{n+k},$$

where $i \in [1, n] \cap \mathbb{Z}$.

So let $\gamma = \pi \circ \phi : \mathbb{R} \to M$ be a curve passing through p, where π is the projection $TM \to M$.

Then

$$\frac{\mathrm{d}^2\gamma^i}{\mathrm{d}t^2} + \Gamma^k_{ij}(\gamma) \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} = 0,$$

$i.e.\gamma(t) = \exp_p(tv)$ is the geodesic.

By Theorem 9.12 of J.Lee's *Introduction to smooth manifolds*, there is a neighborhood $V_{p,v} \subset TM$ around (p, v) such that $1 \in I_{q,w}$ for every $(q, w) \in TM$.

Hence $\mathscr{E} = \bigcup_{v \in \mathscr{E}_p} V_{p,v}$ is open.

And $\exp(p, v) = \exp_p(v) = \gamma_v(1) = \pi(\phi_{(p,v)}(1))$ is smooth.

(2) Let t satisfying that $tv \in \mathscr{E}_p$.

Then $1 \in I_{p,tv}$, $i.e.$ $t \in I_{p,v}$.

So we have

$$I_{p,v} = \{t \in \mathbb{R} | tv \in \mathscr{E}_p\}.$$

(3) By (2), $\forall t \in I_{p,v}, tv \in \mathscr{E}_p$ and $[0, 1] \subset I_{p,v}$.

So $\mathscr{E}_p$ is star-shaped w.r.t. 0.

(4) Given $v \in T_pM$, we also regard $v \in T_0(T_pM)$.

We choose a curve τ in T_pM such that

$$\tau(t) = tv \subset T_pM, i.e.\tau'(0) = v \in T_0(T_pM) \cong T_pM$$

And let $\gamma = \exp_p \circ \tau$

Then $\mathrm{d}\big(\exp_p\big)_0 (v) = \gamma'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (\exp_p \circ \tau(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp_p(tv) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \gamma_v(t) = v$

□

Def 3.6. Let $p \in M$, for $\gamma > 0$, denote

$$B_r(0) = \{v \in T_pM | |v| < r\}.$$

If $r << 1$, then $B_r(0) \subset \mathscr{E}_p$, define

$$B_r(p) = \exp_p(B_r(0))$$

is an open subset of M, and $\exp_p : B_r(0) \to B_r(p)$ is a diffeomorphism.
The supremum of such r is called the injective radius at $p \in M$, denoted by $\mathrm{inj}_p(M, g)$.

Def 3.7. Let $\{e_i\}$ be **any** orthonormal basis of (T_pM, g_p), there exists an isomorphism

$$B : \mathbb{R}^n \to T_pM, (r^1, \cdots, r^n) \mapsto \sum_{i=1}^{n} r^i e_i.$$

By using diffeomorphsim $\exp_p : V \to U$ on small neighborhoods, we obtain a C^∞ coordinate map $\varphi : U \to \mathbb{R}^n$ given by

$$U \xrightarrow{\exp_p^{-1}} T_pM \xrightarrow{B^{-1}} \mathbb{R}^n.$$

Remark 3.3. (1) $B_*^{-1} : T(T_pM) \to T\mathbb{R}^n, B_*^{-1}(0, e_i) = \frac{\partial}{\partial r^i}\big|_0$

(2) $\varphi_* \left(\frac{\partial}{\partial x^i}\big|_p \right) = \frac{\partial}{\partial r^i}\big|_0$

(3) $\mathrm{d}\big(\exp_p\big)_0 : T_0(T_pM) \cong T_pM \to T_pM, v \mapsto v$
So $\frac{\partial}{\partial x^i}\big|_p = e_i$

Coro 3.1. *(U, φ, x) given by the previous setting*

(1) $\varphi(p) = (0, \cdots, 0) \in \mathbb{R}^n$

(2) $g_{ij}(p) = g\left(\frac{\partial}{\partial x^i}\big|_p, \frac{\partial}{\partial x^j}\big|_p\right) = \delta_{ij}$

(3) For every $v = v^i \frac{\partial}{\partial x^i}\big|_p \in T_pM$*, we have*

$$r_v^i(t) = x^i \circ \gamma_v(t) = tv^i, t \in I_{p,v}.$$

(4) $\frac{\partial g_{ij}}{\partial x^k}\Big|_p = 0$*, in particular,* $\Gamma_{ij}^k(p) = 0$.

Proof. (1) $\varphi(p) = B^{-1}(\exp_p^{-1}(p)) = B^{-1}(0) = 0$

(2) Since $e_i = \frac{\partial}{\partial x^i}\big|_p$ is orthonormal basis of (T_pM, g_p).

So $g_{ij}(p) = \delta_{ij}$

(3)

$$\begin{aligned}\gamma_v^i(t) =& x^i \circ \gamma_v(t)\\ =&\gamma^i \circ \varphi \circ \exp_p(tv)\\ =&\gamma^i \circ B^{-1} \circ \exp_p^{-1} \circ \exp_p(tv)\\ =&\gamma^i \circ B(tv^i e_i)\\ =&\gamma^i \circ (tv^1, \cdots, tv^n)\\ =&tv^i\end{aligned}.$$

(4) For any $v \in T_pM$, consider the geodesic equivalent for $\gamma_v(t)$.

$$\frac{\mathrm{d}^2\gamma_v^k}{\mathrm{d}t^2} + \Gamma_{ij}^k(\gamma_v(t))\frac{\mathrm{d}\gamma_v^i}{\mathrm{d}t}\frac{\mathrm{d}r_v^j}{\mathrm{d}t} = 0.$$

So $\Gamma_{ij}^k(r_v(0))v^iv^j = 0, i.e. \Gamma_{ij}^k(p)v^iv^j = 0$.

Hence $\Gamma_{ij}^k(p) = 0$, and

$$\frac{\partial g_{ij}}{\partial x^k}\Big|_p = \frac{\partial}{\partial x^k}\left\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle = \left\langle\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle + \left\langle\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\right\rangle = 0$$

□

Thm 3.3. *Let* (M, g) *be a Riemannian manifold and* (U, φ, x) *be any normal coordinate chart centered at* $p \in M$*.*

Then we can obtain that

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\mathrm{R}_{iklj}(p)x^kx^l + O(|x|^3)$$

$$\det(g_{ij}) = 1 - \frac{1}{3}\mathrm{R}_{kl}(p)x^kx^l + O(|x|^3)$$

Proof. For any $v \in T_pM$, since $\gamma_v(t) = \exp_p(vt)$ is geodesic.

So $\Gamma_{ij}^k(\exp_p(vt))v^iv^j = 0$.

Take a derivative of this equation at p, we obtain

$$\frac{\partial\Gamma_{ij}^k}{\partial x^l}(p)\, v^iv^jv^l \equiv 0.$$

Therefore

$$\frac{\partial\Gamma_{ij}^k}{\partial x^l}+\frac{\partial\Gamma_{il}^k}{\partial x^j}+\frac{\partial\Gamma_{jl}^k}{\partial x^i}=2\frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}+2\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l}+2\frac{\partial^2 g_{kl}}{\partial x^i\partial x^j}-\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}-\frac{\partial^2 g_{il}}{\partial x^j\partial x^k}-\frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}=0.$$

Sum the similar cyclic equations:

$$\frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}+\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l}+\frac{\partial^2 g_{kl}}{\partial x^i\partial x^j}+\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}+\frac{\partial^2 g_{il}}{\partial x^j\partial x^k}+\frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}=0$$

Moreover, by some simple calculation, we have

$$\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}=\frac{\partial^2 g_{kl}}{\partial x^i\partial x^k}.$$

And notice that $\Gamma_{ij}^k(p)=0$ and remark 1.5,

$$\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}=\frac{1}{3}(\mathrm{R}_{ilkj}+\mathrm{R}_{iklj})$$

Hence the proof is complete by Taylor's expansion. □

Coro 3.2. *When r is small enough, show that*

$$\mathrm{Vol}(B(p,r))=\omega_n r^n\left(1-\frac{s(p)}{6(n+2)}r^2+O(r^3)\right)$$

and

$$\mathrm{Area}(S(p,r))=n\omega_n r^{n-1}\left(1-\frac{s(p)}{6}r^2+O(r^3)\right).$$

Proof. Consider a normal coordinate chart (U,φ,x^i) that is centered at p.

$$\begin{aligned}
\mathrm{Vol}(B(p,r))=&\int\sqrt{\det g}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x^n\\
=&\int\sqrt{1-\frac{1}{3}\mathrm{R}_{ij}(p)x^ix^j+O(|x|^3)}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x^n\\
=&\int\left(1-\frac{1}{6}\mathrm{R}_{ij}(p)x^ix^j+O(|x|^3)\right)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x^n\\
=&\mathrm{Vol}(B_0(r))-\frac{\mathrm{R}_{ij}(p)}{6}\int x^ix^j\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x^n+O(r^{n+3})\\
=&\omega_n r^n-\frac{\delta^{ij}\mathrm{R}_{ij}(p)}{6n}\int|x|^2\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x^n+O(r^{n+3})\\
=&\omega_n r^n-\frac{s(p)}{6n}\left(r^2\mathrm{Vol}(B_0(r))-\int_0^r 2t\mathrm{Vol}(B_0(t))\mathrm{d}t\right)+O(r^{n+3})\\
=&\omega_n r^n-\frac{s(p)}{6n}\left(\omega_n r^{n+2}-2\omega_n\int_0^r t^{n+1}\mathrm{d}t\right)+O(r^3)\\
=&\omega_n r^n\left(1-\frac{s(p)}{6(n+2)}r^2+O(r^3)\right)
\end{aligned}$$

$$\mathrm{Area}(S(p,r))=\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{Vol}(B(p,r))=n\omega_n r^{n-1}\left(1-\frac{s(p)}{6n}r^2+O(r^3)\right)$$

□

3.2 Completeness and the Hopf-Rinow theorem

Def 3.8. Let (M, g) be a connected Riemannian manifold, we define the distance function

$$d_g : M \times M \to \mathbb{R}, (p, q) \mapsto \inf_{\gamma \in \mathscr{L}} \int |\gamma'| \mathrm{d}t,$$

where $\mathscr{L}$ is the set of piecewise smooth curves connecting p and q.

Remark 3.4. Actually, piecewise is not necessary, since we are considering the inf.

Lemma 3.1 (Gauss). *Let (M, g) is a Riemannian manifold, and fix $p \in M, r < \mathrm{inj}_p(M, g)$. Let $I \subset \mathbb{R}$ be an open interval and suppose:*

(1) $w : I \to T_pM, |w(s)| \equiv r$

(2) $\alpha(t, s) = \exp_p(tw(s))$ *for* $(t, s) \in \mathbb{R} \times I$ *and* $tw(s) \in \mathscr{E}_p$.

Then

$$\left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial s} \right) \right\rangle \equiv 0.$$

Proof. Let $\hat{\nabla}$ be the induced connection on $\alpha^* TM$.

Fixed any $s \in I$, then $\alpha(t, s)$ is a geodesic.

And we deduce

$$\hat{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial t} \right) = 0.$$

So

$$\begin{aligned} \frac{\partial}{\partial t} \left| \alpha_* \left(\frac{\partial}{\partial t} \right) \right|_g^2 =& \frac{\partial}{\partial t} \left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle \\ =& 2 \left\langle \hat{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle \\ =& 0 \end{aligned}$$

Therefore

$$\begin{aligned} \left| \alpha_* \left(\frac{\partial}{\partial t} \right) \right|^2 (t, s) =& \left| \alpha_* \left(\frac{\partial}{\partial t} \right) \right|^2 (0, s) \\ =& |w(s)|^2 = r^2 \end{aligned}$$

Moreover, we have

$$\begin{aligned} \frac{\partial}{\partial t} \left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial s} \right) \right\rangle =& \left\langle \hat{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial s} \right) \right\rangle + \left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \hat{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s} \right) \right\rangle \\ =& \left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \hat{\nabla}_{\frac{\partial}{\partial s}} \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle \\ =& \frac{1}{2} \frac{\partial}{\partial s} \left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle \\ =& 0 \end{aligned}$$

Hence

$$\left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial s} \right) \right\rangle (t, s) = \left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial s} \right) \right\rangle (0, s) = 0$$

□

Remark 3.5. The geometric intuition behind this lemma is that the meridian and the parallel are perpendicular.

Thm 3.4. *Let M, g be a Riemannian manifold, and fix $p \in M, r < \mathrm{inj}_p(M,g)$.*
Then for every $v \in B_r(0) \subset T_pM$, we have

$$\mathrm{d}_g(p, \exp_p(v)) = |v|.$$

Moreover, for any $q \in B_r(p) \subset M$, a C^∞-curve $\gamma : [0,1] \to M$ has minimal length $L(r) = \mathrm{d}_g(p,q)$ iff there is a C^∞ function $f : [0,1] \to [0,1]$ satisfying $f(0) = 0, f(1) = 1, f' \geqslant 0$ and $\gamma(t) = \exp_p(f(t)v)$.

Proof. Let $q = \exp_p(v), \varepsilon = |v|$.
Suppose $\gamma : [0,1] \to \overline{B_\varepsilon(p)} \subset M$ be a curve connecting p and q.
We claim that $\underline{L(\gamma) \geqslant |v|}$.
Indeed, since $\exp_p : \overline{B_\varepsilon(0)} \to \overline{B_\varepsilon(p)}$ is a diffeomorphism.
So there exists an unique function $v(t) : [0,1] \to \overline{B_\varepsilon(0)}, s.t. \gamma(t) = \exp_p(v(t))$.
Consider

$$I = \{t \in [0,1] | \gamma(t) \neq p\} = \{t \in [0,1] | v(t) \neq 0\} \subset (0,1]$$

and let

$$\beta : [0,1] \to [0,1], t \mapsto \frac{|v(t)|}{\varepsilon}, w : I \to T_pM, t \mapsto \frac{v(t)}{\beta(t)}$$

Then for $t \in I$,

$$|w(t)|_g = \varepsilon, \gamma(t) = \exp_p(\beta(t)w(t)).$$

Consider $\alpha(s,t) = \exp_p(sw(t)) : [0,1] \times I \to M$.
Then $\gamma(t) = \alpha(\beta(t), t)$ and

$$\gamma'(t) = \beta'(t)\ \alpha_* \left(\frac{\partial}{\partial s}\right)\bigg|_{(\beta(t),t)} + \alpha_* \left(\frac{\partial}{\partial t}\right)\bigg|_{(\beta(t),t)}.$$

$$\left|\alpha_* \left(\frac{\partial}{\partial s}\right)\right| = |w(t)| = \varepsilon.$$

By Gauss Lemma,

$$|\gamma'(t)|^2 = |\beta'(t)|^2 \left|\alpha_* \left(\frac{\partial}{\partial s}\right)\right|^2 + \left|\alpha_* \left(\frac{\partial}{\partial t}\right)\right|^2 \geqslant |\beta'(t)|^2 \varepsilon^2.$$

Hence

$$L(\gamma) = \int_0^1 |\gamma'(t)| \mathrm{d}t \geqslant \int_I |\gamma'(t)| \mathrm{d}t \geqslant |v| \int_I |p'(t)| \mathrm{d}t \geqslant |v| \int_I \beta'(t) \mathrm{d}t = |v|$$

And $L(\gamma) = |v|$ iff $\beta'(t) \geqslant 0, \beta(t) = v(t), \beta'(t) = 0$. □

Lemma 3.2. *(M, d_g) is a metric space,* i.e.

(1) $d_g(p,q) = d_g(q,p)$

(2) $d_g(p,q) = 0 \Leftrightarrow p = q$

(3) $d_g(p_1,p_2) \leqslant d_g(p_1,p_3) + d_g(p_3,p_2)$.

Proof. (1) $\forall \gamma \in \mathscr{L}_{(p,q)}, \gamma^{-1} \in \mathscr{L}_{(q,p)}$.
So $d_g(p,q) = d_g(q,p)$.

(2) If $p = q$, then $d_g(p, q) = 0$.

If $p \neq q$, then $\exists r, s.t. q \notin B_r(p)$.

By theorem 3.4, for every $\gamma \in \mathscr{L}_{(p,q)}$,

$$\int |\gamma'| \mathrm{d}t \geqslant d_g(p, \exp_p(v)) = \frac{r}{2},$$

where $|v|_g = \frac{r}{2}$ and $\exp_p(v) \in \operatorname{Im} \gamma$.

So $d(p, q) \geqslant \frac{r}{2}$.

Hence $d(p, q) = 0 \Leftrightarrow p = q$.

□

Remark 3.6. theorem 3.4 is also true for piecewise smooth curve, so the identity of triangle inequality holds iff p_2 is on the minimal geodesic from p_1 to p_3.

In other word, the minimal curve has no corner.

Remark 3.7. The topology determined by local charts is the same as the topology determined by the metric.

Def 3.9. A Riemannian manifold (M, g) is called geodesically complete, if for all $p \in M$, $\exp_p$ is defined for all $v \in T_pM$, *i.e.* every geodesic $\gamma(t)$ starting from p is defined for $t \in \mathbb{R}$.

Lemma 3.3. *Let (M, g) be a Riemannian manifold, if $\exp_p : T_pM \to M$ is well-defined, then $\forall q \in M, \exists a\ C^\infty$ minimal geodesic connecting p and q, such that*

$$\gamma_v(1) = \exp_p(v) = q, \mathrm{d}_g(p, q) = |v| = L(\gamma).$$

Proof. For any $q \in M$, let $r = d_g(p, q)$ and $\varepsilon < r$.

Then if $q \in \bar{B}_\varepsilon(p)$, then this is trivial.

If $q \notin \bar{B}_\varepsilon(p)$, then consider $p_0 \in \partial B_\varepsilon(p)$ such that $d(p_0, q) = d(\partial B_\varepsilon(p), q)$ and a unit-speed geodesic γ passing through p and p_0.

We claim that $\gamma(r) = q$.

Indeed, let

$$A = \{t \in [\varepsilon, r] | d_g(\gamma(t), q) = r - t\},$$

which is closed.

And for $t \in A$, let $p_1 = \gamma(t), p_2 \in \partial B_\varepsilon(p_1)$ such that $d(p_2, q) = d(\partial B_\varepsilon(p_1), q)$.

Then

$$d(p_1, p_2) + d(p_2, q) = d(p_1, q) = r - d(p, p_1),$$

$$d(p, p_2) \geqslant r - d(p_2, q) = d(p, p_1) + d(p_1, p_2).$$

So $\exists t_0, s.t. p_2 = \gamma(t_0), i.e. t_0 \in A$.

Hence A is also open, *i.e.* $A = [\varepsilon, r]$ and $\gamma(r) = q$. □

Thm 3.5 (Hopf-Rinow)**.** *TFAE:*

(1) (M, g) is geodesically complete

(2) $\exists p \in M, \text{s.t.} \exp_p : T_pM \to M$ is well-defined

(3) The closed and bounded sets of M are compact

(4) (M, d_g) is metrically complete

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) are trivial.

(2) $\Rightarrow$ (3): Let K be a closed and bounded set and $r = \sup\limits_{q\in K} d_g(p,q)$.

Then by lemma 3.3, $K \subset \bar{B}_r(p)$, *i.e.* $\exp_p^{-1}(K)$ is closed and bounded in $\mathbb{R}^n$.

So $\exp_p^{-1}(K)$ is compact, *i.e.* K is compact since $\exp_p$ is smooth.

(4) $\Rightarrow$ (1): Let $\gamma : [0,\varepsilon] \to M$ be a C^∞ unit-speed geodesic starting at p, $\gamma(0) = p$.

By ODE theory, the maximal defining interval of γ is an open interval (a,b).

We claim that $a = -\infty, b = +\infty$.

Indeed, if $b < +\infty$, then there exists Cauchy sequence $\{b_i\} \to b$.

So $\forall \varepsilon > 0, \exists N_\varepsilon, s.t. \forall k > l > N_\varepsilon, |b_k - b_l| < \varepsilon$.

On the other hand, $d_g(\gamma(b_k), \gamma(b_l)) \leqslant L\left(\gamma\big|_{[b_l,b_k]}\right) = |b_k - b_l| < \varepsilon$.

Therefore $\{\gamma(b_i)\}$ is Cauchy sequence in (M, d_g), *i.e.* $\gamma(b_k)$ converges to a point $q \in M$.

By the basic proposition of the exponential map, there exists small $\delta \in (0,\varepsilon)$, such that any two points in $\bar{B}_\delta(q)$ can be connected by a unique smooth geodesic.

Choose $\tilde{N}_\varepsilon$ such that if $k \geqslant \tilde{N}_\varepsilon$, we have

$$|b_k - b| < \frac{\delta}{2}, \mathrm{d}_g(\gamma(b_k), q) < \frac{\delta}{2}.$$

So in $B_\delta(r(b_k))$, there is a geodesic $\tilde{\gamma} : \left[b_k, b_k + \frac{\delta}{2}\right] \to M$ such that

$$\tilde{\gamma}(b_k) = \gamma(b_k), \tilde{\gamma}'(b_k) = \gamma'(b_k).$$

Gluing $\gamma\big|_{[a,b_k]}$ and $\tilde{\gamma}\big|_{\left[b_k, b_k+\frac{\delta}{2}\right]}$ together, we extend γ to $b_k + \frac{\delta}{2} > b$, contradiction!

Similarly, we have $a = -\infty$. □

Coro 3.3. *If (M,g) is complete, $\forall p, q \in M$, there exists a minimal geodesic γ, such that*

$$\gamma(t) = \exp_p(tv), \mathrm{d}_g(p,q) = |v|, q = \exp_p(v).$$

Proof. Directly by Hopf-Rinow theorem and lemma 3.3. □

Coro 3.4. *If (M,g) is compact, then (M,g) is complete.*

Proof. When (M,g) is compact, every closed subset of M is compact.

So by Hopf-Rinow theorem, (M,g) is complete. □

Chapter 4

The Hodge decomposition

Lemma 4.1. *$\varphi \in \Omega^k(M) = \Gamma(M, \bigwedge^k T^*M)$ and write φ as*

$$\varphi = \sum_{i_1,\cdots,i_k} f_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

We define

$$\varphi_{i_1\cdots i_k} = \sum_{\sigma\in P_k} (-1)^{|\sigma|} f_{i_{\sigma(1)\cdots\sigma(k)}}.$$

Then $\varphi_{i_1\cdots i_k}$ is skew-symmetric, and

$$\varphi = \sum_{i_1<\cdots<i_k} \varphi_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \frac{1}{k!} \sum_{i_1,\cdots,i_k} \varphi_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots x^{i_k}.$$

Proof.

$$\mathrm{d}x^{i_{\sigma(1)}} \wedge \cdots \wedge \mathrm{d}x^{i_{\sigma(k)}} = (-1)^{|\sigma|} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

So we can obtain that

$$\begin{aligned}
\varphi &= \sum_{i_1,\cdots,i_k} f_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\
&= \sum_{i_1<\cdots<i_k} \sum_{\sigma\in S_k} f_{i_{\sigma(1)}\cdots i_{\sigma(k)}} \mathrm{d}x^{i_{\sigma(1)}} \wedge \cdots \wedge \mathrm{d}x^{i_{\sigma(k)}} \\
&= \sum_{i_1<\cdots<i_k} \varphi_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\
&= \sum_{i_1<\cdots<i_k} \sum_{\sigma\in S_k} (-1)^{\sigma} \frac{\varphi_{i_1\cdots i_k}}{k!} \mathrm{d}x^{i_{\sigma(1)}} \wedge \cdots \wedge \mathrm{d}x^{i_{\sigma(k)}} \\
&= \frac{1}{k!} \sum_{i_1,\cdots,i_k} \varphi_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}
\end{aligned}$$

□

Remark 4.1. We simply denote

$$\sum_I f_I \mathrm{d}x^I \xlongequal{\Delta} \sum_{i_1,\cdots,i_k} f_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Def 4.1. There is a local inner product on $\Omega^k(M)$ such that

$$\langle \varphi, \psi \rangle = \frac{1}{k!} g(\varphi, \psi).$$

Lemma 4.2. *Let $\varphi, \psi \in \Omega^k(M)$ and $\varphi = \sum\limits_I \varphi_I \mathrm{d}x^I, \psi = \sum\limits_J \psi_J \mathrm{d}x^J$.*

Then

$$\langle \varphi, \psi \rangle = g^{IJ} \varphi_I \psi_J,$$

where

$$g^{IJ} = \frac{1}{k!} g(\mathrm{d}x^I, \mathrm{d}x^J) = \begin{vmatrix} g^{i_1 j_1} & \dots & g^{i_1 j_k} \\ \vdots & \ddots & \vdots \\ g^{i_k j_1} & \dots & g^{i_k j_k} \end{vmatrix}.$$

In particular, if $\varphi = \frac{1}{k!} \sum \varphi_I \mathrm{d}x^I, \psi = \frac{1}{k!} \sum \psi_J \mathrm{d}x^J$ and φ_I, ψ_J are skew-symmetric. Then

$$\langle \varphi, \psi \rangle = \frac{1}{k!} \sum g^{i_1 j_1} \cdots g^{i_k j_k} \varphi_{i_1 \cdots i_k} \psi_{j_1 \cdots j_k}$$

Proof.

$$\langle \varphi, \psi \rangle = \frac{1}{k!} g \left(\sum_I \varphi_I \mathrm{d}x^I, \sum_J \psi_J \mathrm{d}x^J \right) = \frac{1}{k!} g(\mathrm{d}x^I, \mathrm{d}x^J) \varphi_I \psi_J = g^{IJ} \varphi_I \psi_J.$$

$$\begin{aligned} \frac{1}{k!} g(\mathrm{d}x^I, \mathrm{d}x^J) =& \frac{1}{k!} g \left(\sum_{\sigma \in S_k} (-1)^{|\sigma|} \mathrm{d}x^{i_{\sigma(1)}} \otimes \cdots \otimes \mathrm{d}x^{i_{\sigma(k)}}, \sum_{\tau \in S_k} (-1)^{|\tau|} \mathrm{d}x^{j_{\tau(1)}} \otimes \cdots \otimes \mathrm{d}x^{j_{\tau(k)}} \right) \\ =& \frac{1}{k!} \sum_{\sigma, \tau \in S_k} (-1)^{|\sigma| + |\tau|} g^{i_{\sigma(1)} j_{\tau(1)}} \cdots g^{i_{\sigma(k)} j_{\tau(k)}} \\ =& \frac{1}{k!} \sum_{\sigma, \tau \in S_k} (-1)^{|\tau \circ \sigma^{-1}|} g^{i_1 j_{\tau \circ \sigma^{-1}(1)}} \cdots g^{i_l j_{\tau \circ \sigma^{-1}(l)}} \\ =& \sum_{\sigma \in S_k} (-1)^{|\sigma|} g^{i_1 j_{\sigma(1)}} \cdots g^{i_k j_{\sigma(k)}} = \det\Big(\big(g^{i_p j_q} \big)_{1 \leqslant p, q \leqslant k} \Big) \end{aligned}$$

In particular, when $\varphi = \frac{1}{k!} \sum \varphi_I \mathrm{d}x^I, \psi = \frac{1}{k!} \sum \psi_J \mathrm{d}x^J$ and φ_I, ψ_J are skew-symmetric,

$$\begin{aligned} \langle \varphi, \psi \rangle =& \frac{1}{(k!)^2} \sum \left(\sum_{\sigma \in S_k} (-1)^{|\sigma|} g^{i_1 j_{\sigma(1)}} \cdots g^{i_k j_{\sigma(k)}} \right) \varphi_{i_1 \cdots i_k} \psi_{j_1 \cdots j_k} \\ =& \frac{1}{(k!)^2} \sum \sum_{\sigma \in S_k} g^{i_1 j_{\sigma(1)}} \cdots g^{i_k j_{\sigma(k)}} \varphi_{i_1 \cdots i_k} \psi_{j_{\sigma(1)} \cdots j_{\sigma(k)}} \\ =& \frac{1}{k!} \sum g^{i_1 j_1} \cdots g^{i_k j_k} \varphi_{i_1 \cdots i_k} \psi_{j_1 \cdots j_k} \end{aligned}$$

□

Def 4.2. $\mathrm{dvol}_g = \sqrt{\det(g_{ij})} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is a volume form.

Prop 4.1. $\langle \mathrm{dvol}_g, \mathrm{dvol}_g \rangle = 1$.

Proof.

$$\langle \mathrm{dvol}_g, \mathrm{dvol}_g \rangle = \det\big(g^{ij}\big) \det(g_{ij}) = 1.$$

□

Def 4.3. Let (M, g) be an oriented Riemannian manifold.

The global inner product on $\Omega^k(M)$ is

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle \mathrm{dvol}_g = \frac{1}{k!} \int_M g(\varphi, \psi) \mathrm{dvol}_g.$$

Def 4.4. Let (V, g) be a vector space with an inner product and $\{e_1, \cdots, e_r\}$ be an orthonormal basis of V.

The Hodge-$*$ star operator is a linear map

$$*: \wedge^k V \to \wedge^{r-k} V, e_I \mapsto \mathrm{sign}(I, I^c) e_{I^c},$$

where $I = (i_1, \cdots, i_k)$ with $1 \leqslant i_1 < \cdots < i_k \leqslant r$.

Def 4.5. Let (M, g) be an oriented Riemannian manifold and $\dim_{\mathbb{R}} M = n$.

Let $\{\xi_1, \cdots, \xi_n\}$ be an orthonormal frame of TM over an open patch U.

The Hodge-* operator on $\Omega^{\cdot}(M)$ is defined as

$$*: \Omega^k(M) \to \Omega^{n-k}(M), v \mapsto \sum_{|I|=k} \mathrm{sign}(I, I^c) v_I \xi^{I^c},$$

where $v = \sum\limits_{|I|=k} v_I \xi^I$.

Prop 4.2. *(1)* $*1 = \mathrm{dvol}_g, *(\mathrm{dvol})_g = 1$.

(2) $**v = (-1)^{k(n-k)} v$ *for* $v \in \Omega^k(M)$

(3) If $u \in \Omega^k(M), v \in \Omega^{n-k}(M)$, *then* $*(u \wedge v) = (-1)^{k(n-k)} \langle u, *v \rangle$

(4) $u, v \in \Omega^k(M)$, *then* $u \wedge *v = v \wedge *u = \langle u, v \rangle \mathrm{dvol}_g$ *and* $\langle *u, *v \rangle = \langle u, v \rangle$.

(5) If $u \in \Omega^k(M), v \in \Omega^{n-k}(M)$, *then* $\langle u, *v \rangle = (-1)^{k(n-k)} \langle *u, v \rangle$.

Proof. (1) $*1 = \xi^1 \wedge \cdots \wedge \xi^n$.

And since $g^{ij} = g(\xi^i, \xi^j) = \delta^{ij}$.

So $*1 = \mathrm{dvol}_g$ and $*(\mathrm{dvol}_g) = *(\xi^1 \wedge \cdots \wedge \xi^n) = 1$.

(2)

$$\begin{aligned} **v &= * \left(\sum_{|I|=k} \mathrm{sign}(I, I^c) v_I \xi^{I^c} \right) \\ &= \sum_{|I|=k} \mathrm{sign}(I, I^c) v_I \cdot \mathrm{sign}(I^c, I) \xi^I \\ &= \sum_{|I|=k} (-1)^{k(n-k)} v_I \xi^I = (-1)^{k(n-k)} v \end{aligned}$$

(3)

$$\begin{aligned} *(u \wedge v) &= * \left(\sum_{|I|=k} \sum_{|J|=n-k} u_I v_J \xi^I \wedge \xi^J \right) \\ &= * \left(\left(\sum_{|I|=k} u_I v_{I^c} \mathrm{sign}(I, I^c) \right) \xi^1 \wedge \cdots \wedge \xi^n \right) \\ &= \sum_{|I|=k} \mathrm{sign}(I, I^c) u_I v_{I^c} \\ &= \left\langle \sum_{|I|=k} u_I \xi^I, \sum_{|I|=k} \mathrm{sign}(I, I^c) v_{I^c} \xi^I \right\rangle \\ &= (-1)^{k(n-k)} \langle u, *v \rangle \end{aligned}$$

(4) $u \wedge *v = (-1)^{k(n-k)}\langle u, **v\rangle \mathrm{dvol}_g = \langle u, v\rangle \mathrm{dvol}_g$
So $v \wedge *u = \langle v, u\rangle \mathrm{dvol}_g = \langle u, v\rangle \mathrm{dvol}_g$.
And $\langle *u, *v\rangle = (-1)^{k(n-k)} * (*u \wedge v) = *(v \wedge *u) = (-1)^{k(n-k)}\langle v, **u\rangle = \langle u, v\rangle$.

(5) $\langle u, *v\rangle = (-1)^{k(n-k)}\langle **u, *v\rangle = (-1)^{k(n-k)}\langle *u, v\rangle$.

□

Coro 4.1. *Let (M, g) be a compact oriented Riemannian manifold, then*

$$(u, v) = \int_M u \wedge *v.$$

Proof.

$$(u, v) = \int_M \langle u, v\rangle \mathrm{dvol}_g = \int_M u \wedge *v.$$

□

Def 4.6. Let (M, g) be a compact oriented Riemannian manifold.
The formal adjoint operator of d is, denoted by d^*, defined as

$$(\mathrm{d}\varphi, \psi) = (\varphi, \mathrm{d}^*\psi)$$

where $\varphi \in \Omega^k(M), \psi \in \Omega^{k+1}(M)$.

Thm 4.1. $\mathrm{d}^* = (-1)^{nk+n+1} * \mathrm{d} *$.

Proof. If $u \in \Omega^{k-1}(M), v \in \Omega^k(M)$, then

$$\mathrm{d}(u \wedge *v) = \mathrm{d}u \wedge *v + (-1)^{k-1} u \wedge \mathrm{d} * v.$$

So by Stokes' theorem,

$$0 = \int_M \mathrm{d}(u \wedge *v) = \int_M \mathrm{d}u \wedge *v + (-1)^{k-1} \int_M u \wedge \mathrm{d} * v.$$

Therefore

$$\begin{aligned}(du, v) = &\int_M \langle \mathrm{d}u, v\rangle \mathrm{dvol}_g \\ = &\int_M \mathrm{d}u \wedge *v \\ =&(-1)^k \int_M u \wedge (\mathrm{d} * v) \\ =&(u, (-1)^{nk+k+1} * \mathrm{d} * v)\end{aligned}$$

□

Coro 4.2. *For $\omega = \omega_i \mathrm{d}x^i \in \Omega^1(M)$,*

$$\mathrm{d}^*\omega = -g^{ij}\left(\frac{\partial \omega_i}{\partial x^j} - \Gamma^k_{ji}\omega_k\right) = -g^{ij}\left(\nabla_j \omega\right)_i = -\left(\nabla^i \omega\right)_i.$$

Proof. WLOG, let $\left\{\frac{\partial}{\partial x^i}\right\}$ be an normal frame.
We then have

$$\begin{aligned}\mathrm{d}^*\omega = &- *\mathrm{d} * \omega = - * \left(\left(\sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i}\right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right) \\ = &- \sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i} = -\left(\nabla^i \omega\right)_i = -g^{ij}\left(\frac{\partial \omega_i}{\partial x^j} - \Gamma^k_{ij}\omega_k\right)\end{aligned}$$

□

Def 4.7. The operator $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} : \Omega^k(M) \to \Omega^k(M)$ is called the Laplacian-Beltrami operator.

Prop 4.3. $(\Delta\varphi, \psi) = (\varphi, \Delta\psi)$.

Proof.

$$\begin{aligned}(\Delta\varphi, \psi) =&(\mathrm{d}\mathrm{d}^*\varphi, \psi) + (\mathrm{d}^*\mathrm{d}\varphi, \psi) = (\mathrm{d}^*\varphi, \mathrm{d}^*\psi) + (\mathrm{d}\varphi, \mathrm{d}\psi)\\ =&(\varphi, \mathrm{d}\mathrm{d}^*\psi) + (\varphi, \mathrm{d}^*\mathrm{d}\psi) = (\varphi, \Delta\psi)\end{aligned}$$

□

Def 4.8. $u \in \Omega^k(M)$ is called harmonic if $\Delta u = 0$ and the space of harmonic k-form is denote $\mathscr{H}^k(M)$.

Remark 4.2. $f \in \Omega^0(M)$, then $\Delta f = -\Delta_g f = -\operatorname{tr}\operatorname{Hess} f$.

Coro 4.3. $\forall u \in \Omega^k(M), \Delta u = 0 \Leftrightarrow \mathrm{d}u = 0, \mathrm{d}^*u = 0.$

Proof.

$$(\Delta u, u) = (\mathrm{d}\mathrm{d}^*u + \mathrm{d}^*\mathrm{d}u, u) = (\mathrm{d}u, \mathrm{d}u) + (\mathrm{d}^*u, \mathrm{d}^*u) = |\mathrm{d}u|^2 + |\mathrm{d}^*u|^2$$

So $\Delta u = 0$ iff $\mathrm{d}u = 0, \mathrm{d}^*u = 0$. □

Thm 4.2 (Hodge decomposition)**.** $\Omega^k(M) = \mathscr{H}^k(M) \oplus \mathrm{d}\big(\Omega^{k+1}(M)\big) \oplus \mathrm{d}^*(\Omega^{k+1}(M))$

Proof. The proof of this theorem need some technique about functional analysis. If you are really interested about this, you can search for the proof on Google by yourself :) □

Def 4.9. If $\alpha \in \Omega^k(M)$ satisfies $\mathrm{d}\alpha = 0$, then α is called a d-closed form.
If $\alpha = \mathrm{d}\alpha_1$ for some $\alpha_1 \in \Omega^{k-1}(M)$, then α is called an exact form.

Def 4.10.

$$\mathscr{Z}^r(M) = \ker(\mathrm{d} : \Omega^r(M) \to \Omega^{r+1}(M)) = \{\text{closed } r\text{-form on } M\},$$

$$\mathscr{B}^r(M) = \operatorname{im}(d : \Omega^{r-1}(M) \to \Omega^r(M)) = \{\text{exact } r\text{-form on } M\}.$$

Then we define the de Rham cohomology group in degree r to be

$$H^r_{dR}(M) = \mathscr{Z}^r(M)/\mathscr{B}^r(M).$$

Exam 4.1. $H^1_{dR}(\mathbb{R}^2 - \{0\}, \mathbb{R}) \neq 0$ *since* $\omega = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}$ *is closed but not exact.*

Thm 4.3 (Hodge)**.** $H^r_{dR}(M, \mathbb{R}) \cong \mathscr{H}^r(M, \mathbb{R})$.

Proof. Let $\alpha \in \Omega^k$ be a closed form and $\mathscr{H}$ be the projection $\Omega^k(M) \to \mathscr{H}^k(M)$.
By Hodge decomposition,

$$\alpha = \mathscr{H}(\alpha) + \mathrm{d}\alpha_1 + \mathrm{d}^*\alpha_2$$

for $\alpha_1 \in \Omega^{r-1}(M), \alpha_2 \in \Omega^{r+1}$.
So $\mathrm{d}\mathrm{d}^*\alpha_2 = \mathrm{d}\alpha = 0, i.e. (\mathrm{d}^*\alpha_2, \mathrm{d}^*\alpha_2) = (\mathrm{d}\mathrm{d}^*\alpha_2, \alpha_2) = 0.$
Therefore $\alpha = \mathscr{H}(\alpha) + \mathrm{d}\alpha_1, i.e. [\alpha] = [\mathscr{H}(\alpha)] \in H^r_{dR}(M, \mathbb{R})$.
Hence $H^r_{dR}(M, \mathbb{R}) \cong \mathscr{H}^r(M, \mathbb{R})$. □

Def 4.11. $X \in \Gamma(M, TM)$, then the divergence of X is

$$\operatorname{div}(X)\mathrm{dvol}_g \overset{\Delta}{=\!=} L_X(\mathrm{dvol}_g).$$

Prop 4.4. $\operatorname{div}(X) = \nabla_i X^i = \displaystyle\sum_{i=1}^n \left(\frac{\partial X^i}{\partial x^i} + \Gamma^i_{ik} X^k\right)$

Proof.

$$\begin{aligned}
L_X(\mathrm{dvol}_g) =&(i_X \circ \mathrm{d} + \mathrm{d} \circ i_X)(\mathrm{dvol}_g) = \mathrm{d}(i_X(\mathrm{dvol}_g)) \\
=&\mathrm{d}\left(X^i i_{\frac{\partial}{\partial x^i}}\left(\sqrt{\det g}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right)\right) \\
=&\frac{\partial}{\partial x^i}\left(X^i\sqrt{\det(g)}\right)\frac{\mathrm{dvol}_g}{\sqrt{\det g}} \\
=&\left(\frac{\partial X^i}{\partial x^i} + \frac{1}{2}\frac{\partial \log \det g}{\partial x^i}X^i\right)\mathrm{dvol}_g \\
\overset{(*)}{=}&\left(\frac{\partial X^i}{\partial x^i} + \Gamma^s_{si}X^i\right)\mathrm{dvol}_g \\
=&(\nabla_i X^i)\mathrm{dvol}(g).
\end{aligned}$$

The $(*)$ is given by

$$\begin{aligned}
\frac{1}{2}\frac{\partial \log \det g}{\partial x^i} =&\frac{1}{2\det g}\frac{\partial \det g}{\partial x^i} = \frac{1}{2\det g}\frac{\partial g_{jk}}{\partial x^i}\left(\det g \cdot g^{jk}\right) \\
=&\frac{1}{2}g^{jk}\frac{\partial g_{jk}}{\partial x^i} = \frac{1}{2}g^{jk}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^j}\right) \\
=&\Gamma^k_{ki}
\end{aligned}$$

□

Coro 4.4. $\Delta_g f = \dfrac{1}{\sqrt{\det g}}\dfrac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det g}\dfrac{\partial f}{\partial x^i}\right).$

Proof.

$$\begin{aligned}
\Delta_g f =&g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right) = \frac{\partial}{\partial x^i}\left(g^{ij}\frac{\partial f}{\partial x^j}\right) - \frac{\partial g^{ij}}{\partial x^i}\frac{\partial f}{\partial x^j} - g^{ij}\Gamma^k_{ij}\frac{\partial f}{\partial x^k} \\
=&\frac{\partial}{\partial x^i}\left(g^{ij}\frac{\partial f}{\partial x^j}\right) + g^{ij}g^{kl}\frac{\partial g_{jl}}{\partial x^i}\frac{\partial f}{\partial x^k} - \frac{1}{2}g^{ij}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)\frac{\partial f}{\partial x^k} \\
=&\frac{\partial}{\partial x^i}\left(g^{ij}\frac{\partial f}{\partial x^j}\right) + \frac{1}{2}g^{ij}g^{kl}\left(\frac{\partial g_{ij}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right)\frac{\partial f}{\partial x^k} \\
=&\frac{\partial}{\partial x^i}\left(g^{ij}\frac{\partial f}{\partial x^j}\right) + \Gamma^i_{il}\left(g^{kl}\frac{\partial f}{\partial x^k}\right) = \mathrm{div}\left(g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}\right) \\
=&\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det g}\frac{\partial f}{\partial x^i}\right)
\end{aligned}$$

□

Prop 4.5 (divergence theorem)**.** $\displaystyle\int_M \mathrm{div}(X)\mathrm{dvol}_g = 0.$

Proof.

$$\int_M \mathrm{div}(X)\mathrm{dvol}_g = \int_M L_X\mathrm{dvol}_g = \int_M \mathrm{d}(i_X\mathrm{dvol}_g) = 0$$

□

Prop 4.6. *If ω is a 1-form, then*

$$\int_M \mathrm{d}^*\omega\mathrm{dvol}_g = 0.$$

Proof. Consider $X^i = g^{ij}\omega_j$.

Then $\mathrm{d}^*\omega = -g^{ij}\nabla_i\omega_j = -\nabla_i X^i = -\mathrm{div}(X)$.

So the integral is 0 by divergence theorem. □

Prop 4.7. $\int_M (\Delta_g f_1)(f_2)\mathrm{dvol}_g = -\int_M g(\nabla f_1, \nabla f_2)\mathrm{dvol}_g = \int_M (\Delta_g f_2) f_1 \mathrm{dvol}_g$

Proof.

$$\mathrm{div}(f_1 \nabla f_2) = g(\nabla f_1, \nabla f_2) + f_1 \Delta_g f_2.$$

□

Chapter 5

Covariant derivatives

Def 5.1.
$$\nabla_X : \Gamma\left(M, (\otimes^r T^*M) \otimes (\otimes^s TM)\right) \to \Gamma(M, (\otimes^r T^*M) \otimes (\otimes^s TM))$$

is defined by

$$\begin{aligned}\nabla_X T(X_1, \cdots, X_r; W_1, \cdots, W_s) =& X(T, X_1, \cdots, X_r; W_1, \cdots, W_s)\\ &- \sum_{i=1}^{r} T(X_1, \cdots, \nabla_X X_i, \cdots, X_r; W_1, \cdots, W_s)\\ &- \sum_{j=1}^{s} T(X_1, \cdots, X_r; W_1, \cdots, \nabla_X W_j, \cdots, W_s)\end{aligned}$$

Def 5.2. Let $T \in \Gamma(M, (\otimes^r T^*M) \otimes (\otimes^s TM))$, the covariant derivative

$$\nabla T \in \Gamma(M, (\otimes^{r+1} T^*M) \otimes (\otimes^s TM))$$

is defined by

$$(\nabla T)(X; \bullet, \cdots, \bullet) = (\nabla_X T)(\bullet, \cdots, \bullet)$$

Prop 5.1. *∇T can be written locally as*

$$\nabla T = W^{j_1 \cdots j_s}_{i i_1 \cdots i_r} \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}.$$

Where
$$W^{j_1 \cdots j_s}_{i i_1 \cdots i_r} = \frac{\partial T^{j_1 \cdots j^s}_{i_1 \cdots i_r}}{\partial x^i} + \Gamma^{j_p}_{mi} T^{j_1 \cdots j_{p-1} m j_{p+1} \cdots j_s}_{i_1 \cdots i_r} - \Gamma^t_{i i_q} T^{j_1 \cdots j_s}_{i_1 \cdots i_{q-1} t i_{q+1} \cdots i_r}$$

Proof.

$$\begin{aligned}W^{j_1 \cdots j_s}_{i i_1 \cdots i_r} =& \frac{\partial}{\partial x^i} T\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \cdots, \mathrm{d}x^{j_s}\right)\\ &- \sum_{m=1}^{s} T\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \cdots, \nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^{j_m}, \cdots, \mathrm{d}x^{j_s}\right)\\ &- \sum_{l=1}^{r} T\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^{i_l}}, \cdots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \cdots, \mathrm{d}x^{j_s}\right)\\ =& \frac{\partial T^{j_1 \cdots j^r}_{i_1 \cdots i_r}}{\partial x^i} - \sum_{m=1}^{s} \left(-\Gamma^{j_m}_{iq}\right) T^{j_1 \cdots j_{m-1} q j_{m+1} \cdots j_s}_{i_1 \cdots i_r} - \sum_{l=1}^{r} \Gamma^p_{i i_l} T^{j_1 \cdots j_s}_{i_1 \cdots i_{l-1} p i_{l+1} \cdots i_r}\end{aligned}$$

□

Remark 5.1. We also write $\nabla_i T^{j_1,\cdots,j_s}_{i_1\cdots i_r} = W^{j_1,\cdots,j_s}_{ii_1\cdots i_r}$.
If you are confused with these tensor notation, you can read the appendix A first.

Def 5.3. we define $\nabla^2 T \xlongequal{\Delta} \nabla(\nabla T) \in \Gamma(M, \otimes^{r+2}T^*M \otimes \otimes^s TM)$.
And we also write

$$\nabla^2 T = \left(\nabla_k \nabla_i T^{j_1\cdots j_s}_{i_1\cdots i_r}\right) \mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}$$

You may think that

$$\nabla_k \nabla_i = \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}}$$

but it is not true. The second covariant derivative ∇_k is actually taken on a $(r+1, s)$-tensor, that is we need to calculus the covariant derivative over the 'i'-component.

Lemma 5.1.

$$\begin{aligned}
\nabla_k \nabla_i T^{j_1\cdots j_s}_{i_1\cdots i_r} =& (\nabla^2 T)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \cdots, \mathrm{d}x^{j_s}\right) \\
=& \left(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}} T\right)\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \cdots, \mathrm{d}x^{j_s}\right) \\
&- \left(\nabla_{\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}} T\right)\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \cdots, \mathrm{d}x^{j_s}\right)
\end{aligned}$$

Proof.

$$\begin{aligned}
\nabla^2 T =& \nabla\left(\nabla_i T^{j_1\cdots j_s}_{i_1\cdots i_r} \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}\right) \\
=& \left(\nabla_k \left(\nabla_{\frac{\partial}{\partial x^i}} T\right)^{j_1\cdots j_s}_{i_1\cdots i_r} - \Gamma^p_{ki} T^{j_1\cdots j_s}_{pi_1\cdots i_r}\right) \mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}
\end{aligned}$$

So

$$\begin{aligned}
\nabla_k \nabla_i T^{j_1\cdots j_s}_{i_1\cdots i_r} &= \left(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}} T - \Gamma^p_{ki} \nabla_{\frac{\partial}{\partial x^p}} T\right)\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \cdots, \mathrm{d}x^{j_s}\right) \\
&= \left(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}} T - \nabla_{\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}} T\right)\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \cdots, \mathrm{d}x^{j_s}\right)
\end{aligned}$$

□

Coro 5.1. *Let* $T \in \Gamma(M, (\otimes^r T^*M) \otimes (\otimes^s TM))$.
Then for $\omega_1, \cdots, \omega_s \in \Gamma(M, T^*M)$ *and* $Y, Z, X_1, \cdots, X_r \in \Gamma(M, TM)$, *we have*

$$(\nabla^2 T)(Y, Z; X_1, \cdots, X_r, \omega_1, \cdots, \omega_s) = (\nabla_Y \nabla_Z - \nabla_{\nabla_Y Z}) T(X_1, \cdots, X_r, \omega_1, \cdots, \omega_s)$$

Proof.

$$\begin{aligned}
&\left(\nabla_Y \nabla_{fZ} T - \nabla_{\nabla_Y fZ} T\right)(X_1, \cdots, X_r, \omega_1, \cdots, \omega_s) \\
=& \left(\nabla_Y f \nabla_Z T - \nabla_{f \nabla_Y Z} T - \nabla_{Y(f)Z} T\right)(X_1, \cdots, X_r, \omega_1, \cdots, \omega_s) \\
=& \left(f \nabla_Y \nabla_Z T + Y(f) \nabla_Z T - f \nabla_{\nabla_Y Z} T - Y(f) \nabla_Z T\right)(X_1, \cdots, X_r, \omega_1, \cdots, \omega_s) \\
=& f\left(\nabla_Y \nabla_Z T - \nabla_{\nabla_Y Z} T\right)(X_1, \cdots, X_r, \omega_1, \cdots, \omega_s)
\end{aligned}$$

So $\nabla_Y \nabla_Z T - \nabla_{\nabla_Y Z} T$ is linear.
Hence by lemma 5.1,

$$(\nabla^2 T)(Y, Z; X_1, \cdots, X_r, \omega_1, \cdots, \omega_s) = \left(\nabla_Y \nabla_Z T - \nabla_{\nabla_Y Z} T\right)(X_1, \cdots, X_r, \omega_1, \cdots, \omega_s).$$

□

Thm 5.1 (Ricci identity)**.**

$$\nabla_k\nabla_l T^{j_1\cdots j_s}_{i_1\cdots i_r} - \nabla_l\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r} = \sum_{m=1}^{s} \mathrm{R}^{j_m}_{klp} T^{j_1\cdots j_{m-1}pj_{m+1}\cdots j_s}_{i_1\cdots i_r} - \sum_{t=1}^{r} \mathrm{R}^{q}_{kli_t} T^{j_1\cdots j_s}_{i_1\cdots i_{t-1}qi_{t+1}\cdots i_r}$$

In particular, we have

$$\nabla_k\nabla_l X^i - \nabla_l\nabla_k X^i = \mathrm{R}^i_{klp}X^p, \nabla_k\nabla_k\omega_i - \nabla_l\nabla_k\omega_i = -\mathrm{R}^q_{kli}\omega_q.$$

Proof.

$$\begin{aligned}
&\nabla_k\nabla_l T^{j_1\cdots j_s}_{i_1\cdots i_r} - \nabla_l\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r}\\
=&\left(\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^l}}T - \nabla_{\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}}T - \nabla_{\frac{\partial}{\partial x^l}}\nabla_{\frac{\partial}{\partial x^k}}T + \nabla_{\nabla_{\frac{\partial}{\partial x^l}}\frac{\partial}{\partial x^k}}T\right)\left(\frac{\partial}{\partial x^{i_1}},\cdots,\frac{\partial}{\partial x^{i_r}},\mathrm{d}x^{j_1},\cdots,\mathrm{d}x^{j_s}\right)\\
=&\left(\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^l}}T - \nabla_{\frac{\partial}{\partial x^l}}\nabla_{\frac{\partial}{\partial x^k}}T\right)\left(\frac{\partial}{\partial x^{i_1}},\cdots,\frac{\partial}{\partial x^{i_r}},\mathrm{d}x^{j_1},\cdots,\mathrm{d}x^{j_s}\right)
\end{aligned}$$

And since

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^l}}T\left(\frac{\partial}{\partial x^{i_1}},\cdots,\frac{\partial}{\partial x^{i_r}},\mathrm{d}x^{j_1},\cdots,\mathrm{d}x^{j_s}\right)\\
=&\sum_{n\neq b}\Gamma^{j_b}_{ld}\Gamma^{j_n}_{kq}T^{j_1\cdots j_{b-1}dj_{b-1}\cdots j_{n-1}qj_{n+1}\cdots j_s}_{i_1\cdots i_r} + \sum_{m\neq a}T^c_{li_a}\Gamma^p_{ki_m}T^{j_1\cdots j_s}_{i_1\cdots i_{a-1}ci_{a+1}\cdots i_{m-1}pi_{m+1}\cdots i_r}\\
&-\sum_{a,n}\Gamma^{j_n}_{kq}\Gamma^c_{li_a}T^{j_1\cdots j_{n-1}qj_{n+1}\cdots j_s}_{i_1\cdots i_{a-1}ci_{a+1}\cdots i_r} - \sum_{b,m}\Gamma^p_{ki_m}T^{j_b}_{ld}T^{j_1\cdots j_{b-1}qj_{b+1}\cdots j_s}_{i_1\cdots i_{m-1}pi_{m+1}\cdots i_r}\\
&+\sum_{n=1}^{s}\Gamma^{j_n}_{kq}\Gamma^q_{ld}T^{j_1\cdots j_{n-1}dj_{n+1}\cdots j_s}_{i_1\cdots i_r} + \sum_{m=1}^{r}\Gamma^p_{ki_m}\Gamma^c_{lp}T^{j_1\cdots j_s}_{i_1\cdots i_{m-1}ci_{m+1}\cdots i_r}\\
&+\frac{\partial^2 T^{j_1\cdots j_s}_{i_1\cdots i_r}}{\partial x^k\partial x^l} + \sum_{n=1}^{s}\Gamma^{j_n}_{kq}\frac{\partial T^{j_1\cdots j_{n-1}qj_{n+1}\cdots j_s}_{i_1\cdots i_r}}{\partial x^l} - \sum_{m=1}^{r}\Gamma^p_{ki_m}\frac{\partial T^{j_1\cdots j_s}_{i_1\cdots i_{m-1}pi_{m+1}\cdots i_r}}{\partial x^l}\\
&+\sum_{b=1}^{s}\Gamma^{j_n}_{ld}\frac{\partial T^{j_1\cdots j_{b-1}dj_{b+1}\cdots j_s}_{i_1\cdots i_r}}{\partial x^k} - \sum_{a=1}^{r}\Gamma^c_{li_a}\frac{\partial T^{j_1\cdots j_s}_{i_1\cdots i_{a-1}ci_{a+1}\cdots i_r}}{\partial x^k}\\
&+\sum_{b=1}^{s}\frac{\partial\Gamma^{j_n}_{ld}}{\partial x^k}T^{j_1\cdots j_{b-1}dj_{b+1}\cdots j_s}_{i_1\cdots i_r} - \sum_{a=1}^{r}\frac{\partial\Gamma^c_{li_a}}{\partial x^k}T^{j_1\cdots j_s}_{i_1\cdots i_{a-1}ci_{a+1}\cdots i_r}
\end{aligned}$$

So

$$\begin{aligned}
&\nabla_k\nabla_l T^{j_1\cdots j_s}_{i_1\cdots i_r} - \nabla_l\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r}\\
=&\sum_{n=1}^{s}\left(\frac{\partial\Gamma^{j_n}_{ld}}{\partial x^k} - \frac{\partial\Gamma^{j_n}_{kd}}{\partial x^l} + \Gamma^{j_n}_{kq}\Gamma^q_{ld} - \Gamma^{j_n}_{lq}\Gamma^q_{kd}\right)T^{j_1\cdots j_{n-1}dj_{n+1}\cdots j_s}_{i_1\cdots i_r}\\
&+\sum_{m=1}^{r}\left(\frac{\partial\Gamma^c_{ki_m}}{\partial x^l} - \frac{\partial\Gamma^c_{li_m}}{\partial x^k} + \Gamma^p_{ki_m}\Gamma^c_{lp} - \Gamma^p_{li_m}\Gamma^c_{kp}\right)T^{j_1\cdots j_s}_{i_1\cdots i_{m-1}ci_{m+1}\cdots i_r}\\
=&\sum_{n=1}^{s}\mathrm{R}^{j_n}_{kld}T^{j_1\cdots j_{n-1}dj_{n+1}\cdots j_s}_{i_1\cdots i_r} + \sum_{m=1}^{r}\mathrm{R}^c_{lki_m}T^{j_1\cdots j_s}_{i_1\cdots i_{m-1}ci_{m+1}\cdots i_r}
\end{aligned}$$

□

Def 5.4. For $S = S_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j, T = T_{kl}\mathrm{d}x^k\otimes\mathrm{d}x^l$, their inner product is

$$\langle S,T\rangle = g(S,T) = S_{ij}T_{kl}g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^l) = S_{ij}T_{kl}g^{ik}g^{jl}.$$

And for $T\in\Gamma(M,T^*M\otimes T^*M)$, the trace operator is defined as

$$\mathrm{tr}_g\,T = g(g,T) = g^{ij}T_{ij}.$$

Lemma 5.2. *For any $S, T \in \Gamma(M, T^*M \otimes T^*M), X \in \Gamma(M, TM)$, we have*

$$Xg(S,T) = g(\nabla_X S, T) + g(S, \nabla_X T).$$

In particular, let $S = g$, then

$$X(\operatorname{tr}_g T) = \operatorname{tr}_g(\nabla_X T).$$

And locally we have

$$\nabla_{\frac{\partial}{\partial x^i}}(g^{kl}T_{kl}) = \nabla_{\frac{\partial}{\partial x^i}}(\operatorname{tr}_g T) = g^{kl}(\nabla_i T_{kl})$$

Proof.

$$\begin{aligned}\nabla_k g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) =& \frac{\partial g_{ij}}{\partial x^k} - \Gamma^p_{ki} g_{pj} - \Gamma^p_{kj} g_{ip} \\ =& \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right) - \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) \\ =& 0\end{aligned}$$

So

$$X(g(S,T)) = (\nabla_X g)(S,T) + g(\nabla_X Y, Z) + g(S, \nabla_X T) = g(\nabla_X Y, Z) + g(S, \nabla_X T).$$

□

Remark 5.2. This lemma tell us if we have some g^{ij} inside the connection, then we can simply exchange its position with ∇.

Thm 5.2 (second Bianchi identity)**.** $\nabla_i \mathrm{R}_{klpq} + \nabla_k \mathrm{R}_{lipq} + \nabla_l \mathrm{R}_{ikpq} = 0$

Proof. WLOG, let $\{\frac{\partial}{\partial x^i}\}$ be a normal local frame.

$$\begin{aligned}&\nabla_i \mathrm{R}_{jkpq} + \nabla_j \mathrm{R}_{kipq} + \nabla_k \mathrm{R}_{ijpq} \\ =& \nabla_i\left(g_{ql}\mathrm{R}^l_{jkp}\right) + \nabla_j\left(g_{ql}\mathrm{R}^l_{kip}\right) + \nabla_k\left(g_{ql}\mathrm{R}^l_{ijp}\right) = g_{ql}\left(\nabla_i \mathrm{R}^l_{jkp} + \nabla_j \mathrm{R}^l_{kip} + \nabla_k \mathrm{R}^l_{ijp}\right) \\ =& g_{ql}\left(\frac{\partial}{\partial x^i}\left(\frac{\partial \Gamma^l_{kp}}{\partial x^j} - \frac{\partial \Gamma^l_{jp}}{\partial x^k}\right) + \frac{\partial}{\partial x^j}\left(\frac{\partial \Gamma^l_{ip}}{\partial x^k} - \frac{\partial \Gamma^l_{kp}}{\partial x^i}\right) + \frac{\partial}{\partial x^k}\left(\frac{\partial \Gamma^l_{jp}}{\partial x^i} - \frac{\partial \Gamma^l_{ip}}{\partial x^j}\right)\right) \\ =& 0\end{aligned}$$

□

Thm 5.3 (Shur)**.** *Suppose (M, g) is a connected Riemannian manifold and* $\dim M \geqslant 3$*. If* $\mathrm{Ric}(g) = fg(n-1)$*, then f is a constant.*

Proof. Let

$$S = \operatorname{tr}_g \mathrm{Ric}(g) = n(n-1)f$$

Then

$$\frac{\partial S}{\partial x^k} = n(n-1)\frac{\partial f}{\partial x^k}.$$

On the other hand,

$$\begin{aligned}\frac{\partial S}{\partial x^k} =& \frac{\partial}{\partial x^k}\left(g^{ij}\mathrm{R}_{ij}\right) = \frac{\partial}{\partial x^k}\left(g^{ij}g^{pq}\mathrm{R}_{pijq}\right) \\ =& g^{ij}g^{pq}\nabla_k \mathrm{R}_{pijq} = g^{ij}g^{pq}\left(-\nabla_p \mathrm{R}_{ikjq} - \nabla_i \mathrm{R}_{kpjq}\right) \\ =& g^{ij}g^{pq}(\nabla_p \mathrm{R}_{ikqj} + \nabla_i \mathrm{R}_{pkjq}) = g^{pq}\nabla_p \mathrm{R}_{kq} + g^{ij}\nabla_i \mathrm{R}_{kj} \\ =& 2g^{pq}\nabla_p \mathrm{R}_{kq} = 2(n-1)\frac{\partial f}{\partial x^k}\end{aligned}$$

Therefore

$$(n-2)\frac{\partial f}{\partial x^k} = 0.$$

Hence f is constant. $\square$

Chapter 6

Curvature and topology of Riemannian manifolds I

6.1 Curvature and Killing vector fields

Lemma 6.1 (Bochner)**.** *If (M, g) is a Riemannian manifold and $f \in C^{\infty}(M, \mathbb{R})$, then*

$$\frac{1}{2}\Delta_g|\nabla f|^2 = |\text{Hess } f|^2 + \text{Ric}(\nabla f, \nabla f) + g(\nabla\Delta_g f, \nabla f).$$

Proof. We denote

$$\nabla_j f \xlongequal{\Delta} \frac{\partial f}{\partial x^j}, \nabla^j f \xlongequal{\Delta} g^{ij}\frac{\partial f}{\partial x^i}$$

Then

$$|\nabla f|^2 = g^{ij}\nabla_i f \nabla_j f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}$$
$$\text{Hess } f = \nabla^2 f = (\nabla_k \nabla_l f)\mathrm{d}x^k \otimes \mathrm{d}x^l$$
$$\Delta_g f = g^{kl}\nabla_k\nabla_l f$$

So

$$\begin{aligned}\frac{1}{2}\Delta_g|\nabla f|^2 =&\frac{1}{2}g^{kl}\nabla_k\nabla_l(g^{ij}\nabla_i f\nabla_j f) = \frac{1}{2}g^{kl}g^{ij}\nabla_k\nabla_l(\nabla_i f\nabla_j f)\\ =&g^{kl}g^{ij}\nabla_k(\nabla_l\nabla_i f\cdot\nabla_j f) = g^{kl}g^{ij}(\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f + \nabla_l\nabla_i f\cdot\nabla_k\nabla_j f)\\ =&g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f + |\text{Hess }(f)|^2\end{aligned}$$

On the other hand,

$$\begin{aligned}g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f =&g^{kl}g^{ij}\nabla_k\nabla_i\nabla_l f\cdot\nabla_j f\\ =&g^{kl}g^{ij}(\nabla_i\nabla_k\nabla_l f - \mathrm{R}^s_{kil}\nabla_s f)\cdot\nabla_j f\\ =&g^{ij}\nabla_i(g^{kl}\nabla_k\nabla_l f)\cdot\nabla_j f + g^{st}g^{ij}\mathrm{R}_{it}\nabla_s f\cdot\nabla_j f\\ =&g(\nabla\Delta_g f, \nabla f) + \text{Ric}(\nabla f, \nabla f)\end{aligned}$$

□

Coro 6.1 (radial curvature equation)**.**

$$\text{Hess }\left(\frac{1}{2}|\nabla f|^2\right)(\bullet,\bullet) = (\nabla_{\nabla f}\text{Hess } f)(\bullet,\bullet) + \mathrm{R}(\bullet,\nabla f,\nabla f,\bullet) + \text{Hess } f(\nabla_\bullet\nabla f,\bullet)$$

Proof. Follow by Bochner formula and

$$g^{ij}\nabla_l\nabla_i f\cdot\nabla_k\nabla_j f = \text{Hess } f\left(\nabla_{\frac{\partial}{\partial x^k}}\nabla f, \frac{\partial}{\partial x^l}\right)$$

$$g^{ij}\nabla_i(\nabla_k\nabla_l f)\cdot\nabla_j f = (\nabla_{\nabla f}\text{Hess } f)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

$$g^{st}g^{ij}\mathrm{R}_{kitl}\nabla_s f\cdot\nabla_j f = \mathrm{R}\left(\frac{\partial}{\partial x^k}, \nabla f, \nabla f, \frac{\partial}{\partial x^l}\right)$$

□

Def 6.1. $X\in\Gamma(M,TM)$ is a called a killing field if $L_X g = 0$.

Lemma 6.2.

$$L_X g(Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y).$$

Proof.

$$\begin{aligned} L_X g(Y,Z) =& Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]) \\ =& g(\nabla_X Y, Z) + g(Y,\nabla_X Z) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) \\ =& g(\nabla_Y X, Z) + g(\nabla_Z X, Y) \end{aligned}$$

□

Coro 6.2. *X is killing, $g(\nabla_\bullet X, \bullet)$ is skew-symmetric.*

Remark 6.1. {parallel vector field} $\subset$ {Killing vector field} $\subset$ {divergence free vector field}.
Since $\nabla X = 0 \Rightarrow \nabla X$ is skew-symmetric$\Rightarrow$ $\mathrm{div}(X) = \mathrm{tr}(\nabla X) = 0$.

Lemma 6.3. *If X is Killing and $f \overset{\Delta}{=\!=} \frac{1}{2}|X|_g^2$, then*

(1) $\nabla f = -\nabla_X X$

(2) For any vector field $V\in\Gamma(M,TM)$,

$$(\text{Hess } f)(V,V) = |\nabla_V X|^2 - \mathrm{R}(V,X,X,V).$$

In particular,

$$\Delta_g f = |\nabla X|_g^2 - \mathrm{Ric}(X,X).$$

Proof. (1)

$$g(\nabla f, V) = Vf = \frac{1}{2}V\langle X,X\rangle = \langle\nabla_V X, X\rangle = -\langle\nabla_X X, V\rangle.$$

(2)

$$\begin{aligned} (\text{Hess } f)(V,V) =& V^pV^q\nabla_p\nabla_q f = \frac{1}{2}V^pV^q\nabla_p\nabla_q(g_{ij}X^iX^j) \\ =& \frac{1}{2}V^pV^q g_{ij}\nabla_p(\nabla_q(X^iX^j)) \\ =& V^pV^q g_{ij}(\nabla_p\nabla_q X^i\cdot X^j + \nabla_q X^i\cdot\nabla_p X^j) \\ =& |\nabla_V X|^2 + V^pV^q g_{ij}\left(\nabla_p\nabla_q X^i\right)X^j \\ =& |\nabla_V X|^2 - V^pV^q g_{iq}\left(\nabla_p\nabla_j X^i\right)X^j \\ =& |\nabla_V X|^2 - V^pV^q g_{iq}\left(\nabla_j\nabla_p X^i + \mathrm{R}^i_{pjs}X^s\right)X^j \\ =& |\nabla_V X|^2 - R(V,X,X,V) \end{aligned}$$

The last step follows from that $g_{ip}\nabla_j\nabla_q X^i = -g_{iq}\nabla_j\nabla_p X^i$.

Alternative proof:

$$\begin{aligned}(\text{Hess } f)(V,V) &= -\,g(\nabla_V(\nabla_X X), V)\\ &= -\,g(\mathrm{R}(V,X)X,V) - g(\nabla_X\nabla_V X, V) - g(\nabla_{[V,X]}X, V)\\ &= -\,\mathrm{R}(V,X,X,V) - g(\nabla_X\nabla_V X, V) + g(\nabla_V X, [V,X])\\ &= -\,\mathrm{R}(V,X,X,V) - g(\nabla_X\nabla_V X, V) + g(\nabla_V X, \nabla_V X) - g(\nabla_V X, \nabla_X V)\\ &= -\,R(V,X,X,V) - X(g(\nabla_V X, V)) + g(\nabla_V X, \nabla_V X)\\ &= -\,R(V,X,X,V) + g(\nabla_V X, \nabla_V X)\end{aligned}$$

□

Coro 6.3. *(M,g) is compact and $\mathrm{Ric}(g) \leqslant 0$, then Killing $\Leftrightarrow$ parallel.*

Proof.

$$\int |\nabla X|^2 \leqslant \int \Delta_g f = 0$$

So $\nabla X = 0$ is parallel. □

Thm 6.1 (Bochner, 1946)**.** *If (M,g) is a compact Riemannian manifold with $\mathrm{Ric}(g) < 0$, then (M,g) has no nontrivial Killing vector field.*

Proof. At the maximal point, $\Delta_g f \leqslant 0$, but

$$|\nabla X|_g^2 - \mathrm{Ric}(X,X) \geqslant 0$$

So $\mathrm{Ric}(X,X) = 0, \nabla X = 0, i.e. X = 0$ is trivial. □

Thm 6.2. *Let (M,g) be a compact Riemannian manifold with $\sec(g) > 0$.*
If $\dim_{\mathbb{R}} M = 2m$, then every Killing vector field on (M,g) has a zero.

Proof. Suppose X is killing and $|X|_g^2$ attains its minimum at $p \in M$ and $X_p \neq 0$.
Let $f = \frac{1}{2}|X|_g^2$.
Then

$$0 = (\nabla f)(p) = (\nabla_X X)(p)$$

We claim $\exists V \in T_pM, s.t.\ \nabla_V X\big|_p = 0$ where $V \notin \mathrm{span}\{X_p\}$.
Let $T_pM = E \oplus \mathrm{span}_{\mathbb{R}}\{x_p\}$, where $\dim E$ is odd.
Consider the skew-symmetric map

$$A : E \to E^*, V \mapsto g(\nabla_V X_p, \bullet)$$

It must have the eigenvalue 0, since $\det A = \det A^T = (-1)^{n-1}\det A = 0$.
Hence we have

$$0 \leqslant (\text{Hess } f)(V,V) = |\nabla_V X|^2 - R(V,X,X,V) = -R(V,X,X,V) < 0,$$

contradiction! □

6.2 Curvature and Betti numbers

Lemma 6.4. *If α is a Harmonic 1-form, then*

$$\frac{1}{2}\Delta_g|\alpha|^2 = |\nabla\alpha|^2 + \mathrm{Ric}(X_\alpha, X_\alpha),$$

where X_α is the dual vector field of α.

Proof.

$$(\nabla_i\alpha_j)\mathrm{d}x^i \wedge \mathrm{d}x^j = \left(\frac{\partial\alpha_j}{\partial x^i} - \Gamma_{ij}^k\alpha_k\right)\mathrm{d}x^i \wedge \mathrm{d}x^j = \mathrm{d}\alpha = 0.$$

So $\nabla_i\alpha_j = \nabla_j\alpha_i$.
And by $\mathrm{d}^*\alpha = 0$, we have

$$g^{ij}\nabla_j\alpha_i = \nabla^i\alpha_i = -\mathrm{d}^*\alpha = 0.$$

Hence

$$\begin{aligned}\frac{1}{2}\Delta_g|\alpha|^2 =& \frac{1}{2}g^{kl}\nabla_k\nabla_l(g^{ij}\alpha_i\alpha_j)\\ =& g^{kl}g^{ij}(\nabla_k\nabla_l\alpha_i \cdot \alpha_j + \nabla_k\alpha_j \cdot \nabla_l\alpha_i)\\ =& |\nabla\alpha|^2 + g^{kl}g^{ij}\nabla_k\nabla_l\alpha_i \cdot \alpha_j\\ =& |\nabla\alpha|^2 + g^{kl}g^{ij}\nabla_k\nabla_i\alpha_l \cdot \alpha_j\\ =& |\nabla\alpha|^2 + g^{kl}g^{ij}(\nabla_i\nabla_k\alpha_l - \mathrm{R}_{kil}^s\alpha_s)\alpha_j\\ =& |\nabla\alpha|^2 + g^{kl}g^{ij}g^{sp}g_{pq}\mathrm{R}_{ikl}^q\alpha_j\alpha_s\\ =& |\nabla\alpha|^2 + \mathrm{Ric}(X_\alpha, X_\alpha)\end{aligned}$$

□

Thm 6.3 (Bochner)**.** *Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric}(g) > 0$, then $b_1(M) = 0$.*

Proof. At the maximal point of a 1-form α,

$$0 \geqslant \frac{1}{2}\Delta_g|\alpha|^2 = |\nabla\alpha|^2 + \mathrm{Ric}(X_\alpha, X_\alpha) \geqslant 0$$

So $\alpha \equiv 0, i.e. H_{dR}^1(M, \mathbb{R}) \cong \mathscr{H}^1(M, \mathbb{R})$ is trivial. □

Lemma 6.5. *If $\mathrm{Ric}(g) \geqslant 0$, then harmonic 1-forms are parallel.*

Proof.

$$0 = \int \frac{1}{2}\Delta_g|\alpha|^2 = \int |\nabla\alpha|^2 + \int \mathrm{Ric}(X_\alpha, X_\alpha) \geqslant \int |\nabla\alpha|^2.$$

So $\nabla\alpha \equiv 0, i.e. \alpha$ is parallel. □

Thm 6.4. *Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric}(g) \geqslant 0$, then $b_1(M) \leqslant \dim_{\mathbb{R}} M$.*

Proof. Consider the evaluation map $\mathscr{H}^1(M, \mathbb{R}) \to T_p^*M, \alpha \mapsto \alpha_p$, it is an embedding.
So $b_1(M) \leqslant \dim_{\mathbb{R}} M$. □

6.3 Harmonic maps between Riemannian manifolds

Def 6.2. For smooth map $f:(M,g)\to(N,h)$, define

$$\Delta f=\operatorname{tr}_g B=g^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i\partial x^j}-\Gamma_{ij}^k\frac{\partial f^\alpha}{\partial x^k}+\Gamma_{\beta\gamma}^\alpha\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\right)\frac{\partial}{\partial y^\alpha}.$$

If $\Delta f=0$, then f is a harmonic map.
And if $B=0$, f is called totally geodesic.

Coro 6.4. *Let $f:(M,g)\to(N,h)$ be a C^∞ maps, TFAE:*

(1) f is totally geodesic i.e. $B=0$.

(2) f maps geodesic in (M,g) to geodesic in (N,h).

Proof. Let $\gamma:[a,b]\to(M,g)$ be a C^∞ curve, $\tilde\gamma=f\circ\gamma$ and $\hat\nabla,\tilde\nabla$ are induced connection on $\gamma^*TM,\tilde\gamma^*TN$ resp.

$$\begin{aligned}
&\tilde\nabla_{\frac{\mathrm d}{\mathrm dt}}\tilde\gamma_*\left(\frac{\mathrm d}{\mathrm dt}\right)=\left(\frac{\mathrm d^2\tilde\gamma^\alpha}{\mathrm dt^2}+\Gamma_{\beta r}^\alpha(\tilde\gamma)\frac{\mathrm d\tilde\gamma^\beta}{\mathrm dt}\frac{\mathrm d\tilde\gamma^r}{\mathrm dt}\right)\frac{\partial}{\partial y^\alpha}\\
&=\left(\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial f^\alpha}{\partial x^i}\frac{\mathrm d\gamma^i}{\mathrm dt}\right)+\Gamma_{\beta r}^\alpha\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^r}{\partial x^j}\frac{\mathrm d\gamma^i}{\mathrm dt}\frac{\mathrm d\gamma^j}{\mathrm dt}\right)\frac{\partial}{\partial y^\alpha}\\
&=\left(\frac{\partial^2 f^\alpha}{\partial x^i\partial x^j}\frac{\mathrm d\gamma^i}{\mathrm dt}\frac{\mathrm d\gamma^j}{\mathrm dt}+\frac{\partial f^\alpha}{\partial x^i}\frac{\mathrm d^2\gamma^i}{\mathrm dt^2}+\Gamma_{\beta r}^\alpha\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^r}{\partial x^j}\frac{\mathrm d\gamma^i}{\mathrm dt}\frac{\mathrm d\gamma^j}{\mathrm dt}\right)\frac{\partial}{\partial y^\alpha}\\
&=\left(\frac{\partial f^\alpha}{\partial x^k}\left(\frac{\mathrm d^2\gamma^k}{\mathrm dt^2}+\Gamma_{ij}^k\frac{\mathrm dr^i}{\mathrm dt}\frac{\mathrm dr^j}{\mathrm dt}\right)+\left(\frac{\partial^2 f^\alpha}{\partial x^i\partial x^j}+\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^r}{\partial x^j}\Gamma_{\beta r}^\alpha-\Gamma_{ij}^k\frac{\partial f^\alpha}{\partial x^k}\right)\frac{\mathrm d\gamma^i}{\mathrm dt}\frac{\mathrm d\gamma^j}{\mathrm dt}\right)\frac{\partial}{\partial y^\alpha}\\
&=f_*\left(\hat\nabla_{\frac{\mathrm d}{\mathrm dt}}\gamma_*\left(\frac{\mathrm d}{\mathrm dt}\right)\right)+\gamma^*B
\end{aligned}$$

So $r^*B=0\Leftrightarrow\gamma$ is geodesic iff $\tilde\gamma$ is geodesic . ∎

Def 6.3. If $f:M\to(N,h)$ is an immersion, consider the induced metric on M, it is

$$g_M=h_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\mathrm dx^i\otimes\mathrm dx^j.$$

Then $f:(M,g_M)\to(N,h)$ is an isometric immersion.
We called $f:M\to(N,h)$ totally geodesic if it is totally geodesic w.r.t. g_M.

Prop 6.1. *$\gamma:[a,b]\to(M,g)$ is a regular curve $\gamma'(t)\neq0$.*
If γ is unit speed curve, then γ is totally geodesic$\Leftrightarrow$ γ is a geodesic.

Proof. consider the induced metric on I, it is

$$g_0=\gamma^*g=g_{ij}\frac{\mathrm d\gamma^i}{\mathrm dt}\frac{\mathrm d\gamma^j}{\mathrm dt}\mathrm dt\otimes\mathrm dt=\left|\gamma'(t)\right|^2\mathrm dt\otimes\mathrm dt$$

So $g_0=g_{can}$, *i.e.*γ is totally geodesic iff $\gamma:([a,b],g_{can})\to(M,g)$ maps geodesic to geodesic.
Hence γ is totally geodesic$\Leftrightarrow$ γ is a geodesic. ∎

We now try to extend the Bochner formula lemma 6.1 to the case that $f:(M,g)\to(N,h)$.

Prop 6.2. *Recall remark 2.7,*

$$B=\tilde\nabla\mathrm df.$$

Proof.

$$\mathrm{d}f = \frac{\partial f^\alpha}{\partial x^i}\mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha} \in \Gamma(M, T^*M \otimes f^*TN).$$

$$\begin{aligned}
\tilde{\nabla}_k \left(\frac{\partial f^\alpha}{\partial x^i}\mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha}\right) &= \left(\nabla^* \otimes \hat{\nabla}\right)_{\frac{\partial}{\partial x^k}} \left(\frac{\partial f^\alpha}{\partial x^i}\mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha}\right) \\
=&\frac{\partial^2 f^\alpha}{\partial x^i \partial x^k}\mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha} + \frac{\partial f^\alpha}{\partial x^i}\left(\nabla^*_{\frac{\partial}{\partial x^k}}\mathrm{d}x^i\right) \otimes \frac{\partial}{\partial y^\alpha} + \frac{\partial f^\alpha}{\partial x^i}\mathrm{d}x^i \otimes \left(\hat{\nabla}_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial y^\alpha}\right) \\
=&\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^k} - \Gamma^s_{ik}\frac{\partial f^\alpha}{\partial x^s} + \Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^k}\right)\mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha} \\
=&\mathrm{d}x^i \otimes B\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}\right)
\end{aligned}$$

□

Thm 6.5. *If $f : (M, g) \to (N, h)$ is C^∞, then*

$$\frac{1}{2}\Delta_g|\mathrm{d}f|^2 = \left|\tilde{\nabla}\mathrm{d}f\right|^2 + \left\langle \hat{\nabla}\Delta_g f, \mathrm{d}f\right\rangle + \mathrm{R}_{ij} f^\alpha_k f^\beta_l h_{\alpha\beta} g^{ik} g^{jl} - \mathrm{R}_{\alpha\beta\gamma\delta} f^\alpha_i f^\beta_j f^\gamma_k f^\delta_l g^{il} g^{jk}.$$

Proof. We denote

$$\mathrm{d}f \xlongequal{\Delta} f^\alpha_i \mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha}.$$

So

$$\begin{aligned}
\frac{1}{2}\Delta_g|\mathrm{d}f|^2 =&\frac{1}{2}g^{kl}\tilde{\nabla}_k\tilde{\nabla}_l\left(g^{ij}h_{\alpha\beta}f^\alpha_i f^\beta_j\right) \\
=&g^{kl}g^{ij}h_{\alpha\beta}\left(\tilde{\nabla}_k f^\alpha_i \cdot \tilde{\nabla}_l f^\beta_j + \tilde{\nabla}_k\tilde{\nabla}_l f^\alpha_i \cdot f^\beta_j\right) \\
=&\left|\tilde{\nabla}\mathrm{d}f\right|^2 + g^{kl}g^{ij}h_{\alpha\beta}\tilde{\nabla}_k\tilde{\nabla}_l f^\alpha_i \cdot f^\beta_j
\end{aligned}$$

And by Ricci identity theorem 5.1,

$$\tilde{\nabla}_k\tilde{\nabla}_l f^\alpha_i = \tilde{\nabla}_k\tilde{\nabla}_i f^\alpha_l = \tilde{\nabla}_i\tilde{\nabla}_k f^\alpha_l - \mathrm{R}^s_{kil}f^\alpha_s + \mathrm{R}^\alpha_{ik\gamma}f^\gamma_l$$

$$g^{kl}g^{ij}h_{\alpha\beta}\tilde{\nabla}_i\tilde{\nabla}_k f^\alpha_l \cdot f^\beta_j = g^{ij}h_{\alpha\beta}\left(\hat{\nabla}\Delta_g f\right)^\alpha_i f^\beta_j = \left\langle \hat{\nabla}\Delta_g f, \mathrm{d}f\right\rangle$$

Hence we obtain

$$\begin{aligned}
\frac{1}{2}\Delta_g|\mathrm{d}f|^2 =&\left|\tilde{\nabla}\mathrm{d}f\right|^2 + \left\langle \hat{\nabla}\Delta_g f, \mathrm{d}f\right\rangle - g^{kl}g^{ij}h_{\alpha\beta}\mathrm{R}^s_{kil}f^\alpha_s f^\beta_j + g^{kl}g^{ij}h_{\alpha\beta}\mathrm{R}^\alpha_{ik\gamma}f^\gamma_l f^\beta_j \\
=&\left|\tilde{\nabla}\mathrm{d}f\right|^2 + \left\langle \hat{\nabla}\Delta_g f, \mathrm{d}f\right\rangle + g^{kl}g^{ij}g^{pq}h_{\alpha\beta}\mathrm{R}_{iklp}f^\alpha_q f^\beta_j - g^{kl}g^{ij}\mathrm{R}_{\mu\delta\gamma\beta}f^\delta_i f^\mu_k f^\gamma_l f^\beta_j \\
=&\left|\tilde{\nabla}\mathrm{d}f\right|^2 + \left\langle \hat{\nabla}\Delta_g f, \mathrm{d}f\right\rangle + \mathrm{R}_{ij} f^\alpha_k f^\beta_l h_{\alpha\beta} g^{ik} g^{jl} - \mathrm{R}_{\alpha\beta\gamma\delta} f^\alpha_i f^\beta_j f^\gamma_k f^\delta_l g^{il} g^{jk}
\end{aligned}$$

□

Coro 6.5. *$f : (M, g) \to (N, h)$ is a harmonic map and*

(1) (M, g) is compact and $\mathrm{Ric}(g) > 0$

(2) (N, h) has non-positive sectional curvature.

Then f is a constant map.

Proof. Consider the maximum point p of $|\mathrm{d}f|^2$.

WLOG, we let $g_{ij}(p) = \delta_{ij}, h_{\alpha\beta}(f(p)) = \delta_{\alpha\beta}$.

Then

$$\mathrm{R}_{ij} f_k^\alpha f_l^\beta h_{\alpha\beta} g^{ik} g^{jl} = \sum_\alpha \mathrm{Ric}_g(V_\alpha, V_\alpha) > 0$$

for $V_\alpha = (f_i^\alpha) \in \Gamma(M, TM)$, and

$$\mathrm{R}_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta g^{il} g^{jk} = \sum_{i,j} R_h(V_i, V_j, V_j, V_i) \leqslant 0$$

for $V_i = (f_i^\alpha) \in \Gamma(N, TN)$.

On the other hand,

$$\Delta_g |\mathrm{d}f|^2 \leqslant 0, \Delta_g f = 0.$$

So $\mathrm{d}f \equiv 0$, *i.e.*f is constant. □

Remark 6.2. These type of theorem is called Liouville-type theorem.

In complex analysis, we have the famous theorem from Liouville:

$f : \mathbb{C} \to \mathbb{C}$ is entire and bounded, then f is constant.

Coro 6.6. *$f : (M, g) \to (N, h)$ is harmonic:*

(1) M is compact and $\mathrm{Ric}(g) \geqslant 0$

(2) (N, h) has non-positive sectional curvature.

Then:

(1) f is totally geodesic, i.e. $\tilde{\nabla}\mathrm{d}f = 0$.

(2) If $\mathrm{Ric}(g)(p) > 0$ *for some point p, then f is constant.*

(3) If (N, h) has negative sectional curvature, then $f(M)$ is a point or a closed geodesic.

Proof. Similar as corollary 6.5, we have

$$0 = \int \frac{1}{2} \Delta_g |\mathrm{d}f|^2 = \int \left|\tilde{\nabla}\mathrm{d}f\right|^2 + \int \sum_\alpha \mathrm{Ric}_g(V_\alpha, V_\alpha) - \int \sum_{i,j} \mathrm{R}_h(V_i, V_j, V_j, V_i) \geqslant \int \left|\tilde{\nabla}\mathrm{d}f\right|^2$$

(1) So $\tilde{\nabla}\mathrm{d}f \equiv 0$.

And $X\left(|\mathrm{d}f|^2\right) = 2\left\langle \tilde{\nabla}_X \mathrm{d}f, \mathrm{d}f \right\rangle = 0$.

Therefore $|\mathrm{d}f|$ is constant.

(2) $V_\alpha(p) = 0$ for every α, *i.e.* $\mathrm{d}f(p) = 0$.

So f is a constant.

(3) $V_i = v_j$ for every i, j.

So $\operatorname{rank} \mathrm{d}f \leqslant 1$, *i.e.* $f(M)$ is a point or a curve.

And since f is totally geodesic and M is compact.

Hence $f(M)$ is a point or a closed geodesic.

□

Chapter 7

Variational formula

7.1 The First Variation

Def 7.1. Given a closed interval $[a,b] \subset \mathbb{R}$ and two points p, q in (M,g), we denote

$$\mathscr{L} = \{\gamma : [a,b] \to M | \gamma(a) = p, \gamma(b) = q, \gamma \text{ is smooth}\}.$$

For any smooth curve $\gamma \in \mathscr{L}$, the length and energy of γ are

$$L(\gamma) = \int_a^b |\gamma'(t)| \mathrm{d}t = \int_a^b \sqrt{g_{ij} \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t}} \mathrm{d}t,$$

$$E(\gamma) = \frac{1}{2} \int_a^b |\gamma'(t)|^2 \mathrm{d}t = \int_a^b \left(g_{ij} \frac{\mathrm{d}r^i}{\mathrm{d}t} \frac{\mathrm{d}r^j}{\mathrm{d}t} \right) \mathrm{d}t.$$

Def 7.2. Given $\gamma \in \mathscr{L}$, a proper variation of γ is a smooth map

$$\alpha : [a,b] \times (-\varepsilon, \varepsilon) \to M$$

such that

(1) $\alpha(t,0) = \gamma(t)$

(2) $\alpha(\bullet, s) \in \mathscr{L}$

Lemma 7.1. *Let X be a C^∞ vector field along γ with $X(a) = X(b) = 0$, then there exists a proper variation α of γ such that*

$$\alpha_* \left(\frac{\partial}{\partial s} \right) \bigg|_{s=0} = X.$$

X is called the variational vector field of α.

Proof. We let

$$\alpha(t,s) = \exp_{\gamma(t)}(sX_{\gamma(t)}).$$

Since $\mathrm{d}\exp_{\gamma(t)}$ is identity around 0, so

$$\alpha_* \left(\frac{\partial}{\partial s} \right) \bigg|_{s=0} = \mathrm{d}\Big(\exp_{\gamma(t)}\Big)_0 \left(X_{\gamma(t)}\right) = X_{\gamma(t)}$$

□

Thm 7.1. *Let $\gamma : [a,b] \to (M,g)$ be a C^∞ curve and α is a proper variation of γ with variational vector field V, then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\alpha(\cdot,s)) = \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma' \right\rangle \mathrm{d}t = -\int_a^b \left\langle V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma' \right\rangle \mathrm{d}t.$$

Proof. We put the canonical metric and trivial connection ∇ on $(a,b) \times (-\varepsilon, \varepsilon)$.

$\bar{\nabla}$ be the induced connection on $\alpha^* TM$, $\bar{g}$ be the induced metric on $\alpha^* TM$.

$\hat{\nabla}$ be the induced connection on $\gamma^* TM$, $\hat{g}$ be the induced metric on $\gamma^* TM$.

$$\begin{aligned}
\frac{\partial}{\partial s}\left\langle \alpha_*\left(\frac{\partial}{\partial t}\right), \alpha_*\left(\frac{\partial}{\partial t}\right)\right\rangle_{\bar{g}} =& 2\left\langle \bar{\nabla}_{\frac{\partial}{\partial s}} \alpha_*\left(\frac{\partial}{\partial t}\right), \alpha_*\left(\frac{\partial}{\partial t}\right)\right\rangle \\
=& 2\left\langle B\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) + \alpha_*\left(\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}\right), \alpha_*\left(\frac{\partial}{\partial t}\right)\right\rangle \\
=& 2\left\langle \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_*\left(\frac{\partial}{\partial s}\right), \alpha_*\left(\frac{\partial}{\partial t}\right)\right\rangle \\
=& 2\frac{\partial}{\partial t}\left\langle \alpha_*\left(\frac{\partial}{\partial s}\right), \alpha_*\left(\frac{\partial}{\partial t}\right)\right\rangle - 2\left\langle \alpha_*\left(\frac{\partial}{\partial s}\right), \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_*\left(\frac{\partial}{\partial t}\right)\right\rangle
\end{aligned}$$

And since

$$\left.\alpha_*\left(\frac{\partial}{\partial s}\right)\right|_{s=0} = V, \left.\bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_*\left(\frac{\partial}{\partial t}\right)\right|_{s=0} = \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t).$$

So we conclude that

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left(\frac{1}{2}\int_a^b \left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^2 \mathrm{d}t\right) &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\left.\left\langle \alpha_*\left(\frac{\partial}{\partial s}\right), \alpha_*\left(\frac{\partial}{\partial t}\right)\right\rangle\right|_{s=0}\right) \mathrm{d}t \\
&\quad - \int_a^b \left.\left\langle \alpha_*\left(\frac{\partial}{\partial s}\right), \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_*\left(\frac{\partial}{\partial t}\right)\right\rangle_{\bar{g}}\right|_{s=0} \mathrm{d}t \\
&= -\int_a^b \left\langle V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma' \right\rangle_{\hat{g}} \mathrm{d}t = \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma' \right\rangle \mathrm{d}t
\end{aligned}$$

□

Coro 7.1. *Let $\gamma : [a,b] \to (M,g)$ be a unit-speed curve and α is a proper variation of γ with vector field V, then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} L(\alpha(\bullet,s)) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\alpha(\bullet,s)) = \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma' \right\rangle \mathrm{d}t$$

Proof.

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_a^b \left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right| \mathrm{d}t =& \frac{1}{2}\int_a^b \left.\left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^{-1}\right|_{s=0} \left.\frac{\partial}{\partial s}\right|_{s=0} \left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^2 \mathrm{d}t \\
=& \frac{1}{2}\int_a^b \frac{1}{|\gamma'(t)|} \left.\frac{\partial}{\partial s}\right|_{s=0} \left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^2 \mathrm{d}t \\
=& \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\alpha(\bullet,s))
\end{aligned}$$

□

Thm 7.2. *Let $[a,b]$ be a compact interval and $\gamma : I \to (M,g)$ be a C^∞ curve, TFAE*

(1) γ is a critical point of $E : \mathscr{L} \to \mathbb{R}$. That is, for any proper variation of γ,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\alpha(\cdot, s)) = 0$$

(2) γ is parametrized proportional to the arclength, $|\gamma'(t)| \equiv c > 0$ and γ is a critical point of $L : \mathscr{L} \to \mathbb{R}$.

(3) γ is a geodesic.

Proof. $(1) \Leftrightarrow (2)$ by corollary 7.1 and $(3) \Rightarrow (1), (2)$ is trivial.
And by $(1), (2)$, for every vector field along γ with $V(a) = V(b) = 0$, we have

$$\int_a^b \left\langle V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma' \right\rangle \mathrm{d}t = 0.$$

Suppose $\exists t_0 \in [a, b], s.t. \gamma''(t_0) \neq 0$.
So there is a neighborhood U around t_0, such that for every $t \in U$, $|\gamma''(t)| > 0$.
Moreover, we let

$$V_{\gamma(t)} = f(t)\gamma''(t)$$

where $f(t)$ is a bump function for $\{t_0\}$ supported in U.
Then

$$\int_a^b \langle V, \gamma'' \rangle \mathrm{d}t = \int_U f(t) \left|\gamma''(t)\right|^2 \mathrm{d}t > 0,$$

contradiction! □

Thm 7.3. *Let $\gamma \in \mathscr{L}$ be a C^∞ curve TFAE:*

(1) γ is parametrized proportional to the arclength and γ minimizes the length, i.e.

$$L(\gamma) = \inf_{\tilde{\gamma} \in \mathscr{L}} L(\tilde{\gamma})$$

(2) γ minimizes E, i.e.

$$E(\gamma) = \inf_{\tilde{\gamma} \in \mathscr{L}} E(\tilde{\gamma})$$

Each statements implies γ is a geodesic, and it is called a minimal geodesic.

Proof. $(1) \Rightarrow (2)$: For any $\tilde{\gamma} \in \mathscr{L}$,

$$L(\tilde{\gamma})^2 \leqslant 2(b - a)E(\tilde{\gamma}).$$

And

$$L(\gamma) = \int_a^b |\gamma'| \mathrm{d}t = (b - a)\left|\gamma'\right|, E(\gamma) = \frac{1}{2} \int_a^b |\gamma'|^2 \mathrm{d}t = \frac{(b - a)}{2} \left|\gamma'\right|^2$$

In this case,

$$2(b - a)E(\gamma) = L(\gamma)^2 = L(\tilde{\gamma})^2 \leqslant 2(b - a)E(\tilde{\gamma}).$$

$(2) \Rightarrow (1)$: Since γ is a critical point of E.
So γ is parametrized proportional to the arclength.
Suppose there is $\tilde{\gamma} \in \mathscr{L}$ such that $L(\tilde{\gamma}) < L(\gamma)$.
WLOG, assume $\tilde{\gamma}$ is regular, and by reparametrize, we can get $\hat{\gamma}$.
Then we conclude that

$$(b - a)\left|\hat{\gamma}'\right| = L(\hat{\gamma}) = L(\tilde{\gamma}) < L(\gamma) = (b - a)|\hat{\gamma}|.$$

But on the other hand,

$$\frac{b - a}{2} \left|\hat{\gamma}'\right|^2 = E(\hat{\gamma}) = E(\tilde{\gamma}) \geqslant E(\gamma) = \frac{b - a}{2} \left|\gamma'\right|^2,$$

contradiction! □

7.2 The Second Variation

Def 7.3. An second variation of a smooth curve $\gamma : [a,b] \to M$ is

$$\alpha(t, s_1, s_2) : [a,b] \times (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \to M, (t,0,0) \mapsto \gamma(t)$$

with variational vector fields V, W such that

$$V = \alpha_* \left(\frac{\partial}{\partial s_1} \right) \bigg|_{s_1 = s_2 = 0}, W = \alpha_* \left(\frac{\partial}{\partial s_2} \right) \bigg|_{s_1 = s_2 = 0}$$

Thm 7.4. *If α is an arbitrary second variation of γ, then*

$$\begin{aligned} \frac{\partial^2}{\partial s_1 \partial s_2}\bigg|_{s_1=s_2=0} E(\alpha(\bullet, s_1, s_2)) &= \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W \right\rangle \mathrm{d}t - \int_a^b \mathrm{R}(V, \gamma', \gamma', W) \mathrm{d}t \\ &+ \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \left(\bar{\nabla}_{\frac{\partial}{\partial s^1}} \alpha_* \left(\frac{\partial}{\partial s_2} \right) \right) \bigg|_{s_1=s_2=0}, \gamma' \right\rangle \mathrm{d}t \end{aligned}.$$

Proof.

$$\frac{\partial}{\partial s_2} \left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle = 2 \left\langle \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_2} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle.$$

$$\begin{aligned} &\frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} \left\langle \alpha_* \left(\frac{\partial}{\partial t} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle = \frac{\partial}{\partial s_1} \left\langle \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_2} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle \\ &= \left\langle \bar{\nabla}_{\frac{\partial}{\partial s_1}} \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_2} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle + \left\langle \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_2} \right), \bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle \\ &= \left\langle \bar{\nabla}_{\frac{\partial}{\partial t}} \bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle + \left\langle \bar{\mathrm{R}} \left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial t} \right) \alpha_* \left(\frac{\partial}{\partial s_2} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle \\ &+ \left\langle \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_2} \right), \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_1} \right) \right\rangle \end{aligned}$$

And since

$$\left\langle \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_1} \right), \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_2} \right) \right\rangle \bigg|_{s_1=s_2=0} = \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W \right\rangle$$

$$\left\langle \bar{\nabla}_{\frac{\partial}{\partial t}} \bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle \bigg|_{s_1=s_2=0} = \left\langle \hat{\nabla}_{\frac{\partial}{\partial t}} \left(\bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2} \right) \right) \bigg|_{s_1=s_2=0}, \gamma' \right\rangle$$

$$\left\langle \bar{R} \left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial t} \right) \alpha_* \left(\frac{\partial}{\partial s_2} \right), \alpha_* \left(\frac{\partial}{\partial t} \right) \right\rangle \bigg|_{s_1=s_2=0} = R(V, \gamma', W, \gamma').$$

So the proof is completed. ☐

Def 7.4. a second variation α of γ is called proper if

$$\alpha(a, s_1, s_2) = \gamma(a), \alpha(b, s_1, s_2) = \gamma(b).$$

Thm 7.5. *Let $\gamma : [a,b] \to (M, g)$ be a geodesic and α is a proper second variation of γ with variational vector fields V and W, then*

$$\frac{\partial^2}{\partial s_1 \partial s_2}\bigg|_{s_1=s_2=0} E(\alpha(\bullet, s_1, s_2)) = \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W \right\rangle \mathrm{d}t - \int_a^b \mathrm{R}(V, \gamma', \gamma', W) \mathrm{d}t$$

Proof. Since α is proper, so

$$\left\langle \bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2}\right)\Big|_{s_1=s_2=0}, \gamma' \right\rangle \Bigg|_a^b = 0$$

By the integration by parts argument,

$$\int_a^b \left\langle \hat{\nabla}_{\frac{\partial}{\partial t}} \left(\bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2}\right)\right)\Big|_{s_1=s_2=0}, \gamma' \right\rangle = \int_a^b \left\langle \left(\bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2}\right)\right)\Big|_{s_1=s_2=0}, \hat{\nabla}_{\frac{\partial}{\partial t}}\gamma' \right\rangle = 0$$

Hence by theorem 7.4, the proof is completed. □

Remark 7.1. In particular, if $\sec(M,g) \leqslant 0$, then

$$\frac{\partial^2}{\partial s^2}\Big|_{s=0} E \geqslant 0.$$

And the integration by parts argument shows that

$$\int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W \right\rangle \mathrm{d}t = -\int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, W \right\rangle \mathrm{d}t$$

So

$$\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} E = -\int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V + \mathrm{R}(V,\gamma')\gamma', W \right\rangle \mathrm{d}t$$

Coro 7.2. *For a unit-speed curve γ and proper second variation α of γ,*

$$\begin{aligned}\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} L(\alpha(\bullet, s_1, s_2)) &= \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W \right\rangle \mathrm{d}t - \int_a^b \mathrm{R}(V,\gamma',\gamma',W)\mathrm{d}t \\ &\quad - \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma' \right\rangle \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W, \gamma' \right\rangle \mathrm{d}t\end{aligned}$$

Moreover, if γ is a geodesic,

$$\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} L(\alpha(\bullet, s_1, s_2)) = -\int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V^\perp + \mathrm{R}\left(V^\perp, \gamma'\right)\gamma', W^\perp \right\rangle \mathrm{d}t,$$

where $V^\perp, W^\perp$ are the normal components of V, W w.r.t. γ'.

Proof.

$$\frac{\partial^2 L(\alpha(\bullet,s_1,s_2))}{\partial s_1 \partial s_2} = \int_a^b \left(\frac{1}{2\left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|} \frac{\partial^2 \left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^2}{\partial s_1 \partial s_2} - \frac{1}{4\left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^3} \frac{\partial \left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^2}{\partial s_1} \cdot \frac{\partial \left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^2}{\partial s_2} \right)$$

And since γ is unit-speed and

$$2\left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma' \right\rangle = \frac{\partial}{\partial s_1}\Big|_{s_1=s_2=0} \left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^2, 2\left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W, \gamma' \right\rangle = \frac{\partial}{\partial s_2}\Big|_{s_1=s_2=0} \left|\alpha_*\left(\frac{\partial}{\partial t}\right)\right|^2$$

we can obtain that

$$\begin{aligned}\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} L(\alpha(\bullet, s_1, s_2)) &= \frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} E(\alpha(\bullet, s_1, s_2)) \\ &\quad - \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma' \right\rangle \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W, \gamma' \right\rangle \mathrm{d}t\end{aligned}$$

And if γ is a geodesic, notice that

$$\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \langle V, \gamma' \rangle \gamma' = \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle V, \gamma' \rangle\right)\gamma' = \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma' \right\rangle \gamma',$$

$$\mathrm{R}(V^\perp, \gamma', \gamma', W^\perp) = \mathrm{R}(V, \gamma', \gamma', W^\perp).$$

Hence we can easily obtain the desired formula. □

7.3 Jacob Field and Conjugate points

We now want to consider the variation α through geodesic with variation field V. Recall that the geodesic has locally minimal energy, so for any variation field W, the variation of energy along geodesic $\alpha(\bullet, s)$ is zero. Then if we take a variation along α, the second variation of energy is still zero. By remark 7.1,

$$\left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V + \mathrm{R}(V,\gamma')\gamma', W \right\rangle \equiv 0.$$

Hence we obtain the definition of Jacobi field:

Def 7.5. Let $\gamma : [a,b] \to (M,g)$ be a geodesic, a vector field $V \in \Gamma(\gamma^* TM)$ is called a Jacobi field if

$$\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V + \mathrm{R}(V,\gamma')\gamma' = 0.$$

A Jacobi-field V with $\langle V, \gamma' \rangle = 0$ is called normal Jocobi field.

Exam 7.1. *$V = \gamma', t\gamma'$ are Jacobi field, but $V = t^2\gamma'$ is not Jacobi field.*

Prop 7.1. *Let $\gamma : [a,b] \to (M,g)$ be a geodesic*

(1) If $\alpha : [a,b] \times (-\varepsilon, \varepsilon) \to (M,g)$ be a family of geodesics and $\alpha(t,0) = \gamma(t)$, then

$$V = \alpha_* \left(\frac{\partial}{\partial s} \right) \bigg|_{s=0}$$

is a Jacobi field.

(2) Given $v_1, v_2 \in T_{\gamma(a)}M$, $\exists!$Jacobi field J around γ, such that

$$J(a) = v_1, J'(a) = v_2.$$

Proof. (1)

$$\bar{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \bar{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \alpha_* \left(\frac{\partial}{\partial s} \right) = \bar{\nabla}_{\frac{\partial}{\partial t}} \bar{\nabla}_{\frac{\partial}{\partial s}} \alpha_* \left(\frac{\partial}{\partial t} \right) = \hat{\nabla}_{\frac{\partial}{\partial s}} \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial t} \right) + \bar{\mathrm{R}} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) \alpha_* \left(\frac{\partial}{\partial t} \right).$$

And since $\alpha(\bullet, s)$ is a geodesic for any s.

So when $t = 0$,

$$\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V = \mathrm{R}(\gamma', V)\gamma'.$$

(2) By ODE.

□

Thm 7.6. *Let (M,g) be a Riemannian manifold and $\gamma : [0,1] \to (M,g)$ be a geodesic , then the Jacobi field J along γ with $J(0) = 0, J'(0) = v$ is the variational vector field of α, where*

$$\alpha(t,s) = \exp_{\gamma(0)}(t(\gamma'(0) + sv))$$

Proof.

$$J(t) = \alpha_* \left(\frac{\partial}{\partial s} \right) \bigg|_{s=0} = \left(\mathrm{d} \exp_{\gamma(0)} \right)_{t\gamma'(0)} (tv)$$

So we can obtain

$$J(0) = (\mathrm{d} \exp_{\gamma(0)})_0(0) = 0$$

$$J'(0) = \bar{\nabla}_{\frac{\partial}{\partial s}} \alpha_* \left(\frac{\partial}{\partial t} \right) \bigg|_{s=0} = \bar{\nabla}_{\frac{\partial}{\partial s}} (\mathrm{d} \exp_{\gamma(0)})_0 (\gamma'(0) + sv) \bigg|_{s=0} = \bar{\nabla}_{\frac{\partial}{\partial s}} (\gamma'(0) + sv) \bigg|_{s=0} = v.$$

□

Coro 7.3. *Prove the Taylor expansion formula for Riemannian metric in theorem 3.3.*

Proof. Consider the geodesic

$$\gamma(t) = (tx^1, \cdots, tx^n)$$

and the Jacobi field J along γ with $J(0) = 0, J'(0) = v$ with norm $f = \|J\|^2$. We denote

$$\mathrm{R}X = \mathrm{R}(\gamma', X)\gamma', \mathrm{R}'X = \left(\hat{\nabla}_t \mathrm{R}\right)(\gamma', X)\gamma'$$

So we can conclude that

$$J(0) = 0, J'(0) = W, J''(0) = \mathrm{R}J(0) = 0,$$

$$f(0) = 0, f'(0) = 0, f''(0) = 2\|W\|^2, f^{(3)}(0) = 0$$

$$\begin{aligned} f^{(4)}(0) =& 8\langle(\mathrm{R}J)'(0), W\rangle = 8\langle(\mathrm{R}'J)(0), W\rangle + 8\langle \mathrm{R}W, W\rangle \\ =& 8\langle \mathrm{R}'W, J(0)\rangle + 8\langle \mathrm{R}W, W\rangle = 8\langle \mathrm{R}W, W\rangle \end{aligned}$$

On the other hand,

$$f = \|J\|^2 = \langle tW, tW\rangle_g(\gamma(t)) = t^2 g_{ij}(tx)W^iW^j.$$

So by Taylor expansion,

$$t^2 g_{ij}(tx) = t^2\delta_{ij} - \frac{t^4}{3}\mathrm{R}_{iklj}(p)x^kx^l + o(t^4)$$

Hence we complete the proof. $\square$

Remark 7.2. Using this method, we can compute higher degree terms of $g_{ij}(x)$, the first four terms are

$$\ell_{ij} - \frac{1}{3}\mathrm{R}_{iklj}(p)x^kx^l - \frac{1}{6}\nabla_m \mathrm{R}_{iklj}x^kx^lx^m + \left(\frac{2}{45}\mathrm{R}^k_{ilm}\mathrm{R}_{jpqk} - \frac{1}{20}\nabla_q\nabla_p \mathrm{R}_{ilmj}\right)x^lx^mx^px^q$$

Prop 7.2. *Let J be a Jacobi-field along geodesic $\gamma : [a, b] \to (M, g)$.*

(1) $\langle J(t), \gamma'(t)\rangle = t\langle J'(0), \gamma'(0)\rangle + \langle J(0), \gamma'(0)\rangle$.

(2) If γ is a unit-speed geodesic, then

$$J^\perp = J - \langle J, \gamma'\rangle\gamma' = J(t) - t\langle J'(0), \gamma'(0)\rangle\gamma' + \langle J(0), \gamma'(0)\rangle\gamma'$$

is a normal Jacobi field.

Proof. (1)

$$\langle J, \gamma'\rangle' = \langle J', \gamma'\rangle + \langle J, \gamma''\rangle = \langle J', \gamma'\rangle$$

$$\langle J', \gamma'\rangle' = \langle J'', \gamma'\rangle + \langle J', \gamma''\rangle = \langle J'', \gamma'\rangle = -\langle \mathrm{R}(J, \gamma')\gamma', \gamma'\rangle = 0$$

So $\langle J', \gamma'\rangle$ is a constant, this concludes the desired formula.

(2)

$$\left(J^\perp\right)'' = J'' - \langle J, \gamma'\rangle''\gamma' = J'', \mathrm{R}\left(J^\perp, \gamma'\right) = \mathrm{R}(J, \gamma')$$

So $J^\perp$ is also a Jacobi field and moreover, it is normal.

$\square$

Prop 7.3. *Let (M, g) be a Riemannian manifold with constant sectional curvature K and $\gamma : [0, b] \to M$ be a unit-speed geodesic, then a normal Jacobi field J with $J(0) = 0$ is of the form*

$$J(t) = \begin{cases} mtE(t) & K = 0 \\ \frac{m\sin(\sqrt{K}t)}{\sqrt{K}}E(t) & K > 0 \\ \frac{m\sinh(\sqrt{-K}t)}{\sqrt{-K}}E(t) & K < 0 \end{cases}$$

Where $E(t)$ is any parallel vector field along γ such that

(1) $\langle E(t), \gamma'(t)\rangle = 0$

(2) $|E(t)| = 1$

(3) $\left(\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} J\right)\Big|_{t=0} = mE(0)$

Proof.

$$R(J, \gamma', \gamma', W) = K(g(J, W)g(\gamma', \gamma') - g(J, \gamma')g(W, \gamma')) = Kg(J, W).$$

So $\mathrm{R}(J, \gamma')\gamma' = KJ$, and the Jacobi field equation reduces to

$$J'' + KJ = 0.$$

Let $E(t)$ be a parallel vector field along γ satisfying (1) and (2).
Suppose $J(t) = u(t)E(t)$ for some function u.
Then we have the differential equation

$$u'' + Ku = 0, u(0) = 0$$

And the solution are

$$u(t) = \begin{cases} mt & K = 0 \\ \frac{m\sin(\sqrt{K}t)}{\sqrt{K}} & K > 0 \\ \frac{m\sinh(\sqrt{-K}t)}{\sqrt{-K}} & K < 0 \end{cases}$$

where m is given by

$$m = \frac{J'(0)}{E(0)}.$$

And since the dimension of space of normal Jacobi field with $J(0) = 0$ is $n - 1$.
Hence every normal Jacobi field with $J(0) = 0$ is of the form we desired. □

Def 7.6. Let $\gamma : [a, b] \to (M, g)$ be a geodesic with $\gamma(a) = p, \gamma(b) = q$ for some $a, b \in I$.

(1) We say p and q are conjugate along γ if there exists some non-trivial Jacobi field J, such that $J(a) = J(b) = 0$.

(2) The maximum of such linearly independent Jacobi field is called the multiplicity of the conjugate point q, denoted by $m_\gamma(q)$.

(3) the conjugate set of p is $\mathrm{conj}(p) = \{q \in M | \exists\gamma(a) = p, \gamma(b) = q \text{ is geodesic}, J(a) = J(b) = 0\}$.

Lemma 7.2. *Let $\gamma : [a, b] \to (M, g)$ be a geodesic with $\gamma(a) = p, \gamma(b) = q$ for some $a, b \in I$, then there are at most $(n - 1)$ linearly independent Jacobi fields along γ with $J(a) = J(b) = 0$.*

Proof. By ODE theory, there are at most n linearly independent Jacobi fields with $J(a) = 0$.
But the Jacobi field $J(t) = (t - a)\gamma'(t)$ does not vanish at b. □

Exam 7.2. *Consider the n-sphere $(\mathbb{S}^n, g_{can})$ and let $\gamma : [0, +\infty) \to \mathbb{S}^n$ be a unit-speed geodesic in $\mathbb{S}^n$.*

Since $\mathbb{S}^n$ has constant sectional curvature 1.

So the normal Jacobi field with $J(0) = 0$ is of the form

$$J(t) = m \sin(t) E(t).$$

Hence the antipodal point $\gamma(\pi)$ is conjugate to $\gamma(0)$ along γ with multiplicity $n-1$.

Moreover, $\operatorname{conj}(p) = \{-p\}$ for every $p \in \mathbb{S}^n$.

Exam 7.3. *Consider the n-torus $(\mathbb{T}^n, g_{can})$ and let $\gamma : [0, +\infty) \to \mathbb{S}^n$ be a unit-speed geodesic in $\mathbb{T}^n$.*

Since $\mathbb{T}^n$ is flat.

So the normal Jacobi field with $J(0) = 0$ is of the form

$$J(t) = mtE(t).$$

Hence $\operatorname{conj}(p) = \varnothing$ for every $p \in \mathbb{T}^n$.

Thm 7.7. *Let (M, g) be a Riemannian mainfold, fix $p \in M$ and let $v \in \mathscr{E}_p \subset T_pM, \gamma_v(t) = \exp_p(tv) : [0,1] \to M$ and $q = \gamma_v(1)$.*

Then v is a critical point of $\exp_p : \mathcal{E}_p \to M$ iff q is conjugate to p along γ_v.

Proof. For any $w \in T_v(T_pM)$, let $\alpha(t,s) = \exp_p(t(v + sw))$.

Then the variation vector field is

$$\tilde{J}(t) = \left(\mathrm{d}\exp_p\right)_{tv}(tw), \tilde{J}(0) = 0, \left(\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \tilde{J}\right)(0) = w.$$

If $w \neq 0$ and w lies in the kernel $(\mathrm{d}\exp_p)_v$, then

$$\tilde{J}(1) = (\mathrm{d}\exp_p)_v(w) = 0.$$

So p and q are conjugate along γ.

Conversely, suppose p and q are conjugate along γ by some Jacobi field J such that $J(0) = J(1) = 0$, let

$$w = \left(\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} J\right)(0).$$

Then by the uniqueness of Jacobi field with initial conditions, $J = \tilde{J}$.

Hence $(\mathrm{d}\exp_p)_v(w) = 0$, *i.e.*$v$ is a critical point of $\exp_p : \mathcal{E}_p \to M$. □

Coro 7.4. *Let (M, g) be a complete Riemannian manifold. Fix $p \in M$ and $v \in T_pM$, TFAE:*

(1) v is a critical point of $\exp_p : T_pM \to M$.

(2) $q = \exp_p(v)$ is conjugate to p along $r_v(t) = \exp_p(tv) : [0,1] \to M$.

Coro 7.5. *Let (M, g) be a complete Riemannian manifold, $p \in M$.*

If $q \in \operatorname{conj}(p)$, then $\exists v \in T_pM$ such that $\exp_p(v) = q$ and

(1) v is a critical point of $\exp_p : T_pM \to M$.

(2) q is conjugate to p along $\gamma_v(t) = \exp_p(tv) : [0,1] \to M$.

Prop 7.4. *If (M, g) is a complete Riemannian manifold with non-positive sectional curvature, then for every $p \in M$, $\operatorname{conj}(p) = \varnothing$.*

Moreover, the exponential map $\exp_p : T_pM \to M$ is a local diffeomorphism.

Proof. Suppose $q \in \mathrm{conj}(p)$ is conjugate to p along geodesic $\gamma : [0,1] \to M$ by Jacobi field J such that $J(0) = J(1) = 0$.

Consider $f = ||J||^2$, then

$$f''(t) = \langle J'', J\rangle + \left\|J'\right\|^2 = -\mathrm{R}(J, \gamma', \gamma', J) + \left\|J'\right\|^2 \geqslant 0.$$

And since $f(t) \geqslant 0, f(0) = f(1) = 0$.

So $f \equiv 0$, i.e. J is trivial, contradiction! □

Remark 7.3. Further more, we can prove that the exponential map is covering map, we will discuss these in next chapter.

7.4 Index form

Def 7.7. Suppose $\gamma : [a, b] \to (M, g)$ is a non-trivial geodesic, the index form $I_\gamma : \Gamma(\gamma^*TM) \times \Gamma(\gamma^*TM) \to \mathbb{R}$, is defined as

$$I_\gamma(V, W) = \int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W \right\rangle \mathrm{d}t - \int_a^b \mathrm{R}(V, \gamma', \gamma', W)\mathrm{d}t$$

Lemma 7.3. *V is a Jacobi field iff*

$$I_\gamma(V, W) = 0$$

for every variational vector field W.

Proof. If V is a Jacobi field, then it is trivial.

If $I_\gamma(V, W) = 0$ for every variational vector field W, then

$$\int_a^b \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V + \mathrm{R}(V, \gamma')\gamma', W \right\rangle \mathrm{d}t \equiv 0.$$

So using the similar method in theorem 7.2, we can obtain

$$\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V + \mathrm{R}(V, \gamma')\gamma' = 0.$$

Hence V is the Jacob field. □

Def 7.8. If V is piecewise C^∞ over $[a, b]$, *i.e.* $a \leqslant t_0 \leqslant \cdots \leqslant t_{k+1} = b, V_i = V\big|_{[t_i,t_{i+1}]}$ is smooth, we define

$$I_\gamma(V, W) = \sum_{i=0}^{k} \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i, W \right\rangle \Bigg|_{t_i}^{t_{i+1}} - \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i + \mathrm{R}(V_i, \gamma')\gamma', W \right\rangle \mathrm{d}t.$$

Prop 7.5. *If $I_\gamma(V, W) = 0$ for any piecewise C^∞ variation vector field W, then V is a C^∞ Jacobi field.*

Proof. Let W be a piecewise C^∞ variational vector field such that $W(t_i) = 0$ for every i, then

$$\sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i + \mathrm{R}(V_i, \gamma')\gamma', W \right\rangle \mathrm{d}t = I_\gamma(V, W) = 0.$$

And since every $W\big|_{(t_i,t_{i+1})}$ are independent, so

$$\int_{t_i}^{t_{i+1}} \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i + \mathrm{R}(V_i, \gamma')\gamma', W \right\rangle \mathrm{d}t = 0$$

for all $i \in [0,k]$, otherwise we can times a constant on $W\big|_{(t_i,t_{i+1})}$.

Therefore $\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V_i + \mathrm{R}(V_i,\gamma')\gamma' \equiv 0$ for every i,*i.e.* V_i is Jacobi field.

Moreover, fix $i \in \{0,\cdots,k\}$ and let

$$W(t_j) = \begin{cases} 0 & i = j \\ \left(\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V_j\right)(t_j) - \left(\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V_{j-1}\right)(t_j) & i \neq j \end{cases}$$

Then

$$0 = I_\gamma(V,W) = \left|\left(\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V_i\right)(t_i) - \left(\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V_{i-1}\right)(t_i)\right|^2$$

Hence by gluing up every $\{V_i\}$, we obtain the Jacobi field V. □

Coro 7.6. *Let $\gamma : [a,b] \to (M,g)$ be a unit-speed curve, if γ is a local minimal geodesic, then $I_\gamma(V,V) \geqslant 0$ for every variational vector field V.*

Proof. Since γ is locally minimal, for every proper variation α with variational vector field V,

$$\frac{\partial^2}{\partial s^2}\bigg|_{s=0} E(\alpha(\bullet,s)) = I_\gamma(V,V) \geqslant 0.$$

□

Lemma 7.4. *Let $\gamma : [a,b] \to (M,g)$ be a unit-speed geodesic. If γ has no conjugate point, then there exist Jacobi fields $J_2,\cdots,J_n$ along γ such that*

(1) $J_i(a) = 0, i \geqslant 2$ and $\{\gamma'(b), J_2(b),\cdots,J_n(b)\}$ is an orthonormal basis of of $T_{\gamma(b)}M$.

(2) $\langle J_i(t),\gamma'(t)\rangle \equiv 0$ for $t \in [a,b], i \geqslant 2$.

(3) $\{\gamma'(t), J_2(t),\cdots,J_n(t)\}$ be linearly independent for $t \in (a,b]$.

Proof. (1) Let $\{\gamma'(b), e_2,\cdots,e_n\} \in T_{\gamma(b)M}$ be an orthonormal basis.

Since there is no conjugate points along γ.

There exist a unique Jacobi field J_i such that

$$J_i(a) = 0, J_i(b) = e_i,$$

for every $2 \leqslant i \leqslant n$.

(2) Since

$$\langle J_i(t),\gamma'(t)\rangle'' = 0$$

And J_i,γ' are normal at a and b.

So J_i are normal Jacobi field on γ.

Moreover, $\mathrm{span}\{J_2(t),\cdots,J_n(t)\}$ is linearly independent with γ'.

(3) Suppose $\exists c \in (a,b), s.t.\{J_2(c),\cdots,J_n(c)\}$ are linearly dependent, let

$$W(t) = \sum_{i=2}^{n}\lambda_i J_i(t)$$

such that $W(c) = 0$.

So $W\big|_{[a,c]}$ is a Jacobi field with $W(a) = W(c) = 0$.

Therefore $W(t) \equiv 0$ over $[a,c]$.

Hence $W(b) = 0, i.e.\lambda_2 = \cdots\lambda_n = 0$, contradiction!

□

Def 7.9. $\mathcal{V}_0 = \{V \in \Gamma(\gamma^*TM) | V(a) = V(b) = 0\}, \mathcal{N}_0 = \{v \in \mathcal{V}_0 | \langle v, \gamma' \rangle \equiv 0\}$.

Thm 7.8. *Let $\gamma : [a, b] \to (M, g)$ be a unit speed geodesic, then*

(1) If γ has no conjugate points, then I_γ is positive definite on $\mathcal{N}_0, \mathcal{V}_0$.

(2) If $\gamma(a)$ and $\gamma(b)$ are the only conjugate points along γ, then $I_\gamma(V, V) \geqslant 0$ on $\mathcal{N}_0, \mathcal{V}_0$, moreover, $I_\gamma(V, V) = 0 \Leftrightarrow V$ is a Jacobi field and $V \in \mathcal{N}_0$.

(3) γ has an interior conjugate point$\Leftrightarrow \exists$some $V \in \mathcal{N}_0$, s.t.$I_\gamma(V, V) < 0$. In particular, γ is not a local minimal geodesic.

Proof. (1) Suppose $V \in \mathcal{N}_0$, let $\{\gamma'(b), e_2, \cdots, e_n\}$ be an orthonormal basis of $T_{\gamma(b)}M$ and $J_i(t)$ are Jacobi fields such that $J_i(a) = 0, J_i(b) = e_i$.

Since J_i are perpendicular to γ', so let

$$V = \sum_{i=2}^{n} f^i(t) J_i(t),$$

where $f^i(b) = 0$ for each i and

$$\begin{aligned}
I_\gamma(V, V) &= \int_a^b \left\langle \left(f^i J_i\right)', \left(f^j J_j\right)' \right\rangle \mathrm{d}t - \int_a^b f^i f_j \mathrm{R}(J_i, \gamma', \gamma', J_j) \mathrm{d}t \\
&= \int_a^b \left(\left(f^i\right)' f^j \langle J_i, J_j' \rangle + \left(f^j\right)' f^i \langle J_i', J_j \rangle + f^i f^j \langle J_i', J_j' \rangle \right) \mathrm{d}t \\
&+ \int_a^b \left(\left(f^i\right)' \left(f^j\right)' \langle J_i, J_j \rangle - f^i f^j \mathrm{R}(J_i, \gamma', \gamma', J_j) \right) \mathrm{d}t
\end{aligned}$$

On the other hand,

$$\left(\langle J_i, J_j' \rangle - \langle J_i', J_j \rangle\right)' = \langle J_i, J_j'' \rangle - \langle J_i'', J_j \rangle = \langle \mathrm{R}(J_i, \gamma')\gamma', J_j \rangle - \langle \mathrm{R}(J_j, \gamma')\gamma', J_i \rangle = 0$$

Therefore

$$\langle J_i, J_j' \rangle = \langle J_i', J_j \rangle,$$

$$\begin{aligned}
I_\gamma(V, V) &= \int_a^b \left(\left(f^i f^j \langle J_i', J_j \rangle\right)' - f^i f^j \langle J_i'', J_j \rangle \right) \mathrm{d}t \\
&+ \int_a^b \left(\left(f^i\right)' \left(f^j\right)' \langle J_i, J_j \rangle - f^i f^j \mathrm{R}(J_i, \gamma', \gamma', J_j) \right) \mathrm{d}t \\
&= f^i f^j \langle J_i', J_j \rangle \Big|_a^b + \int_a^b \left(f^i\right)' \left(f^j\right)' \langle J_i, J_j \rangle \mathrm{d}t \\
&= \int_a^b \left\| \left(f^i\right)' J_i \right\|^2 \mathrm{d}t \geqslant 0
\end{aligned}$$

The identity holds iff $\left(f^i\right)' J_i = 0, i.e. f_i$ are constant.

And since $f_i(b) = 0$.

So $f_i \equiv 0, i.e. V$ is trivial.

Hence I_γ is positive definite on $\mathcal{N}_0$.

Now we suppose $V \in \mathcal{V}_0$ and define

$$V^\perp = V - \langle V, \gamma' \rangle \gamma' \in \mathcal{N}_0.$$

So

$$
\begin{aligned}
I_r(V,V) =& I_r\left(V^{\perp},V^{\perp}\right)+2I_\gamma\left(V,\langle V,\gamma'\rangle\gamma'\right)-I_\gamma\left(\langle V,\gamma'\rangle\gamma',\langle V,\gamma'\rangle\gamma'\right)\\
=& I_r\left(V^{\perp},V^{\perp}\right)+2\int_a^b\left\langle V',\langle V',\gamma'\rangle\gamma'\right\rangle\mathrm{d}t-\int_a^b\left|\langle V',\gamma'\rangle\right|^2\mathrm{d}t\\
=& I_r\left(V^{\perp},V^{\perp}\right)+\int_a^b\left|\langle V',\gamma'\rangle\right|^2\mathrm{d}t\geqslant 0
\end{aligned}
$$

The identity holds iff $V^{\perp}=0, \langle V',\gamma'\rangle=0, i.e. V=0$.

Hence I_γ is positive definite on $\mathcal{V}_0$.

(2) Define $\gamma^c:[a,c]\to(M,g), \gamma^c=\gamma$.

Then I_{γ^c} is positive define.

Consider a parallel frame $\{\gamma'=e_1,e_2,\cdots,e_n\}$ along γ.

If $V\in\mathcal{V}_0$, then let

$$
V=\sum_{i=1}^n f^i(t)e_i(t).
$$

So there is an induced variational vector field along γ^c given by

$$
V^c(t)=\sum f^i\left(\frac{(b-a)t+a(c-b)}{c-a}\right)e_i(t)
$$

Therefore $I_{\gamma^c}(V^c,V^c)\geqslant 0$ and

$$
I_\gamma(V,V)=\lim_{c\to b}I_{\gamma^c}(V^c,V^c)\geqslant 0.
$$

If $I_\gamma(V,V)=0$, then $I_\gamma(V,W)=0$ for every $W\in\mathcal{V}_0$ since I_γ is semi-positive definite.

So V is Jacobi field with $V(a)=V(b)=0, i.e. V\in\mathcal{N}_0$.

(3) If $\gamma(a)$ is conjugate to $\gamma(c)$ for some $c\in(a,b)$, then there exists a Jacobi field J_1 along γ^c with $J_1(a)=J_1(c)=0$.

Consider

$$
J(t)=\begin{cases}J_1(t) & a\leqslant t\leqslant c\\ 0 & c\leqslant t\leqslant b\end{cases}
$$

Then it is easy to see that $I_\gamma(J,J)=0$.

Let W be a C^∞ variational vector field with

$$
W(c)=-\left(\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J\right)(c^-).
$$

So for sufficiently small ε,

$$
I_\gamma(J,W)=\left\langle\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J,W\right\rangle\Big|_a^c=-|W(c)|^2<0
$$

$$
I_\gamma(J+\varepsilon W,J+\varepsilon W)=2\varepsilon I_\gamma(J,W)+\varepsilon^2 I_\gamma(W,W)<0
$$

Hence $\exists V\in\mathcal{V}_0, s.t. I_\gamma\left(V^{\perp},V^{\perp}\right)\leqslant I_\gamma(V,V)<0$.

$\square$

Coro 7.7. *Let γ be a unit-speed geodesic that has no conjugate point, and V, W are vector field along γ such that*

$$V(a) = W(a), V(b) = W(b).$$

If V is a Jacobi field, then

$$I_\gamma(V, V) \leqslant I_\gamma(W, W)$$

and the identity holds iff $V = W$.

Proof.

$$I_\gamma(V - W, V - W) = I_\gamma(V, V) + I_\gamma(W, W) - 2I_\gamma(V, W) \geqslant 0$$

$$I_\gamma(V, V) = \langle V, V' \rangle \Big|_a^b = \langle V', W \rangle \Big|_a^b = I_\gamma(V, W)$$

So $I_\gamma(V, V) \leqslant I_\gamma(W, W)$ and the identity holds iff $V - W = 0$. $\square$

Def 7.10. Let $B : V \times V \to \mathbb{R}$ be an symmetric bilinear form.

(1) The index of B is defined to be the maximal dimension of all subspaces W of V such that $B : W \times W \to \mathbb{R}$ is negative definite, denoted as $\mu(B)$.

(2) The nullity of B is the dimension of the null space

$$\mathcal{N}(B) = \{v \in V | B(v, w) = 0 \text{ for all } w \in V\}$$

B is said to be degenerate if the nullity is positive.

Coro 7.8. *If $\gamma(b)$ is a conjugate point of $\gamma(0)$ along a unit-speed geodesic $\gamma : [0, b] \to (M, g)$, then*

$$m_\gamma(\gamma(b)) = \dim \mathcal{N}(I_\gamma).$$

Proof. By theorem 7.8(2), $\mathcal{N}(I_\gamma) = \{V \in \mathcal{N}_0 | V \text{ is a Jacobi field}\}$.
So by definition $m_\gamma(\gamma(b)) = \dim \mathcal{N}(I_\gamma)$. $\square$

Thm 7.9 (Morse index theorem)**.** *Let $\gamma : [0, b] \to (M, g)$ be a unit-speed geodesic, then the index of I_γ is finite and equals to the number of points $\gamma(t)$ with $0 < t < b$ that conjugate to $\gamma(0)$, each counted with its multiplicity,* i.e.

$$\mu(I_\gamma) = \sum_{\substack{0<t<b \\ \gamma(t)\in\mathrm{conj}(\gamma(0))}} \dim \mathcal{N}(I_{\gamma^t}).$$

Exam 7.4. *Consider $(\mathbb{S}^1 \times \mathbb{R}, g_{can})$, which is flat.*
Let $\gamma : [a, b] \to M$ be a unit-speed geodesic, we claim that γ has no conjugate point.
Otherwise, there exists Jacobi field J along γ such that $J(a) = J(c) = 0$, so

$$J'' + R(J, \gamma')\gamma' = 0, \text{i.e.} J'' = 0.$$

For $J(t) = J^i(t)e_i(t)$ where $\{e_i(t)\}$ is a parallel basis, we have

$$J''(t) = \left(J^i\right)''(t)e_i(t) = 0, \text{i.e.} \left(J^i\right)'' = 0.$$

So $J \equiv 0$ is trivial, contradiction!
Hence γ is a locally minimal geodesic, while it may not be the minimal geodesic.

7.5 The cut locus and injective radius

Def 7.11. Suppose (M, g) is a complete Riemannian manifold and $p \in M, v \in T_pM$.
The cut time of point (p, v) is defined as

$$t_{cut}(p, v) = \sup \left\{ b > 0 \middle| \gamma_v\big|_{[0,b]} \text{ is a minimal geodesic} \right\}.$$

Suppose $t_{cut}(p, v) < +\infty$, the cut point of p along γ_v is $\gamma_v(t_{cut}(p, v)) \in M$.

(1) The cut locus of p, denoted by $\mathrm{cut}(p)$, is the set of all $q \in M$, such that q is the cut point of p along some geodesic.

(2) The tangent cut locus of p is defined as

$$T_{cut(p)} = \left\{ v \in T_pM \middle| |v| = t_{cut}\left(p, \frac{v}{|v|}\right) \right\}.$$

(3) the injective radius domain of p is

$$\Sigma(p) = \left\{ v \in T_pM \middle| |v| < t_{cut}\left(p, \frac{v}{|v|}\right) \right\}.$$

Prop 7.6. *The cut point(if exists) occur at or before the first conjugate point along every geodesic.*

Proof. Suppose $\gamma : [0, b] \to M$ is a geodesic such that $\gamma(c) \in \mathrm{conj}(\gamma(0))$ with $c \in [0, b)$.
Then by theorem 7.8(3), γ is not local minimal.
So the cut point of $\gamma(0)$ along γ is $< b$, contradiction! □

Exam 7.5. *(1) For* $(\mathbb{S}^n, g_{can}), \mathrm{cut}(p) = \mathrm{conj}(p) = \{-p\}$.

(2) For $(\mathbb{S}^1 \times \mathbb{R}, g_{can}), \mathrm{conj}((p, x)) = \varnothing, \mathrm{cut}((p, x)) = \{p\} \times \mathbb{R}$.

Thm 7.10. *Let* (M, g) *be a complete Riemannian manifold and* $p \in M, v \in T_pM$ *with* $|v| = 1$, $t_{cup}(p, v) = c \in (0, +\infty]$

(1) If $0 < b < c$, *then*

(a) $\gamma_v\big|_{[0,b]}$ *has no conjugate points*

(b) $\gamma_v\big|_{[0,b]}$ *is the unique minimal unit-speed geodesic connecting* $\gamma_v(0)$ *and* $\gamma_v(b)$.

(2) If $c < +\infty$, *then* $\gamma_v\big|_{[0,c]}$ *is a minimal geodesic connection* $\gamma_v(0)$ *and* $\gamma_v(c)$.

Moreover, one or both the following statements hold.

(a) $\gamma_v(c)$ *is conjugate to* p *along* γ_v

(b) There are two or more unit speed minimal geodesics connecting $\gamma_v(0), \gamma_v(c)$.

Proof. (1) By proposition 7.6, $\gamma_v(t)$ can not be conjugate to p for $0 < t \leqslant b$.

And by the definition of $t_{cut}(p, v)$, there exists some b' such that $b < b' < c$ such that $\gamma_v\big|_{[0,b']}$ is minimal.

Suppose the exists a unit-speed curve $\mu : [0, c] \to M$ with $\mu(0) = p, \mu(c) = \gamma(b)$ and $c \leqslant b$.

Then $b' = d(p, \gamma(b')) \leqslant d(p, \gamma(b)) + d(\gamma(b), \gamma(b')) \leqslant c + b' - b$.

So $c = b$.

And consider the curve $\tilde{\mu}\big|_{[0,b]} = \mu$, $\tilde{\mu}\big|_{[b,b']} = \gamma_v\big|_{[b,b']}$.

Then $L(\tilde{\mu}) = d(p,q)$, $i.e.\tilde{\mu}$ is smooth geodesic.

Hence $\mu'(b) = \gamma\big|_v' (b)$, $i.e.\gamma_v = \mu$ is unique.

(2) We assume $\gamma_v(c)$ is not conjugate to p along γ_v.

There exists a sequence $\{b_i\}$ decreasing to c such that $\gamma_v : [0, b_i] \to M$ is well-defined but not a minimal geodesic by definition.

Let $\gamma_i : [0, a_i] \to M$ be a unit-speed minimal geodesic connection p and $\gamma_v(b_i)$.

Suppose $w_i = \gamma_i'(0) \in T_pM, \gamma_i(t) = \exp_p(tw_i)$.

By compactness and passing to a subsequence, we assume $a_i \to a, w_i \to w$, then

$$\exp_p(aw) = \lim \exp_p(a_i w_i) = \lim \gamma_i(a_i) = \lim \gamma_v(b_i) = \gamma_v(c).$$

So $\gamma_w : [0, a] \to M, t \mapsto \exp_p(tw)$ is a unit-speed geodesic connecting p and $\gamma_v(c)$.

Moreover,

$$c = d(p, \gamma_v(c)) = \lim d(p, \gamma_i(a_i)) = d(p, \exp_p(aw)) = a$$

Therefore γ_w is a minimal geodesic, we shall show that $w \neq v$.

Since cv is not a critical point of $\exp_p : T_pM \to M$ and so it is locally injective in a small neighborhood U of cv.

But $\exp_p(b_i v) = \exp_p(a_i w_i)$.

So for large i, $b_i v \in U$ and $a_i w_i \notin U$.

Hence $cv \neq aw$, $i.e.v \neq w$.

□

Remark 7.4. In many usual examples, there seems to have more than one minimal geodesics passing through p and q when p, q are conjugate along γ.

However, this is not necessary.

We consider the meridian γ of parabolic starting at $p = (x_0, y_0, z_0)$ for $z_0 > 1$.

Then γ is obviously minimal around 0.

but when $\gamma(t) = (-x_0, -y_0, z_0)$, the length of latitude is $\pi\sqrt{z_0} < 2\left(z_0 + \sqrt{z_0}\right) < L\left(\gamma\big|_{[0,t]}\right)$.

So while going through γ, there is first only one minimal geodesic, and at some point, the single minimal geodesic bifurcates into two, which are symmetrical about the plane of γ.

That point is called bifurcation point, and obviously its a conjugate point of p along γ with only one minimal geodesic γ.

This type of point has some good properties, you can search for them if you are interested.

Thm 7.11. *(M, g) is complete, $p \in M$, then*

(1) $t_{cut} : STM \to (0, +\infty]$ *is continuous.*

(2) $T_{cut}(p)$ *is the boundary* $\Sigma(p)$.

(3) $\exp_p(T_{cut}(p)) = \operatorname{cut}(p)$.

(4) $\operatorname{cut}(p)$ *is closed subset in* M *of measure* 0.

(5) If M *is compact, then* $\operatorname{cut}(p)$ *is compact.*

(6) $\exp_p : \Sigma(p) \to M \backslash \mathrm{cut}(p)$ *is a diffeomorphism.*

(7) $\exp_p : \overline{\Sigma(p)} \to M$ *is surjective.*

(8) $\mathrm{inj}_p(M, g) = d(p, \mathrm{cut}(p))$.

(9) $\mathrm{inj} : M \to (0, +\infty]$ *is continuous.*

Proof. (1) Consider a sequence (p_i, v_i) converges to (p, v) in STM.

Let $c_i = t_{cut}(p_i, v_i)$ and $C = t_{cut}(p, v)$, assume a subsequence of c_i converges to c, we shall proof that $c \leqslant C$ and $C \leqslant c$.

We first assume $c < +\infty$.

By continuity of distance function, $d(p, \exp_p(cv)) = \lim d(p_i, \exp_{p_i}(c_i v_i)) = \lim c_i = c$.

So γ_v is minimizing on $[0, b]$, $i.e. C \geqslant c$.

By theorem 7.10, there must be infinite (p_i, v_i) such that $\gamma_{v_i}(c_i)$ is conjugate to p_i along γ_{v_i}, or infinite (p_i, v_i) such that for some $w \neq v \in ST_{p_i}M$, $\exp_{p_i}(tw_i)$ is minimizing and $\exp_{p_i}(c_i w_i) = \exp_{p_i}(c_i v_i)$.

For the first case, $(p_i, c_i v_i)$ are critical points of the exponential map.

By continuity, (p, cv) is also a critical points of the exponential map, *i.e.* $\gamma_v(c)$ is the conjugate point of p and so $C \leqslant c$.

For the second case, by passing to a subsequence, we assume $w_i \to w$.

And WLOG, we may assume that $\gamma_v(c)$ is not a conjugate point of p.

Then exp is locally injective in a small neighborhood of (p, cv).

So similar to the proof of theorem 7.10, we can conclude that $w \neq v$, $i.e. C \leqslant c$.

Now suppose $c = +\infty$.

Then for $c_0 > 0$, γ_{v_i} is minimizing on $[0, c_0]$ for sufficiently large i.

By continuity, $d(p, \exp(p, c_0 v)) = \lim c_0 = c_0$, $i.e. \gamma_v$ is minimizing on $[0, c_0]$.

So $c \leqslant C$, and $C \leqslant c$ is trivial.

Hence $c \equiv C$, *i.e.* $\lim\limits_{i \to +\infty} c_i = C$.

(2) since t_{cut} is continuous.

So $\Sigma(p)$ is open and $T_{cut}(p) = \partial\Sigma(p)$.

(3) For every $q \in \mathrm{cut}(p)$, assume q is the cut point of p along γ_v for $v \in ST_pM$, then

$$\exp_p(t_{cut}(p, v)v) = \gamma_v(t_{cut}(p, v)) = q.$$

And since $|t_{cut}(p, v)v| = t_{cut}(p, v)$.

So $\exp_p(T_{cut}(p)) \supset \mathrm{cut}(p)$.

And $\exp_p(T_{cut}(p)) \subset \mathrm{cut}(p)$ by definition.

(4) Consider a sequence $\{q_i\}$ converges to q in M such that $q_i \in \mathrm{cut}(p)$.

Let $q_i = \exp(t_{cut}(p, v_i)v_i)$ and by passing to a subsequence, we assume $v_i \to v$.

Then $t_{cut}(p, v_i)$ must be bounded, so

$$\lim t_{cut}(p, v_i) = t_{cut}(p, v).$$

By the continuity, $q = \exp_p(t_{cut}(p,v)v) \in \mathrm{cut}(p), i.e.\mathrm{cut}(p)$ is closed.

And since $T_{cut}(p)$ is the graph of $t_{cut}(p, \bullet) : ST_pM \to (0, +\infty]$.

So $T_{cut}(p)$ is measure zero, so is $\mathrm{cut}(p) = \exp_p(T_{cut}(p))$.

(5) Follows by the fact that closed subset of compact set is compact.

(6) For $v \in \Sigma(p)$, there is one and only one unit-speed minimal geodesic passing through p and $\exp_p(v)$ given by $\exp_p\left(t\frac{v}{|v|}\right)$.

So $\exp_p$ is a smooth bijection from $\Sigma(p)$ onto its image.

And by theorem 7.10 and (3), $\exp_p(\Sigma(p)) = M\backslash\mathrm{cut}(p)$.

Hence $\exp_p : \Sigma(p) \to M\backslash\mathrm{cut}(p)$ is diffeomorphic.

(7) Follows by $(2), (3), (6)$.

(8) Since $B_{d(p,\mathrm{cut}(p))}(p) \subset \Sigma(p)$.

So $\mathrm{inj}_p(M, g) \geqslant d(p, \mathrm{cut}(p))$.

And for $q \in B_{\mathrm{inj}_p(M,g)}(p)$, q is not critical value of $\exp_p$ and has a unique preimage.

Therefore $q \notin \mathrm{cut}(p), i.e.\mathrm{inj}_p(M, g) \leqslant d(p, \mathrm{cut}(p))$, this completes the proof.

(9) $\mathrm{inj}_p(M, g) = \inf\{t_{cut}(p, v)|v \in ST_pM\}$.

So consider a sequence p_i converges to p in M.

Let $r_i = \mathrm{inj}_{p_i}(M, g) = t_{cut}(p_i, v_i), R = \mathrm{inj}_p(M, g) = t_{cut}(p, V)$, assume a subsequence of r_i converges to r, we shall proof that $r \leqslant R$ and $R \leqslant r$.

By passing to a subsequence, we assume $v_i \to v$.

Then $r_i = t_{cut}(p_i, v_i) \to t_{cut}(p, v) \geqslant R, i.e.r \geqslant R$.

And by continuity, for the sequence (p_i, w_i) converges to (p, V) in STM, $r_i \leqslant t_{cut}(p_i, w_i) = t_{cut}(p, V) = R$.

So $r \leqslant R, i.e.R \equiv r = \lim\limits_{i\to+\infty} r_i$.

□

Chapter 8

Topological properties of Riemannian manifold

8.1 Local isometries and isometries

Def 8.1. If $\varphi : (M, g_M) \to (N, g_N)$ is a C^∞ map,

(1) φ is called a local isometry if

$$\mathrm{d}\varphi_p : T_pM \to T_{g(p)}N$$

is a linear isometry, *i.e.* $\varphi^* g_N = g_M$

(2) φ is called an isometry if φ is a diffeomorphism(or surjective) and

$$d_N(\varphi(p), \varphi(q)) = d_M(p, q).$$

Thm 8.1. *If $\varphi : (M, g_M) \to (N, g_N)$ is bijective, TFAE*

(1) φ is an isometry

(2) φ is a diffeomorphism and a local isometry

(3) φ is a diffeomorphism and for every C^∞ curve $\gamma : [a, b] \to M$,

$$L(\varphi \circ \gamma) = L(\gamma).$$

Proof. (1) $\Rightarrow$ (2): For $r < \min\{\mathrm{inj}(M, p), \mathrm{inj}(N, q)\}$, consider the unit-speed minimal geodesic

$$\gamma_v : [0, r] \to M, t \mapsto \exp_p(tv).$$

Then $d_N(\varphi(\gamma_v(s)), \varphi(\gamma_v(t))) = |s - t|$.
And let $\tilde{\gamma_v}$ be the unique unit-speed minimal geodesic from q to $\varphi(\gamma_v(r))$.
Suppose $\varphi(\gamma_v(t)) \neq \tilde{\gamma_v}(t)$ for some $t \in (0, r)$.
Then the composition of two minimal geodesics from q to $\varphi(\gamma_v(t))$, from $\varphi(\gamma_v(t))$ to $\tilde{\gamma_v}(r)$ resp. is different from $\tilde{\gamma_v}$ but has the length r, contradiction!
So $\varphi \circ \gamma_v = \tilde{\gamma_v}$ is an unit-speed minimal geodesic and

$$|\mathrm{d}\varphi_p(v)| = \left|\frac{\mathrm{d}\tilde{\gamma_v}}{\mathrm{d}t}(0)\right| = 1.$$

Hence $\mathrm{d}\varphi_p$ is isometric,*i.e.* φ is a local isometry.

(2) $\Rightarrow$ (3):

$$L(\varphi\circ\gamma)=\int_a^b\left|(\varphi\circ\gamma)'\right|\mathrm{d}t=\int_a^b\left|\mathrm{d}\varphi(\gamma')\right|=\int_a^b\left|\gamma'\right|=L(\gamma).$$

(3) $\Rightarrow$ (1):

$$d_N(\varphi(p),\varphi(q))=\inf_{\gamma\in\mathscr{L}_{\varphi(p),\varphi(q)}}L(\gamma)=\inf_{\gamma\in\mathscr{L}_{\varphi(p),\varphi(q)}}L(\varphi^{-1}\circ\gamma)=\inf_{\gamma\in\mathscr{L}_{p,q}}L(\gamma)=d_M(p,q).$$

So φ is an isometry. □

Thm 8.2. *Let (M,g_M) and (N,g_N) be two Riemannian manifolds and $\varphi:M\to N$ be a smooth map, TFAE:*

(1) φ is a local isometry.

(2) For every $p\in M$, there are open neighborhoods U of p in M and V of $\varphi(p)$ in N such that $\varphi\big|_U:U\to V$ is an isometry.

Proof. (2) $\Rightarrow$ (1) is trivial by theorem 8.1.
(1) $\Rightarrow$ (2): Since linear isometry is invertible.
By inverse function theorem, φ is local diffeomorphism.
So the proof is completed by theorem 8.1. □

Coro 8.1. *Let $\varphi:(M,g_M)\to(N,g_N)$ be a local isometry, if $\gamma:[a,b]\to M$ is a C^∞ curve and $\tilde{\gamma}=\varphi\circ\gamma$, then*

(1) φ is totally geodesic.

(2) γ is a geodesic in $(M,g_M)\Leftrightarrow\tilde{\gamma}$ is a geodesic in (N,g_N).

Proof. Since φ is a local isometry.
So $\varphi^*g_N=g_M, i.e. B(X,Y)=\hat{\nabla}_X\varphi_*Y-\varphi_*(\nabla_XY)=0$.
And by corollary 6.4, φ is totally geodesic and γ is geodesic iff $\tilde{\gamma}$ is geodesic. □

8.2 covering map

Def 8.2. A Riemannian covering map $\pi:(M,g_M)\to(N,g_N)$ if

(1) π is a covering map

(2) π is C^∞

(3) π is a local isometry

Thm 8.3. *Suppose $\pi:(\tilde{M},g)\to(M,g)$ is a local isometry.*

(1) If $\tilde{M}$ is complete$\Rightarrow$ M is complete and π is a Riemannian covering map

(2) If π is a covering map, (M,g) is complete$\Leftrightarrow$ $(\tilde{M},\tilde{g})$ is complete

Proof. (1) For every $p \in M, \tilde{p} \in \pi^{-1}(p)$ and a geodesic γ starting at p, there is a unique geodesic $\tilde{\gamma}$ starting at $\tilde{p}$ with

$$\tilde{\gamma}'(0) = \mathrm{d}\varphi_p^{-1}\left(\gamma'(0)\right).$$

Then $\pi \circ \tilde{\gamma}$ is a geodesic starting at p with

$$(\pi \circ \tilde{\gamma})'(0) = \left(\mathrm{d}\varphi_p \circ \mathrm{d}\varphi_p^{-1}\right)\left(\gamma'(0)\right) = \gamma'(0).$$

So $\pi \circ \tilde{\gamma} = \gamma$ is well-defined all over $\mathbb{R}$, *i.e.* M is complete, and

$$\pi\left(\exp_{\tilde{p}}\left(\mathrm{d}\varphi_p^{-1}(v)\right)\right) = \exp_p(v),$$

i.e. π is surjective.

Define $U_p = B_\varepsilon(p), U_{\tilde{p}} = B_\varepsilon(\tilde{p})$, we now prove that U_p is evenly covered by $\{U_{\tilde{p}}\}$.

For another $\tilde{q} \in \pi^{-1}(p)$, the geodesic from $\tilde{p}$ to $\tilde{q}$ maps to a geodesic loop from p to p.

And the loop must go outside U_p and get back, *i.e.* $d(\tilde{p}, \tilde{q}) \geqslant 2\varepsilon$.

So $U_{\tilde{p}} \cap U_{\tilde{q}} = \varnothing$.

On the other hand, for every $q \in \pi^{-1}(U_p), d(\pi(q), p) < \varepsilon, i.e. d(q, \tilde{p}) < \varepsilon$ for some $\tilde{p}$.

Therefore $q \in U_{\tilde{p}}$ and

$$\pi(U_p) = \bigsqcup_{\tilde{p} \in \pi^{-1}(p)} U_{\tilde{p}}.$$

Hence π is a Riemannian covering map.

(2) By Hopf-Rinow theorem, Riemannian manifold is geodesic complete iff it is metric complete.

So (M, g) is complete iff $(\tilde{M}, \tilde{g})$ is complete since π is a covering map.

□

Prop 8.1. *Suppose (M, g) is a Riemannian manifold and Γ is a discrete Lie group acting smoothly, freely, properly, and isometrically on M.*

Then M/Γ ha s a unique Riemannian metric such that the quotient map $\pi : M \to M/\Gamma$ is a normal Riemannian covering.

8.3 Deck transformation

Prop 8.2. *Suppose $\tilde{M} \xrightarrow{\pi} M$ is a covering map*

(1) (Unique lifting proposition)If B is a connected topology space, $f : B \to M$ is continuous, then any two lifts of f that agree at one point are identical.

(2) (Path lifting proposition)Suppose $f : [0,1] \to M$ is a continuous path, then for any $\tilde{p} \in \pi^{-1}(f(0))$, $\exists!$lift $\tilde{f} : [0,1] \to \tilde{M}, \tilde{f}(0) = \tilde{p}$.

(3) (Monodromy theorem)Suppose $f, g : [0,1] \to M$ are path-homotopic and $\tilde{f}, \tilde{g} : [0,1] \to M$ are their lifts starting at the same point, then $\tilde{f}, \tilde{g}$ are path homotopic and $\tilde{f}(1) = \tilde{g}(1)$.

Proof. These are some easy topology proposition, so we will use them directly without proving.

□

Def 8.3. Let $\pi : \tilde{M} \to M$ be the universal cover of M, deck transformation $F : \tilde{M} \to \tilde{M}$ is a homeomorphism with $\pi \circ F = \pi$, the set of deck transformation is denoted by $\mathrm{Aut}_\pi\left(\tilde{M}\right)$.

Prop 8.3. *(1)* $\pi_1(M) \cong \mathrm{Aut}_\pi\left(\tilde{M}\right)$

(2) $\mathrm{Aut}_\pi\left(\tilde{M}\right)$ *acts* C^∞*, freely and properly on* $\tilde{M}$

(3) $\mathrm{Aut}_\pi\left(\tilde{M}\right)$ *acts transitively on each fiber of* π*.*

Proof. These are some easy topology proposition, so we will use them directly without proving. □

Prop 8.4. *Deck transformation is isometry.*

Proof. $F^*\tilde{g} = F^*\pi^*g = (\pi \circ F)^*g = \pi^*g = \tilde{g}$. □

Def 8.4. Let $\gamma_0, \gamma_1 : [0,1] \to M$ be two loops, they are said to be free homotopic if they are homotopic through closed paths, *i.e.* there exists a homotopy $H : [0,1] \times [0,1] \to M$ such that

$$H(0,t) = \gamma_0(t), H(1,t) = \gamma_1(t), H(s,0) = H(s,1).$$

The equivalence class of loops in M given by freely homotopic is called free homotopy class.

Def 8.5. Let (M,g) be a complete Riemannian manifold and $F : M \to M$ be an isometry.

A geodesic $\gamma : \mathbb{R} \to M$ is an axis of F if $F \circ \gamma$ is a nontrivial translation of γ, *i.e.* there exists a nonzero constant c, such that

$$(F \circ \gamma)(t) = \gamma(t+c).$$

An isometry with no fixed points that has an axis is said to be axial.

Lemma 8.1. *Let F be an isometry of a complete manifold (M,g). If $\delta_F(p) = d(p, F(p))$ has a positive minimum, then it is axial.*

Proof. Suppose δ_F has a minimum at $p \in M$, and let $\gamma : [0,1] \to M$ be a minimal geodesic connecting p and $F(p)$.

Then $\tilde{\gamma} = F \circ \gamma$ is a minimal geodesic connecting $F(p)$ and $F^2(p)$.

We claim γ and $\tilde{\gamma}$ form an angle π at $F(p)$ and thus fit together an extension of γ to $[0,2]$.

For any $t \in [0,1]$, we have

$$\begin{aligned}
\delta_F(p) =& d(p, F(p)) \leqslant d(\gamma(t), \tilde{\gamma}(t)) \\
\leqslant& d(\gamma(t), \gamma(1)) + d(\gamma(1), \tilde{\gamma}(t)) \\
=& d(\gamma(t), \gamma(1)) + d(\tilde{\gamma}(0), \tilde{\gamma}(t)) \\
=& d(\gamma(t), \gamma(1)) + d(\gamma(0), \gamma(t)) \\
=& d(\gamma(0), \gamma(1)) = d(p, F(p))
\end{aligned}$$

So $d(\gamma(t), \tilde{\gamma}(t)) = d(\gamma(t), \gamma(1)) + d(\gamma(1), \tilde{\gamma}(t))$, this proves the claim.

Hence $(F \circ \gamma)(t) = \gamma(t+1)$ and repeating this process we can define γ all over $\mathbb{R}$. □

Lemma 8.2. *If (M,g) is compact Riemannian manifold and $F : \tilde{M} \to \tilde{M}$ be a nontrivial deck transformation on the universal cover $\pi : \tilde{M} \to M$, then*

(1) $\delta_F(p) = d(p, F(p))$ *has a positive infimum,* $\delta_F(p) \geqslant 2\mathrm{inj}(M)$, i.e. *$F$ is axial*

(2) The axis $\tilde{\gamma}$ corresponding to this minimum is mapped to a closed geodesic in M whose length is minimal in its free homotopy class.

Proof. (1) is directly follow by theorem 8.3 and lemma 8.1, we now proof (2).

It is obvious that $\tilde{\gamma} : [0,1] \to \tilde{M}, \tilde{\gamma}(0) = \tilde{x}, \tilde{\gamma}(1) = F(\tilde{x})$ projects to a loop in M.

We first claim that if $\tilde{\gamma}_0, \tilde{\gamma}_1 : [0,1] \to \tilde{M}$ are curves connecting $\tilde{x}_0, F(\tilde{x}_0)$ and $\tilde{x}_1, F(\tilde{x}_1)$ resp. with, then γ_0, γ_1 are freely homotopic.

We define a homotopy $\tilde{H}(s,t) : [0,1] \times [0,1] \to M$ with

$$\tilde{H}(s,1) = F\left(\tilde{H}(s,0)\right), \tilde{H}(0,t) = \tilde{\gamma}_0(t), \tilde{H}(1,t) = \tilde{\gamma}_1(t).$$

Then $\tilde{H}$ is a free homotopy of $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$.

And let $H = \pi \circ \tilde{H}$, we have

$$H(0,t) = \left(\pi \circ \tilde{H}\right)(0,t) = (\pi \circ \tilde{\gamma}_0)(t) = \gamma_0(t), H(1,t) = \gamma_1(t),$$

$$H(s,0) = \left(\pi \circ \tilde{H}\right)(s,0) = \left(\pi \circ F \circ \tilde{H}\right)(s,0) = \left(\pi \circ \tilde{H}\right)(s,1) = H(s,1).$$

So H is a free homotopy of γ_0 and γ_1.

We then claim that let $\tilde{\gamma}_0 : [0,1] \to M$ be a path with $\tilde{\gamma}_0(0) = \tilde{x}_0, \tilde{\gamma}_0(1) = F(\tilde{x}_0)$, if γ_1 is freely homotopic to γ_0 by H, then $F(\tilde{\gamma}_1(0)) = \tilde{\gamma}_1(1)$ for some lifting $\tilde{\gamma}_1$ of γ_1.

Let $\tilde{H} : [0,1] \times [0,1] \to \tilde{M}$ be the lift of H with $\tilde{H}(0,0) = \tilde{x}_0$.

Then both $\tilde{\gamma}_0(t), \tilde{H}(0,t)$ are lifts of $\gamma_0(t)$ and $\tilde{\gamma}_0(0) = \tilde{H}(0,0)$.

So $\tilde{\gamma}_0(t) = \tilde{H}(0,t)$.

And both $\left(F \circ \tilde{H}\right)(s,0), \tilde{H}(s,1)$ are lifts of $H(s,0)$ with

$$\left(F \circ \tilde{H}\right)(0,0) = F \circ \tilde{\gamma}_0(0) = \tilde{\gamma}_0(1) = \tilde{H}(0,1).$$

Therefore $\left(F \circ \tilde{H}\right)(s,0) = \tilde{H}(s,1)$ and $\tilde{\gamma}_1 = \tilde{H}(1,t)$ is a lift of $\gamma_1(t)$ since

$$(\pi \circ \tilde{\gamma}_1)(t) = \left(\pi \circ \tilde{H}\right)(1,t) = H(1,t) = \gamma_1(t).$$

By lemma 8.1, the axis of F in $[0,c]$ is a minimal geodesic $\tilde{\gamma}$ from $\tilde{p}$ to $F(\tilde{p})$.

So for $\gamma_1 \sim \pi \circ \tilde{\gamma}$ w.r.t. freely homotopic, we have

$$L(\tilde{\gamma}_1) \geqslant d(\tilde{\gamma}_1(0), \tilde{\gamma}_1(1)) = \delta_F(\tilde{\gamma}_1(0)) \geqslant \delta_F(\tilde{p}) = L(\tilde{\gamma}).$$

Hence γ has the minimal length in $[\gamma]$. □

Coro 8.2. *Let M be a compact connected Riemannian manifold, then every nontrivial free homotopy class in M is represented by a closed geodesic that has minimum length among all admissible loops in its free homotopy class.*

Proof. For an arbitrary loop γ_0 start at p, we want to find a closed geodesic γ that is freely homotopic to γ_0.

Consider the deck transformation such that $F(\tilde{\gamma}_0(0)) = \tilde{\gamma}_0(1)$.

By lemma 8.2, F has an axis $\tilde{\gamma}$ and $\pi \circ \tilde{\gamma}\big|_{[0,c]}$ is a closed geodesic in M whose length is minimal in its free homotopy class.

And by the proof of lemma 8.2, $\pi \circ \tilde{\gamma}\big|_{[0,c]}$ is freely homotopic to γ_0. □

8.4 Manifold with Non-positive Sectional Curvature

Thm 8.4 (Cartan-Hadamard)**.** *Let (M, g) be a complete Riemannian manifold with* $\sec(g) \leqslant 0$.
Then $\forall p \in M, \exp_p : T_pM \to M$ is a covering map.
In particular, the universal covering $\tilde{M}$ of M is $\cong \mathbb{R}^n$.

Proof. Since $\sec(g) \leqslant 0$.
By proposition 7.4, $\exp_p : T_pM \to M$ is a local diffeomorphism.
Let $\tilde{g} = \exp_p^*(g)$.
Then $\exp_p : (T_pM, \tilde{g}) \to (M, g)$ is a local isometry.
Consider $\tilde{\gamma}_v(t) = tv : \mathbb{R} \to T_pM$.
Then $\exp_p(\tilde{\gamma}_v(t)) = \exp_p(tv)$ is a geodesic in (M, g).
Therefore $\tilde{\gamma}_v(t)$ is a geodesic in $(T_pM, \tilde{g})$.
By Hopf-Rinow, $(T_pM, \tilde{g})$ is complete.
Hence by theorem 8.3, $\exp_p$ is a Riemannian covering map and $\tilde{M} \cong T_pM \cong \mathbb{R}^n$. □

Remark 8.1. $\mathbb{S}^n$ has no Riemannian metric with $\sec(g) \leqslant 0$.
But for $n \geqslant 3$, $\mathbb{S}^n$ has Riemannian metric with $\mathrm{Ric}(g) \leqslant 0$ (HIGHLY NONTRIVIAL).

Coro 8.3. *Suppose M, N are compact C^∞ manifolds, if one of them is simply connected, then $M \times N$ does not admit a Riemannian metric with non-positive sectional curvature.*

Proof. Suppose M is simply connected and $\tilde{N}$ is the universal covering of N.
Then $\pi : M \times \tilde{N} \to M \times N$ is the universal covering.
If g is a Riemannian metric with $\sec(g) \leqslant 0$ on $M \times N$, then the pull back metric $\pi^* g$ is a complete Riemannian metric with $\sec(\pi^* g) \leqslant 0$, and $M \times \tilde{N} \cong \mathbb{R}^N$.
Let $p_1 : M \times \tilde{N} \to M$ be the canonical projection and μ be a volume form on M.
Then $\mathrm{d}p_1^*(\mu) = p_1^*(\mathrm{d}\mu) = 0$, *i.e.*$p_1^*(\mu)$ is a closed form on $M \times \tilde{N} \cong \mathbb{R}^N$.
So $p_1^*\mu$ is exact.
Fix a point $y_0 \in \tilde{N}$ and let $\iota : M \to M \times \tilde{N}, x \mapsto (x, y_0)$ be a embedding, then

$$\int_{\iota(M)} p_1^*\mu = \int_M \iota^*(p_1^*\mu) = \int_M (p_1 \circ \iota)^*\mu = \int_M \mu > 0.$$

On the other hand, $p_1^*\mu$ is exact on $M \times \tilde{N}$, *i.e.* $\int_{\iota(M)} p_1^*\mu = 0$, contradiction! □

Def 8.6. A simply connected complete Riemannian manifold (M, g) with $\sec(g) \leqslant 0$ is called a Cartan-Hadamard manifold.

Thm 8.5. *Let (M, g) be a simply connected complete Riemannian manifold, TFAE:*

(1) (M, g) *has* $\sec(g) \leqslant 0$

(2) for all $p \in M$ and $v, \hat{v} \in T_pM$,
$$\left|(\mathrm{d}\exp_p)_v(\hat{v})\right| \geqslant |\hat{v}|.$$

(3) for all $p \in M$ and $v_0, v_1 \in T_pM$
$$d(\exp_p(v_0), \exp_p(v_1)) \geqslant |v_1 - v_0|.$$

Moreover, if one of them is satisfied, then

(a) $\exp_p : T_pM \to M$ *is a diffeomorphism*

(b) Any two points in M are connected by a unique geodesic.

Proof. (1) $\Rightarrow$ (2): Consider the variation

$$\alpha(t,s) = \exp_p\left(t\left(v + s\hat{v}\right)\right)$$

with variational vector field

$$J(t) = \alpha_*\left(\frac{\partial}{\partial s}\right)\bigg|_{s=0} = (\mathrm{d}\exp_p)_{t(v+s\hat{v})}\left(t\hat{v}\right) = (\mathrm{d}\exp_p)_{tv}\left(t\hat{v}\right).$$

So $J(0) = 0, J'(0) = \hat{v}$.
Let $f(t) = |J(t)|$, then

$$f'(t) = |\eta|^{-1}\langle \eta, J'\rangle \text{ with } \eta = \begin{cases} t^{-1}J(t) & t > 0 \\ J'(0) & t = 0 \end{cases}$$

Therefore we have $|f'(0)| = |J'(0)| = |\hat{v}|$ and

$$f''(t) = \frac{1}{|J|^3}\left(J^2\langle J', J'\rangle - \langle J, J'\rangle^2\right) - \frac{\mathrm{R}(J, \gamma', \gamma', J)}{|J|} \geqslant 0$$

Set $F(t) = f(t) - t|\hat{v}|$.
Then $F'' \geqslant 0, F'(0) = 0, F(0) = 0$.
Hence $F(t)$ is increasing, *i.e.*

$$F(1) = \left|(\mathrm{d}\exp_p)_v\left(\hat{v}\right)\right| - |\hat{v}| \geqslant 0.$$

(2) $\Rightarrow$ (a): $\exp_p$ is a local diffeomorphism.
So similar to the proof of Cartan-Hadamard theorem, $\exp_p : (T_pM, \tilde{g}) \to (M, g)$ is a Riemannian covering map, where $\tilde{g} = \exp_p^* g$.
And T_pM, M are simply connected.
Hence $\exp_p : T_pM \to M$ is diffeomorphism.
(2) $\Rightarrow$ (3): Consider the geodesic γ from $\exp_p(v_0)$ to $\exp_p(v_1)$ and let $\exp_p(v(t)) = \gamma(t)$.
Then by using (a), we have

$$\begin{aligned} d(\exp_p(v_0), \exp_p(v_1)) &= \int_0^1 \left|\gamma'(t)\right|\mathrm{d}t = \int_0^1 \left|(\mathrm{d}\exp_p)_{v(t)}(v'(t))\right|\mathrm{d}t \\ &\geqslant \int_0^1 \left|v'(t)\right|\mathrm{d}t \geqslant \left|\int_0^1 v'(t)\mathrm{d}t\right| \\ &= |v_1 - v_0| \end{aligned}$$

(3) $\Rightarrow$ (2): Fix $p \in M, v \in T_pM$ and let $q = \exp_p(v)$.
Then $\exp_q : T_qM \to M$ is bijective since (M, g) is complete and so for every $w \in T_qM$,

$$d(\exp_q(0), \exp_q(w)) = d(q, \exp_q(w)) = |w|.$$

Define

$$\varphi = \exp_q^{-1} \circ \exp_p : T_pM \to T_qM.$$

Then $\varphi(v) = 0$ and it is differentiable around v,

$$(\mathrm{d}\varphi)_v = (\mathrm{d}\exp_q^{-1})_q \circ (\mathrm{d}\exp_p)_v = (\mathrm{d}\exp_p)_v.$$

We choose $w = \varphi\left(v + \hat{v}\right)$ for some $\hat{v} \in T_pM$, then

$$|\varphi\left(v + \hat{v}\right)| = d(q, \exp_q(\varphi(v + \hat{v}))) = d(\exp_p(v), \exp_p(v + \hat{v})) \geqslant |\hat{v}|.$$

Hence

$$\left|(\mathrm{d}\exp_p)_v(\hat{v})\right| = |(\mathrm{d}\varphi)_v(\hat{v})| = \lim_{t\to 0}\frac{|\varphi(v+t\hat{v})-\varphi(v)|}{t} \geqslant \lim_{t\to 0+}\frac{|t\hat{v}|}{t} = |\hat{v}|.$$

(2) $\Rightarrow$ (1): Suppose $R(\hat{v}, v, v, \hat{v}) > 0$, consider the Jacobi field

$$J(t) = (\mathrm{d}\exp_p)_{tv}(t\tilde{v}).$$

Then $J(0) = 0, J'(0) = \tilde{v}$.
And let $f(t) = |J(t)|^2$, by theorem 3.3,

$$f(t) = |\tilde{v}|^2 t^2 - 8R(\tilde{v}, v, v, \tilde{v})t^4 + O(t^5).$$

So $f(t) - |\tilde{v}|^2 t^2 < 0$ around 0, but

$$f(t) = |J(t)|^2 = \left|(\mathrm{d}\exp_p)_{tv}(t\tilde{v})\right|^2 \geqslant |t\tilde{v}|^2,$$

contradiction! □

Prop 8.5. *Let (M, g) be a Cartan-Hadamard manifold, then for every $p \in M, v_0, v_1 \in T_pM$ and $0 < t \leqslant T$,*

$$|v_0 - v_1| \leqslant \frac{d(\exp_p(tv_0), \exp_p(tv_1))}{t} \leqslant \frac{d(\exp_p(Tv_0), \exp_p(Tv_1))}{T}.$$

Proof. By theorem 8.5,

$$|tv_0 - tv_1| \leqslant d(\exp_p(tv_0), \exp_p(tv_1)), |Tv_0 - Tv_1| \leqslant d(\exp_p(Tv_0), \exp_p(Tv_1)).$$

So we only need to show that

$$\frac{d(\exp_p(tv_0), \exp_p(tv_1))}{t} \leqslant \frac{d(\exp_p(Tv_0), \exp_p(Tv_1))}{T}.$$

Consider the Jacobi field

$$J(t) = \left(\mathrm{d}\exp_p\right)_{tv}(t\hat{v})$$

and $f(t) = |J(t)|$.
Then f is convex by theorem 8.5.
Therefore

$$f(t) \leqslant \frac{(T-t)f(0) + tf(T)}{T}, \textit{i.e.} \frac{\left|(\mathrm{d}\exp_p)_{tv}(t\hat{v})\right|}{t} \leqslant \frac{\left|(\mathrm{d}\exp_p)_{Tv}(T\hat{v})\right|}{T}.$$

Consider the geodesic γ from $\exp_p(Tv_0)$ to $\exp_p(Tv_1)$ and let $\exp_p(Tv(s)) = \gamma(s)$, then

$$\begin{aligned}\frac{d(\exp_p(Tv_0), \exp_p(Tv_1))}{T} &= \int_0^1 \frac{\left|(\mathrm{d}\exp_p)_{Tv(s)}(Tv'(s))\right|}{T}\mathrm{d}s \\ &\geqslant \int_0^1 \frac{\left|(\mathrm{d}\exp_p)_{tv(s)}(tv'(s))\right|}{t}\mathrm{d}s \\ &\geqslant \frac{d(\exp_p(tv_0), \exp_p(tv_1))}{t}\end{aligned}$$

□

Coro 8.4. *Let (M, g) be a Cartan-Hadamard manifold, then for a geodesic triangle ABC,*

(1) $\angle A + \angle B + \angle C \leqslant \pi$

(2) $c^2 \geqslant a^2 + b^2 - 2ab\cos\angle C$.

If $\sec(g) < 0$, *inequalities are strict.*

Proof. (1) Let $A = \exp_C(v_0), B = \exp_C(v_1)$, then

$$\begin{aligned} c^2 \geqslant & |v_0 - v_1|^2 = |v_0|^2 + |v_1|^2 - 2|v_0||v_1|\cos\angle(v_0, v_1) \\ = & d(C, A)^2 + d(C, B)^2 - 2d(C, A)d(C, B)\cos\angle ACB \\ = & a^2 + b^2 - 2ab\cos\angle C \end{aligned}$$

(2)

$$\angle A + \angle B + \angle C \leqslant \arccos\frac{a^2 + b^2 - c^2}{2ab} + \arccos\frac{b^2 + c^2 - a^2}{2bc} + \arccos\frac{c^2 + a^2 - b^2}{2ca} = \pi.$$

If M has negative sectional curvature, then $d(\exp_p(v_0), \exp_p(v_1)) > |v_0 - v_1|$.
So the inequalities are strict. □

Lemma 8.3. *Let* (M, g) *be a Cartan-Hadamard manifold, for every* $p, q \in M, v \in T_pM$, *we define* $p_0 = \exp_p(-v), p_1 = \exp_p(v)$, *then*

$$d^2(p_0, q) + d^2(p_1, q) \geqslant d^2(p_0, p) + d^2(p_1, p) + 2d^2(p, q).$$

Proof. let $q = \exp_p(w)$
Then $d(p, q) = |w|, d(p_0, q) \geqslant |w + v|, d(p_1, q) \geqslant |w - v|$.
And $d(p, q) = |w|, d(p_0, p) = d(p_1, p) = |v|$, so

$$\begin{aligned} d^2(p_0, q) + d^2(p_1, q) \geqslant & |w + v|^2 + |w - v|^2 = 2|v|^2 + 2|w|^2 \\ = & d^2(p_0, p) + d^2(p_1, p) + 2d^2(p, q) \end{aligned}$$

□

Lemma 8.4 (Serre)**.** *Let* (M, g) *be a Cartan-Hadamard manifold, for every* $p \in M, r \geqslant 0$, *let* $\bar{B}(p, r) \subset M$ *be the closed ball of radius* r *centered at* p.
Let $\Omega \subset M$ *be a non-empty bounded set, and define*

$$r_\Omega = \inf\left\{r > 0 \middle| \exists p \in M, \text{s.t.} \Omega \subset \bar{B}(p, r)\right\}.$$

Then there exists a unique point $p_\Omega \in M, \text{s.t.}\Omega \subset \bar{B}(p_\Omega, r_\Omega)$.

Proof. Existence: let (p_i, r_i) be a sequence such that $\Omega \subset \bar{B}(p_i, r_i)$ and $r_i \to r_\Omega$.
Then for $q \in \Omega$, $d(q, p_i) \leqslant r_i, i.e.\{p_i\}$ is bounded.
So $\{p_i\}$ has a convergence subsequence converges at some point p_Ω.
Hence $\Omega \subset \bar{B}(p_\Omega, r_\Omega)$.
Uniqueness: Suppose there exist $p_0, p_1 \in M$ such that $\Omega \subset \bar{B}(p_0, r_\Omega) \cap \bar{B}(p_1, r_\Omega)$.
Let $p_1 = \exp_{p_0}(v_0)$ and take

$$p = \exp_{p_0}\left(\frac{v_0}{2}\right).$$

Then by lemma 8.3, for any $q \in \Omega$ we have

$$d^2(p, q) \leqslant \frac{d^2(p_0, q) + d^2(p_1, q)}{2} - \frac{d^2(p_0, p_1)}{4} \leqslant r_\Omega^2 - \frac{d^2(p_0, p_1)}{4} \leqslant r_\Omega^2.$$

So $d(p_0, p_1) = 0, i.e. p_0 = p_1$. □

Thm 8.6 (Cartan fixed point theorem)**.** *Suppose that (M,g) is a Cartan-Hadamard manifold and G is a compact Lie group acting smoothly and isometrically on M, then G has a fixed point in M,*i.e. *$\exists p_0 \in M$ such that $gp_0 = p_0$ for all $g \in G$.*

Proof. Let $p \in M$ and consider the group orbit

$$\Omega_p = \{gp | g \in G\}.$$

Since G is compact, Ω is bounded.
So $p_\Omega, r_\Omega > 0$ is defined by lemma 8.4 and

$$\Omega = g\Omega \subset \bar{B}(gp_\Omega, r_\Omega).$$

Hence $p_\Omega = gp_\Omega$ by the uniqueness of p_Ω. □

Thm 8.7 (Cartan torsion theorem)**.** *If (M,g) is a complete Riemannian manifold with* $\sec(g) \leqslant 0$*, then $\pi_1(M)$ is torsion free,* i.e. *all nontrivial elements have infinite order.*

Proof. Let $\pi : \left(\tilde{M}, \tilde{g}\right) \to (M,g)$ be the universal cover and $\Gamma = \mathrm{Aut}_\pi\left(\tilde{M}\right) \cong \pi_1(M)$.
Then by proposition 8.1, $M \cong \tilde{M}/\Gamma$ and the induced metric is isometric.
Suppose φ is a torsion free element in Γ.
Then the subgroup of Γ generated by φ is a compact Lie group and acts smoothly and isometrically on $\tilde{M}$.
So by Cartan fixed point theorem, φ must have a fixed point.
But Γ act freely on $\tilde{M}$, contradiction! □

8.5 Manifold with negative sectional curvature

Thm 8.8 (Preissmann)**.** *Let (M,g) be a compact Riemannian manifold with* $\sec(g) < 0$.

(1) Any nontrivial abelian subgroup of $\pi_1(M)$ is $\cong \mathbb{Z}$

(2) $\pi_1(M)$ is not abelian

Coro 8.5. *If M, N are two compact manifolds, then $M \times N$ can not support* $\sec(g) < 0$.

Proof. Suppose $M \times N$ has $\sec(g) < 0$.
By corollary 8.3, M, N are not simply connected.
Then $\pi_1(M \times N) \cong \pi_1(M) \times \pi_1(N)$.
Consider nontrivial elements $g_1 \in \pi_1(M), g_2 \in \pi_1(N)$, then g_1, g_2 must be torsion free.
So $\langle (g_1, e), (e, g_2) \rangle \cong \mathbb{Z} \times \mathbb{Z}$ is a nontrivial abelian subgroup of $\pi_1(M \times N)$, contradiction! □

Lemma 8.5. *Suppose (M,g) is a Cartan-Hadamard manifold with negative sectional curvature.*
If $\varphi : M \to M$ is an axial isometry, then its axis is unique up to reparametrization.

Proof. Suppose γ_1 and γ_2 are both axis but do not intersect.
Let $A = \gamma_1(0), B = \gamma_1(1) = \varphi(A), C = \gamma_2(0), D = \gamma_2(1) = \varphi(C)$.
Consider the geodesic σ from A to C, then $\varphi \circ \sigma$ is a geodesic from B to D.
So the geodesic quadrilateral $ABDC$ has angle sum 2π.
On the other hand, the geodesic triangles ABC, BCD have angle sums strictly less than π.
And B is inside the angle ACD since γ_1 and γ_2 do not intersect.
Therefore $\angle ACB + \angle DCB = \angle ACD$ and similarly $\angle ABC + \angle DBC = \angle ABD$.
These are contradiction, *i.e.* γ_1, γ_2 must intersect at some point $p = \gamma_1(t_1) = \gamma_2(t_2)$, then

$$\varphi(p) = \gamma_1(t_1 + c_1) = \gamma_2(t_2 + c_2).$$

is another intersection point.
Hence $\gamma_1 = \gamma_2$ since (M,g) is a Cartan-Hadamard manifold. □

proof of theorem 8.8(1). Let $\pi : \left(\tilde{M}, \tilde{g}\right) \to (M, g)$ be the universal covering.

Then $\left(\tilde{M}, \tilde{g}\right)$ is a Cartan-Hadamard manifold with negative sectional curvature.

It suffices to show that every nontrivial abelian subgroup H of $\mathrm{Aut}_\pi\left(\tilde{M}\right) \cong \pi_1(M)$ is isomorphic to $\mathbb{Z}$.

Let φ be a nontrivial element in H and γ is its unique unit-speed axis.
If ψ is another nontrivial element in H, then
$\varphi(\psi(\gamma(t))) = \psi(\varphi(\gamma(t))) = \psi(\gamma(t+c))$.
So $\psi \circ \gamma$ is also an axis for φ.
By lemma 8.5, $\psi \circ \gamma$ is a unit-speed reparametrization of γ, *i.e.* $\psi(\gamma(t)) = \gamma(t+a)$.
Suppose $\frac{a}{c}$ is irrational, then $\mathrm{span}_{\mathbb{Z}}(a, c)$ is dense in $\mathbb{R}$.
Consider $x, y \in \mathbb{Z}$ such that $0 < xa + yc < \mathrm{inj}(M)$.
Then $(\psi^x \circ \varphi^y)(\gamma(0)) = \gamma(xa + yc)$.
Therefore $\pi \circ \gamma\big|_{[0, xa+yc]}$ is a unit-speed geodesic loop, contradiction!
So $\frac{a}{c} = \frac{x}{y}$ for some $x, y \in \mathbb{Z}$, *i.e.*$\varphi^x = \psi^y$.
Hence $H \cong \mathbb{Z}$. □

Lemma 8.6. *Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature and $\pi : \tilde{M} \to M$ be the universal covering.*

If the geodesic $\tilde{\gamma} : \mathbb{R} \to \tilde{M}$ is a common axis for all elements of $\mathrm{Aut}_\pi\left(\tilde{M}\right)$, then M is not compact.

Proof. For any point $\tilde{p} = \tilde{\gamma}(s)$ and $k > 0$, we consider the unit-speed geodesic

$$\tilde{\beta} : [0, k] \to \tilde{M}, \tilde{\beta}(0) = \tilde{p}, \left\langle \tilde{\beta}'(0), \tilde{\gamma}'(s) \right\rangle = 0.$$

Let α_k be the minimal geodesic in M passing through $\beta(k)$ and $p = \beta(0)$.
Then $\beta \cdot \alpha_k$ is a loop at p, *i.e.* $L(\alpha_k) \leqslant L(\beta) = k$.
We claim that $L(\alpha_k) = k$, so that M is not bounded.
Let $\tilde{\alpha}_k$ be the lift of α_k starting from $\tilde{\beta}(k)$ and F is the deck transformation w.r.t. $\beta \cdot \alpha_k$.
Then $F(\tilde{p})$ is the end point of $\tilde{\alpha}_k$ and

$$F(\tilde{p}) = F(\tilde{\gamma}(s_0)) = \tilde{\gamma}(s_0 + c).$$

By corollary 8.4, $L(\alpha_k) = L(\tilde{\alpha}_k) \geqslant L\left(\tilde{\beta}\right) = k$ in right geodesic triangle $\left(\tilde{p}, \tilde{\beta}(k), F(\tilde{p})\right)$. □

proof of theorem 8.8(2). Suppose $\pi_1(M)$ is abelian.

Let $\tilde{\gamma}$ be the axis of the generator φ of $\mathrm{Aut}_\pi\left(\tilde{M}\right)$.

Then $\varphi^n(\tilde{\gamma}(t)) = \tilde{\gamma}(t + nc)$, *i.e.*$\gamma$ is the common axis for all elements of $\mathrm{Aut}_\pi\left(\tilde{M}\right)$.
By lemma 8.6, M is not compact, contradiction! □

There are some advance theorem for the fundamental group of compact Riemannian manifold with non-positive sectional curvature, we will not proof them.

Thm 8.9 (Byers)**.** *If $\sec(g) < 0$, then any nontrivial solvable subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$ and $\pi_1(M)$ is not solvable.*

Thm 8.10. *If $\sec(g) < 0$, then any subgroup of $\pi_1(M)$ that contains a nontrivial abelian normal subgroup is isomorphic to $\mathbb{Z}$.*

Thm 8.11 (Yau,1971)**.** *If $\sec(g) \leqslant 0$ and $\pi_1(M)$ is solvable, then M is flat,*i.e. *isometric to $\mathbb{R}^n/\Gamma$.*

Thm 8.12 (Lawson-Yau,Wolf)**.** *If $\sec(g) \leqslant 0$ and A is a free abelian subgroup of $\pi_1(M)$ of rank $k \geqslant 2$, then M admits a totally geodesic and isometrically immersed flat k-torus.*

Prob 8.1. *Open problem: does $\mathbb{S}^4 \times \mathbb{S}^1$ have Ricci-flat metric?*

8.6 Manifold with non-negative curvature

Thm 8.13 (Myers)**.** *Let (M, g) be a complete Riemannian manifold with*

$$\mathrm{Ric}(g) \geqslant \frac{(n-1)g}{R^2},$$

where $R \in \mathbb{R}^+, n \geqslant 2$, then

(1) $\mathrm{diam}(M, g) \leqslant \pi R$*,*

(2) M is compact,

(3) $|\pi_1(M)|$ is finite.

Proof. (1) Suppose $\mathrm{diam}(M, g) > \pi R$.

Since (M, g) is complete.

So there exists a unit-speed minimal geodesic $\gamma : [0, l] \to M$ with $l > \pi R$.

Let $(e_1(t) = \gamma'(t), e_2(t), \cdots, e_n(t))$ be an orthonormal basis of $T_{\gamma(t)}M$ where $e_2, \cdots, e_n$ are parallel vector field along γ, consider the variational vector field

$$V_i(t) = \sin\left(\frac{\pi t}{l}\right) e_i(t).$$

Then $V_i(0) = V_i(l) = 0$ and so by theorem 7.5,

$$\begin{aligned}
0 \leqslant \sum_{i=2}^{n} \frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0} E(\alpha_i(\cdot, s)) &= \sum_{i=2}^{n} \int_0^l \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i \right\rangle \mathrm{d}t - \sum_{i=2}^{n} \int_0^l \mathrm{R}(V_i, \gamma', \gamma', V_i)\mathrm{d}t \\
&= (n-1)\frac{\pi^2}{l^2} \int_0^l \cos^2\left(\frac{\pi t}{l}\right) \mathrm{d}t - \int_0^l \sin^2\left(\frac{\pi t}{l}\right) \mathrm{Ric}(\gamma', \gamma')\mathrm{d}t \\
&\leqslant (n-1)\frac{\pi^2}{l^2} \int_0^l \cos^2\left(\frac{\pi t}{l}\right) \mathrm{d}t - \frac{(n-1)}{R^2} \int_0^l \sin^2\left(\frac{\pi t}{l}\right) \mathrm{d}t \\
&< \frac{n-1}{R^2} \int_0^l \left(\cos^2\left(\frac{\pi t}{R^2}\right) - \sin^2\left(\frac{\pi t}{l}\right)\right) \mathrm{d}t = 0
\end{aligned}$$

This is contradiction!

So $\mathrm{diam}(M, g) \leqslant \pi R$.

(2) Since $\mathrm{diam}(M, g) \leqslant \pi R$ is bounded.

So M is compact.

(3) Let $\pi : \left(\tilde{M}, \tilde{g}\right) \to (M, g)$ be the universal cover.

Then $\left(\tilde{M}, \tilde{g}\right)$ is also compact.

So π is proper and must be a finite cover, *i.e.* $|\pi_1(M)|$ is finite.

∎

Remark 8.2. S.Y. Cheng proved that if the identity holds, then (M, g) is isometric to the constant sectional curvature manifold $(\mathbb{S}^n(R), g_{can})$, we will proof this in the last chapter.

Coro 8.6. *Let (M, g) be a complete Riemannian manifold.*

(1) If it has sectional curvature $K \geqslant \frac{1}{R^2} > 0$, then $\operatorname{diam}(M, g) \leqslant \pi R$.

(2) If it is compact and has positive Ricci curvature, then $|\pi_1(M)|$ is finite.

(3) If it is an Einstein manifold with positive scalar curvature, then M is compact and $|\pi_1(M)|$ is finite.

Proof. All are directly follows from Myers theorem. □

Lemma 8.7. *Let $A : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be an orthonormal transformation, if $\det A = (-1)^n$, then there exists some $v \in \mathbb{R}^{n-1} \backslash \{0\}$ such that $Av = v$.*

Proof. Let the eigenvalues and eigenvectors of A be λ_i, v_i resp, then

$$|v_i|^2 = v_i^* v_i = v_i^* A^* A v_i = |\lambda_i|^2 |v_i|^2$$

So $|\lambda_i| = 1$
And since $|A| = (-1)^n, \dim A = n - 1$.
Hence there is some i with $\lambda_i = 1, i.e. Av_i = v_i$. □

Thm 8.14 (Synge)**.** *Let (M, g) be a compact Riemannian manifold with positive sectional curvature.*

(1) If $\dim M$ is even and M is orientable, then $\pi_1(M) = \{e\}$.

(2) If $\dim M$ is odd, then M is orientable.

Proof. Suppose the conclusions are not correct.
Then $\pi_1(M) \neq \{e\}$.
Let $\pi : \left(\tilde{M}, \tilde{g}\right) \to (M, g)$ be the universal cover.
We endow $\tilde{M}$ with

(1) the pullback orientation when $\dim M$ is even.

(2) an arbitrary orientation when $\dim M$ is odd.

There is a nontrivial deck transformation $F : \tilde{M} \to \tilde{M}$ and

(1) F is orientation preserving when $\dim M$ is even

(2) F is orientation reversing when $\dim M$ is odd

So by lemma 8.2, there exists an axis $\tilde{\gamma} : \mathbb{R} \to \tilde{M}$ for F and $\gamma = \pi \circ \tilde{\gamma}$ is a closed geodesics in M that minimizes curve length in its free homotopy class and WLOG, we assume

$$F(\tilde{\gamma}(t)) = \tilde{\gamma}(t + 1).$$

We claim that there exists a variation α of γ in the free homotopy class of γ,*i.e.*

$$\alpha(t, 0) = \gamma(t), \alpha(0, s) = \alpha(1, s)$$

such that the variational vector field V satisfies that

$$V(0) = V(1), \langle V, \gamma' \rangle = 0, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V = 0.$$

Assuming the claim, we have

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\bigg|_{s=0} E(\alpha(\bullet,s)) &= \int_0^1 \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V \right\rangle \mathrm{d}t - \int_0^1 \mathrm{R}(V,\gamma',\gamma',V)\mathrm{d}t \\ &+ \int_0^1 \left\langle \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \left(\nabla_{\frac{\partial}{\partial s}} \alpha_* \left(\frac{\partial}{\partial s} \right) \right) \bigg|_{s=0}, \gamma' \right\rangle \mathrm{d}t \\ &= -\int_0^1 \mathrm{R}(V,\gamma',\gamma',V)\mathrm{d}t + \left\langle \left(\nabla_{\frac{\partial}{\partial s}} \alpha_* \left(\frac{\partial}{\partial s} \right) \right) \bigg|_{s=0}, \gamma' \right\rangle \bigg|_{t=0}^{t=1} \\ &= -\int_0^1 \mathrm{R}(V,\gamma',\gamma',V) < 0\end{aligned}$$

This is contradiction!

Proof of claim:

Let $\tilde{x_0} = \tilde{\gamma}(0), \tilde{x_1} = \tilde{\gamma}(1) = F(\tilde{x}_0), p = \pi(\tilde{x}_0) = \pi(\tilde{x}_1)$ and

$$\tilde{P} = P_{0,1,\tilde{\gamma}} : T_{\tilde{x}_0}\tilde{M} \to T_{\tilde{x}_1}\tilde{M}, P = P_{0,1,\gamma} : T_pM \to T_pM.$$

be the parallel transports along $\tilde{\gamma}\big|_{[0,1]}$ in $\tilde{M}$ and loop $\gamma\big|_{[0,1]}$ in M resp.

Since local isometries preserve parallelisms.

So the following diagram commutes and all these maps are linear isometries

$$\begin{tikzcd} T_{\tilde{x}_0}\tilde{M} \arrow[r, "\tilde{P}"] \arrow[d, "d\pi_{\tilde{x}_0}"'] & T_{\tilde{x}_1}\tilde{M} \arrow[d, "d\pi_{\tilde{x}_1}"] \\ T_pM \arrow[r, "P"] & T_pM \end{tikzcd}$$

And since $\tilde{M}$ is simply connected and it is oriented.

Therefore $\tilde{P}$ is orientation preserving, moreover,

(1) when n is even, the vertical maps are both orientation-preserving, so is P

(2) when n is odd, F is orientation-reversing and

$$\mathrm{d}\pi_{\tilde{x}_0} = \mathrm{d}\pi_{\tilde{x}_1} \circ \mathrm{d}F - \tilde{x}_0$$

implies that the two vertical maps induce opposite orientations on T_pM, so P is orientation reversing.

Since γ is closed geodesic, we have

$$P(\gamma'(0)) = \gamma'(1) = \gamma'(0).$$

On the other hand, P is an isometry and it induces a linear map

$$A = P\bigg|_W : W \to W$$

where W is the orthogonal complement of $\gamma'(0)$ in T_pM.

Notes that $\dim W = n-1$ and $\det(A) = (-1)^n$, by lemma 8.7, there exists nontrivial $v \in W$ such that $A(v) = v$.

Consider the parallel vector field

$$V(t) = P_{0,t,\gamma}(v)$$

along γ with $V(0) = V(1) = v, V(t) \perp \gamma'(t)$ and the variation

$$\alpha(t,s) = \exp_{\gamma(t)}(sV(t)).$$

Then $\alpha(t,0) = \gamma(t), \alpha(0,s) = \alpha(1,s)$ and the variational field of α is $V(t)$. □

Coro 8.7. *Let (M, g) be a compact Riemannian manifold with positive sectional curvature, if $\dim M$ is even, not orientable, then $\pi_1(M) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $\tilde{M}$ be a two-sheet orientable cover of M.

By theorem 8.14, $\tilde{M}$ is simply connected and so it is the universal cover of M.

Hence $|\pi_1(M)| = 2, i.e. \pi_1(M) = \mathbb{Z}/2\mathbb{Z}$. □

Coro 8.8. *the product metric of $\mathbb{RP}^2 \times \mathbb{RP}^2$ has positive Ricci curvature, but it cannot support a metric with positive sectional curvature.*

Proof. $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2) = (\mathbb{Z}/2\mathbb{Z})^2$.

So the conclusion follows by corollary 8.7. □

Conj 8.1 (Hopf)**.** *The product metric of $\mathbb{S}^2 \times \mathbb{S}^2$ has positive Ricci curvature, does it admit a metric with positive sectional curvature?*

Thm 8.15 (Weinstein-Synge)**.** *Let (M, g) be an n-dimensional compact oriented Riemannian manifold with positive sectional curvature.*

Given an isometry $F : (M, g) \to (M, g)$ that F preserves the orientation when n is even, or F reverses the orientation when n is odd, then F has a fixed point.

Proof. Suppose f has no fixed point.

Then there is a point $p \in M$ such that $d(p, f(p))$ is minimal since M is compact.

by lemma 8.1, let γ be the axis of f such that $\gamma(0) = p, \gamma(1) = f(p)$.

So $f_*(\gamma'(0)) = \gamma'(1), i.e. f_*$ is isometric from $\gamma'(0)^\perp \to \gamma'(1)^\perp$, consider

$$W = \gamma'(0)^\perp, A = f_*^{-1} \circ P_{0,1,\gamma}\Big|_W : W \to W.$$

Then $\det(A) = (-1)^n$ and by lemma 8.7, there exists $v \in W \backslash \{0\}$ such that $Av = v$.

Define

$$V(t) = P_{0,t,\gamma}(v), \alpha(t, s) = \exp_{\gamma(t)}(sV(t)),$$

then we have

$$V(0) = v, V(1) = f_*(v),$$

$$\alpha(t, 0) = \gamma(t), \alpha(1, s) = \exp_{f(p)}(f_*(sv)) = f(\exp_p(sv)) = f(\alpha(0, s)).$$

Similar to Synge theorem, we have

$$\frac{\partial^2}{\partial s^2}\Big|_{s=0} E(\alpha(\cdot, s)) = -\int_0^1 R(V, \gamma', \gamma', V) \mathrm{d}t < 0,$$

contradiction!

Hence f has fixed point. □

8.7 Constant sectional curvature

Exam 8.1. *(1) $(\mathbb{R}^n, g_{can})$ has sectional curvature zero.*

(2) $(\mathbb{S}^n(r), g_{can})$ has constant sectional curvature $\frac{1}{r^2}$.

(3) $(\mathbb{B}^n(r), g_{can})$ with

$$g_{can} = \frac{4r^4}{r^2 - |x|^2} \sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i$$

has constant sectional curvature $-\frac{1}{r^2}$.

(4) $(\mathbb{H}^n, g_{can})$ *with*

$$g_{can} = \frac{r^2}{y^2}\left(\sum_{i=1}^{n-1} \mathrm{d}x^i \otimes \mathrm{d}x^i + \mathrm{d}y \otimes \mathrm{d}y\right)$$

has constant sectional curvature $-\frac{1}{r^2}$.

Thm 8.16 (Local Cartan-Ambrose-Hicks theorem)**.** *Let* (M, g) *and* $(\tilde{M}, \tilde{g})$ *be two Riemannian manifolds and*

$$\Phi_0 : T_pM \to T_{\tilde{p}}\tilde{M}$$

be some fixed linear isometry.
Suppose $\delta \in \left(0, \min\left\{\mathrm{inj}_p M, \mathrm{inj}_{\tilde{p}}\tilde{M}\right\}\right)$, *for any* $v \in T_pM$ *with* $|v| < \delta$ *we define*

$$\gamma(t) = \exp_p(tv), \tilde{\gamma}(t) = \exp_{\tilde{p}}(t\Phi_0(v))$$

and a linear isometry

$$\Phi_t = p_{0,t,\tilde{\gamma}} \circ \Phi_0 \circ p_{0,t,\gamma}^{-1} : T_{\gamma(t)}M \to T_{\tilde{\gamma}}(t)\tilde{M}.$$

TFAE:

(1) There is an isometry $\varphi : B(p, \delta) \to B(\tilde{p}, \delta)$ *with* $\varphi(p) = \tilde{p}, (\varphi_*)_p = \Phi_0$

(2) Φ_t *preserves the curvature,* i.e. *for all* $t \in [0, 1]$ *and* $u, v, w, z \in T_{\gamma(t)}M$, *then*

$$\mathrm{R}(U, V, W, Z) = \tilde{\mathrm{R}}(\Phi_t U, \Phi_t V, \Phi_t W, \Phi_t Z).$$

Moreover, if one of these conditions hold, the isometry is precisely given by

$$\varphi = \exp_{\tilde{p}} \circ \Phi_0 \circ \exp_p^{-1} : B(p, \delta) \to B(\tilde{p}, \delta)$$

and $\Phi_t = \varphi_{*\gamma(t)}$.

$$\begin{tikzcd} T_pM \arrow[r, "\Phi_0"] \arrow[d, "P_{0,t,\gamma}"'] & T_{\tilde{p}}\tilde{M} \arrow[d, "P_{0,t,\tilde{\gamma}}"] \\ T_{\gamma(t)}M \arrow[r, "\Phi_t"] & T_{\tilde{\gamma}(t)}\tilde{M} \end{tikzcd} \qquad \begin{tikzcd} T_pM \arrow[r, "\Phi_0"] \arrow[d, "\exp_p"'] & T_{\tilde{p}}\tilde{M} \arrow[d, "\exp_{\tilde{p}}"] \\ B(p,\delta) \arrow[r, "\varphi"] & B(\tilde{p},\delta) \end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2): we only need to show that $\Phi_t = (\varphi_*)_{\gamma(t)}$.
Since $(\varphi \circ \gamma)'(0) = \Phi_0(v) = \tilde{\gamma}'(0)$.
So $\tilde{\gamma}(t) = \varphi(\gamma(t))$.
Consider an orthonormal basis $\left(e_1 = \frac{\gamma'(0)}{|\gamma'(0)|}, e_2, \cdots, e_n\right)$ of T_pM.
By parallel transport, let $(e_1(t), \cdots, e_n(t))$ be a orthonormal basis of $T_{\gamma(t)}M$, then

$$\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \Phi_t(e_i(t)) = \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} (\varphi_*)_{\gamma(t)}(e_i(t)).$$

Hence $\Phi_t = (\varphi_*)_{\gamma(t)}$.
(2) $\Rightarrow$ (1): we define

$$\varphi = \exp_{\tilde{p}} \circ \Phi_0 \circ \exp_p^{-1} : B(p, \delta) \to B(\tilde{p}, \delta).$$

Then $\varphi(p) = \tilde{p}, (\varphi_*)_p = \Phi_0$.

We want to show that φ is an isometry, *i.e.* for any $w \in T_qM$,

$$|(\varphi_*)_q w| = |w|.$$

Consider a geodesic $\gamma : [0,1] \to M$ from p to q and Jacobi field J with $J(0) = 0, J(1) = w$ and let $J = \sum y_i(t)e_i(t)$.

We claim that $\tilde{J} = \Phi_t(J(t))$ is a Jacobi field along $\tilde{\gamma}$.

By the Jacobi field equation,

$$y_j'' + \sum y_i \left|\gamma'(0)\right|^2 \mathrm{R}(e_i, e_1, e_1, e_j) = 0.$$

If we set $\tilde{e_i}(t) = \Phi_t(e_i(t))$, then

$$\tilde{J}(t) = \sum_{i=1}^{n} y_i(t)\tilde{e_i}(t).$$

Since Φ_t preserves curvatures and $|\gamma'(0)| = |\tilde{\gamma}'(0)|$, so we obtain

$$y_j'' + \sum y_i \left|\tilde{\gamma}'(0)\right|^2 \tilde{\mathrm{R}}(\tilde{e}_i, \tilde{e}_1, \tilde{e}_1, \tilde{e}_j) = 0,$$

i.e. $\tilde{J}(t)$ is a Jacobi field.

On the other hand,

$$\tilde{J}'(0) = \Phi_0(J'(0)), \tilde{J}(t) = (\mathrm{d}\exp_{\tilde{p}})_{t\tilde{\gamma}'(0)}\left(t\tilde{J}'(0)\right).$$

So we have

$$\begin{aligned}\tilde{J}(1) =&(\mathrm{d}\exp_{\tilde{p}})_{\tilde{\gamma}'(0)}\left(\tilde{J}'(0)\right)\\ =&(\mathrm{d}\exp_{\tilde{p}})_{\tilde{\gamma}'(0)} \circ \Phi_0 \circ (\mathrm{d}\exp_p)^{-1}_{\gamma'(0)}(J(1))\\ =&(\varphi_{*q})(J(1))\end{aligned}$$

Hence $|(\varphi_{*q})(w)| = \left|\tilde{J}(1)\right| = |\Phi_1(J(1))| = |J(1)|$ and φ is a isometry. □

Thm 8.17 (Cartan-Ambrose-Hicks theorem)**.** *Let (M, g_M) be a connected Riemannian manifold, suppose φ and ψ are two local isometries from M to (N, g_N).*

If there exists $p \in M$ such that $\varphi(p) = \psi(p)$ and $\varphi_{*p} = \psi_{*p}$, then $\varphi = \psi$.

Proof. Since M is connected.

So there is a path $\gamma : [0,1] \to M$ from p to q and let

$$A = \left\{t \in [0,1] \middle| \varphi(\gamma(t)) = \psi(\gamma(t)), (\varphi_*)_{\gamma(t)} = (\psi_*)_{\gamma(t)}\right\}.$$

Consider a small neighborhood V of p, such that $\varphi\big|_V, \psi\big|_V$ are isometry.

Then $f : \varphi^{-1} \circ \psi : V \to W$ is an isometry with $f(p) = p$, $f_{*p} = \mathrm{Id}$.

By Local Cartan-Ambrose-Hicks theorem, we have

$$f(q) = \exp_p \circ \mathrm{Id} \circ \exp_p^{-1}(q) = q.$$

Therefore $f = \mathrm{Id}$, *i.e.*A is open, and the closeness of A is obvious since φ, ψ are smooth.

Hence $A = [0,1]$, *i.e.*$\varphi \equiv \psi$. □

Thm 8.18 (Uniformization)**.** *Let (M, g) be a complete Riemannian manifold with constant sectional curvature K, then (M, g) is isometric to $\tilde{M}/\Gamma$, where $\tilde{M}$ is one of the model spaces $\mathbb{R}^n, \mathbb{S}^n(r), \mathbb{B}^n(r)$ and $\Gamma \subset \mathrm{Iso}(\tilde{M}, g_{can})$ is discrete and acts freely.*

Proof. Let $\left(\tilde{M}, \tilde{g}\right)$ be the universal cover of M and M_0 be the corresponding model space.

If $M_0 = \mathbb{R}^r$ or $\mathbb{B}^n(r)$ and $K \leqslant 0$, fix $p \in M_0, \tilde{p} \in \tilde{M}$ and linear isometry $\Phi_0 = T_p M_0 \to T_{\tilde{p}}\tilde{M}$. Consider the map

$$\varphi = \exp_{\tilde{p}} \circ \Phi \circ \exp_p^{-1} : M_0 \to \tilde{M}$$

By Cartan-Hadamard theorem, φ is well-defined.
And By local Cartan-Ambrose-Hicks theorem, φ is local isometric.
Moreover, φ is isometric by theorem 8.3 and theorem 8.1.
If $M_0 = S^n(r)$ and $K > 0$, fix $p \in M_0, \tilde{p} \in \tilde{M}$ and linear isometry $\Phi_0 : T_p M_0 \to T_{\tilde{p}}\tilde{M}$. Consider the map

$$\varphi_1 = \exp_{\tilde{p}} \circ \Phi_0 \circ \exp_p^{-1} : \mathbb{S}^n(r)\backslash\{-p\} \to \tilde{M}.$$

Then by local Cartan-Ambrose-Hicks theorem, φ_1 is well-defined local isometry.
Choose another point $m \in \mathbb{S}^n(r)\backslash\{\pm p\}$ and set $\tilde{m} = \varphi_1(m)$.
Define $\Psi_0 = (\mathrm{d}\varphi_1)_m : T_m M_0 \to T_{\tilde{m}}\tilde{M}$ and

$$\varphi_2 = \exp_{\tilde{m}} \circ \Psi_0 \circ \exp_m^{-1} : \mathbb{S}^n(r)\backslash\{-m\} \to \tilde{M}.$$

Since $W = \mathbb{S}^n(r)\backslash\{-m, -p\}$ is connected and

$$\varphi_1(m) = \varphi_2(m) = \tilde{m}, (\mathrm{d}\varphi_2)_m = \psi_0 = (\mathrm{d}\varphi_1)_m.$$

So by Cartan-Ambrose-Hicks theorem, $\varphi_1 = \varphi_2$ on $S^n(r)\backslash\{-p, -m\}$.
Now we define

$$\varphi(x) = \begin{cases} \varphi_1(x) & x \in \mathbb{S}^n(r)\backslash\{-p\} \\ \varphi_2(x) & x \in \mathbb{S}^n(r)\backslash\{-m\} \end{cases}$$

Hence $\varphi : \mathbb{S}^n(r) \to \tilde{M}$ is local isometry and by theorem 8.3 and theorem 8.1, φ is isometry. $\square$

Coro 8.9. *Let (M, g) be a complete Riemannian manifold with constant sectional curvature $K \equiv 1$. If $\dim_{\mathbb{R}} M = 2024$, then $M \cong \mathbb{S}^{2024}$ or $M \cong \mathbb{RP}^{2024}$.*

Proof. By the theorem, $\tilde{M} \cong \mathbb{S}^{2024}$.
And if M is orientable, then $\pi_1(M) = \{e\}, i.e. M = \tilde{M}$.
If M is not orientable, then $\pi_1(M) = \mathbb{Z}_2, i.e. M \cong \mathbb{S}^{2024}/\mathbb{Z}_2 \cong \mathbb{RP}^{2024}$. $\square$

Chapter 9

Radial distance function

9.1 Elementary computations

Prop 9.1. *Let (M, g) be a complete Riemannian manifold, $p \in M, U = M \backslash \mathrm{cut}(p)$.*
For any $q \in U$, we define the distance function $r : U \to \mathbb{R}, r(q) = d(p, q)$, then

(1) r is continuous.

(2) r is C^∞ over $U \backslash \{p\}$.

(3) r^2 is C^∞ over U.

(4) $r(q) = \left|\exp_p^{-1}(q)\right|$.

(5) ∇r is defined intrinsically on $U \backslash \{p\}$.

Proof. In normal coordinate $(x^1, \cdots, x^n)$ centered at p, the unique geodesic γ with $\gamma(0) = p$ and $\gamma'(0) = v$ is given by

$$\gamma_v(t) = (tv^1, \cdots, tv^n).$$

So for $q = (q^1, \cdots, q^n) \in U$,

$$r(q) = \left|\exp_p^{-1}(q)\right| = \sqrt{\sum_{i=1}^{n} (q^i)^2}.$$

□

Def 9.1. In normal coordinates $(x^1, \cdots, x^n)$ around p, the radial vector field over $U \backslash \{p\}$ is

$$\partial_r = \sum \frac{\partial r}{\partial x^i} \frac{\partial}{\partial x^i} = \sum \frac{x^i}{r} \frac{\partial}{\partial x^i}.$$

Remark 9.1. ∂_r is invariant under the orthogonal transform, but it is not defined in general coordinate system.

Thm 9.1. *On $U \backslash \{p\}$,*

(1) $|\partial_r| = 1$

(2) $\nabla r = \partial_r$.

Proof. (1) In normal coordinates $(x^1, \cdots, x^n)$ centered at $p \in M$, for $q = (q^1, \cdots, q^n)$,

$$\exp_p^{-1}(q) = \sum q^i \left.\frac{\partial}{\partial x^i}\right|_p, r(q) = \sqrt{\sum (q^i)^2}.$$

Let $b = r(q)$, then

$$\gamma_v(t) = \left(\frac{tq^1}{b}, \cdots, \frac{tq^n}{b}\right), \left.\partial_r\right|_q = \tfrac{q^i}{b} \left.\tfrac{\partial}{\partial x^i}\right|_q = \gamma_v'(b).$$

And since γ_v is unit-speed geodesic.
So $|\partial_r| = |\gamma_v'(b)| = 1$.

(2) We shall show ∂_r is orthogonal to the level set of r, so that ∇r is parallel to ∂_r.
Let $\Sigma_b = \exp_p(\partial B(0, b) \cap \Sigma(p)) \subset M$.
We show that for any $q \in \Sigma_b$ and $w \in T_qM$ tangent to Σ_b at p,

$$g\left(\left.\partial_r\right|_q, w\right) = 0.$$

Let $c : (-\varepsilon, \varepsilon) \to \Sigma_b$ be the smooth curve with $c(0) = q, c'(0) = w$ and $|c(s)| = b$, consider

$$w(s) = \left.\partial_r\right|_{c(s)} = \sum \tfrac{c^i(s)}{b} \left.\tfrac{\partial}{\partial x^i}\right|_{c(s)}.$$

And we define a family of geodesic $\alpha : [0, b] \times (-\varepsilon, \varepsilon) \to U$ by

$$\alpha(t, s) = \exp_p(t\omega(s)) = \left(\frac{tc^1(s)}{b}, \cdots, \frac{tc^n(s)}{b}\right).$$

By Gauss lemma 3.1, since $|w(s)| = 1$, we can obtain

$$\left\langle \alpha_*\left(\frac{\partial}{\partial t}\right), \alpha_*\left(\frac{\partial}{\partial s}\right)\right\rangle \equiv 0.$$

Note that

$$\alpha_*\left(\frac{\partial}{\partial s}\right)(b, 0) = \left.\left(\frac{t(c^1(s))'}{b}, \cdots, \frac{t(c^n(s))'}{b}\right)\right|_{(b,0)} = \left.\left(\frac{\mathrm{d}c^1}{\mathrm{d}s}, \cdots, \frac{\mathrm{d}c^n}{\mathrm{d}s}\right)\right|_0 = c'(0) = w,$$

$$\alpha_*\left(\frac{\partial}{\partial t}\right)(b, 0) = \left(\frac{c^1(0)}{b}, \cdots, \frac{c^n(0)}{b}\right) = w(0) = \left.\partial_r\right|_q.$$

Hence $g\left(\left.\partial_r\right|_q, w\right) = 0$.
On the other hand,

$$g\left(\left.\partial_r\right|_q, \nabla r\right) = g\left(\frac{x^i}{r}\frac{\partial}{\partial x^i}, g^{kl}\frac{\partial r}{\partial x^k}\frac{\partial}{\partial x^l}\right) = \frac{x^i}{r}\frac{\partial r}{\partial x^k} g^{kl} g_{il} = \sum_k \frac{x^k}{r}\frac{x^k}{r} = 1 = g\left(\left.\partial_r\right|_q, \left.\partial_r\right|_q\right).$$

Hence $\left.\partial_r\right|_q - \nabla r = 0$.

□

Coro 9.1. *In normal coordinates $\{x^1, \cdots, x^n\}$ centered at $p \in M$*

(1) $\sum_j g_{ij}x^j = x^i$

(2) $g_{im} = \delta_{im} - \sum_j \frac{\partial g_{ij}}{\partial x^m} x^j$

(3) $\sum_j \frac{\partial g_{ij}}{\partial x^m} x^j = \sum_j \frac{\partial g_{mj}}{\partial x^i} x^j$

(4) $\sum_{i,j} \frac{\partial g_{ij}}{\partial x^m} x^i x^j = \sum_{i,j} \frac{\partial g_{mj}}{\partial x^i} x^i x^j = 0$

(5) $\sum_{i,j} \Gamma^k_{ij} x^i x^j = 0$

(6) $\nabla_{\partial_r}\partial_r = 0$ *over* $U\backslash\{p\}$.

Proof. (1)

$$g^{ij}\frac{x^i}{r}\frac{\partial}{\partial x^j} = \nabla r = \partial_r = \frac{x^i}{r}\frac{\partial}{\partial x^i}.$$

$$\sum_i g^{ij}x^i = x^j, i.e. \sum_j g_{ij}x^j = x^i.$$

(2)

$$\delta_{im} = \frac{\partial x^i}{\partial x^m} = \sum_j \frac{\partial g_{ij}}{\partial x^m} x^j + g_{im},$$

$$\sum_j \frac{\partial g_{ij}}{\partial x^m} x^j = \delta_{im} - g_{im} = \delta_{mi} - g_{mi} = \sum_j \frac{\partial g_{mj}}{\partial x^i} x^j.$$

(3)

$$\sum_{i,j} \frac{\partial g_{ij}}{\partial x^m} x^j x^i = \sum_i (\delta_{im} - g_{im}) x^i = x^m - \sum_i g_{mi}x^i = 0 = \sum_{i,j} \frac{\partial g_{mj}}{\partial x^i} x^j x^m.$$

(4)

$$\sum_{i,j} \Gamma^k_{ij} x^i x^j = \frac{g^{kl}}{2} \sum_{i,j} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) x^i x^j = 0.$$

(5)

$$\nabla_{\partial_r}\partial_r = \sum_{i,j} \frac{x^i}{r} \left(\frac{\partial}{\partial x^i}\left(\frac{x^j}{r}\right)\frac{\partial}{\partial x^j} + \frac{x^j}{r}\Gamma^k_{ij}\frac{\partial}{\partial x^k} \right) = \sum_{i,j} \frac{x^i(\delta_{ij}r^2 - x^j x^i)}{r^4}\frac{\partial}{\partial x^j} = 0.$$

□

Thm 9.2. *Let* (M, g) *be a complete Riemannian manifold,* $p \in M, U = M\backslash \mathrm{cut}(p)$ *and* $v \in \Sigma(p)\backslash\{0\}$*, then*

(1) The differential of $\exp_p$ *satisfies*

$$(\mathrm{d}\exp_p)_{tv}(v) = \gamma'_v(t),$$

where $v \in T_{tv}(T_pM) \cong T_pM, \gamma'_v(t) \in T_{\gamma(t)}M$.

(2) For any $\xi \in T_pM \cong T_{tv}(T_pM)$, we have

$$g\left((\mathrm{d}\exp_p)_{tv}(v), (\mathrm{d}\exp_p)_{tv}(\xi)\right) = g(v,\xi)$$

In particular,

(1) For any $t \in [0,1]$, we have $\left|(\mathrm{d}\exp_p)_{tv}(v)\right| = |v|$.

(2) If $\xi \perp v$, then $(\mathrm{d}\exp_p)_{tv}(v) \perp (\mathrm{d}\exp_p)_{tv}(\xi)$.

Proof. (1)

$$(\mathrm{d}\exp_p)_{tv}(v) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp_p(tv, sv) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \gamma_v(t+s) = \gamma_v'(t).$$

(2) Consider the variation

$$\alpha(t,s) = \exp_p(t(v+s\xi))$$

Then its variation field is the Jacobi field

$$J(t) = \alpha_*\left(\frac{\partial}{\partial s}\right)\Big|_{b=0} = (\mathrm{d}\exp_p)_{tv}(t\xi)$$

with $J(0)=0, J'(0)=\xi$, so

$$\langle J(t), \gamma'(t)\rangle = t\langle J'(0), \gamma'(0)\rangle = t\langle \xi, v\rangle.$$

Hence the proof is completed.

□

Prop 9.2. *(M,g) is complete, $p \in M, U = M\backslash \mathrm{cut}(p)$.*
Let $J(t)$ be the Jacobi field with $J(0)=0, J'(0)=a$, then in normal coordinates $(x^1,\cdots,x^m)$,

$$J(t) = \alpha_*\left(\frac{\partial}{\partial s}\right)\Big|_{s=0} = \sum_{i=1}^n ta^i \left.\frac{\partial}{\partial x^i}\right|_{\gamma_v(t)}.$$

And $\alpha(t,s) = \exp_p(t(v+sa)) = (t(v^1+sa^1),\cdots,t(v^n+sa^n))$.

Proof. Trivial. □

Def 9.2. Define $\mathscr{H}_f \in \Gamma(M, T^*M \otimes TM)$ as

$$g(\mathscr{H}_f(X), Y) = (\mathrm{Hess}\ f)(X,Y).$$

Prop 9.3. *For $X \in \Gamma(M,TM)$,*

$$\mathscr{H}_f(X) = \nabla_X \nabla f.$$

In particular, we have

$$\mathscr{H}_r(\partial_r) = \nabla_{\partial_r}\nabla r = \nabla_{\partial_r}\partial_r = 0.$$

Proof. For any $X, Y \in \Gamma(M,TM)$,

$$Xg(\nabla f, Y) = g(\nabla_X \nabla f, Y) + g(\nabla f, \nabla_X f).$$

So

$$g(\nabla_X\nabla f, Y) = X(Yf) - (\nabla_X Y)f = (\mathrm{Hess}\ f)(X,Y) = g(\mathscr{H}_f(X), Y).$$

□

Prop 9.4. *Locally,*

$$\mathscr{H}_f = g^{jk}\nabla_i\nabla_j f \mathrm{d}x^i \otimes \frac{\partial}{\partial x^k}.$$

Proof. let

$$\mathscr{H}_f = M_i^j \mathrm{d}x^i \otimes \frac{\partial}{\partial x^j}.$$

Then by definition,

$$M_i^l g_{lj} = g\left(\mathscr{H}_f\left(\frac{\partial}{\partial x^i}\right), \frac{\partial}{\partial x^j}\right) = \nabla_i\nabla_j f.$$

This concludes the desired formula. □

Prop 9.5. *Let (M,g) be a complete Riemannian manifold, $p \in M, U = M\backslash\mathrm{cut}(p)$ and r is the radial distance function on U, suppose J is a normal Jacobi field along γ_v with $J(0) = 0$, then*

(1) For any $t \in (0,b]$,

$$\mathscr{H}_r(J(t)) = J'(t), \mathscr{H}_r(\gamma_v'(t)) = 0.$$

(2) In particular, for any $W(t)$ along γ with $W(0) = 0$,

$$(\text{Hess } r)(J(s), W(s)) = \left(I_{\gamma_v}\right)_s(J,W) = \int_0^s \langle J', W'\rangle - R(J,\gamma',\gamma',W)\mathrm{d}t.$$

Proof. (1) Suppose $J'(0) = a = a^i \left.\frac{\partial}{\partial x^i}\right|_p$.

In normal coordinates $\{x^1,\cdots,x^n\}$ centered at $p \in M$,

$$J(t) = ta^i \left.\frac{\partial}{\partial x^i}\right|_{\gamma_v(t)}.$$

Then we have

$$J'(t) = \hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J(t) = a^i \left.\frac{\partial}{\partial x^i}\right|_{\gamma_v(t)} + ta^i\hat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\left.\frac{\partial}{\partial x^i}\right|_{\gamma_v(t)} = \left(a^k + ta^i\Gamma_{ij}^k\frac{\mathrm{d}\gamma_v^j}{\mathrm{d}t}\right)\left.\frac{\partial}{\partial x^k}\right|_{\gamma_v(t)}.$$

Since $\gamma_v(t) = (tv^1,\cdots,tv^n)$ and

$$r(\gamma_v(t)) = d(p,\gamma_v(t)) = \left|\exp_p^{-1}(\gamma_v(t))\right| = |tv| = t.$$

So

$$\begin{aligned}\mathscr{H}(J(t)) =& \nabla_{J(t)}\partial_r = \nabla_{ta^i\frac{\partial}{\partial x^i}}\left(\frac{x^j}{r}\frac{\partial}{\partial x^j}\right)\\ =& \left(\frac{ta^i}{r} - \frac{ta^ix^ix^j}{r^3}\right)\frac{\partial}{\partial x^j} + ta^i\frac{x^j}{r}\Gamma_{ij}^k(\gamma_v(t))\frac{\partial}{\partial x^k}\end{aligned}$$

And since $J(t)$ is normal Jacobi field

So $\langle J(t),\gamma'(t)\rangle = t\langle a,v\rangle = 0$.

Hence we have

$$\mathscr{H}_r(J(t)) = a^i \left.\frac{\partial}{\partial x^j}\right|_{\gamma_v(t)} + ta^iv^j\Gamma_{ij}^k \left.\frac{\partial}{\partial x^k}\right|_{\gamma_v(t)} = J'(t),$$

$$\mathscr{H}_r(\gamma'(t)) = \frac{x^ix^j}{t^2}\Gamma_{ij}^k(\gamma(t))\frac{\partial}{\partial x^k} = 0.$$

(2)

$$
\begin{aligned}
(\text{Hess } r)(J(s), W(s)) =& \langle \mathscr{H}_r(J(s)), W(s)\rangle = \langle J'(s), W(s)\rangle - \langle J'(0), W(0)\rangle \\
=& \int_0^s \left(\langle J'(t), W(t)\rangle\right)' \mathrm{d}t \\
=& \int_0^s \langle J''(t), W(t)\rangle + \langle J'(t), W'(t)\rangle \mathrm{d}t \\
=& I_{\gamma^s}(J, W)
\end{aligned}
$$

□

9.2 Metrics on manifolds with constant sectional curvature

Thm 9.3. *Let U be a geodesic ball around $p \in \mathbb{S}_k^n$, r be the radial distance function, $q \in U$ and $(x^1, \cdots, x^n)$ is normal coordinates centered at p, then*

$$g_k = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r)\hat{g},$$

where $\hat{g}$ is the induced form by $U \backslash \{p\} \to \mathbb{R}^n \backslash \{0\} \to \mathbb{S}^{n-1}$.

Proof. We set

$$g_c = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r)\hat{g}.$$

For $q \in U \backslash \{p\}$ and $w \in T_qM$, we shall show that

$$g_k(w, w) = g_c(w, w).$$

For $w = \nabla r = \partial r$, we have

$$g_k(w, w) = 1, g_c(w, w) = (\mathrm{d}r \otimes \mathrm{d}r)(\partial_r, \partial_r) + \mathrm{sn}_k^2(r)\hat{g}(\partial_r, \partial_r) = 1.$$

For $w \perp \partial_r\big|_q$, $(\mathrm{d}r)(w) = g(\partial_r, w) = 0$ and so $(\mathrm{d}r \otimes \mathrm{d}r)(w, w) = 0$.
Let $b = d(p, q)$, then

$$g_c(w, w) = \mathrm{sn}_k^2(b)\hat{g}(w, w) = \mathrm{sn}_k^2(b)\frac{\bar{g}(w, w)}{b^2} = \frac{\mathrm{sn}_k^2(b)}{b^2}\sum_i \left|w^i\right|^2.$$

Assume $\gamma : [0, b] \to U$ is a unit-speed geodesic connecting p and q and J is a Jacobi field along γ with $J(0) = 0, J(b) = w$.
By proposition 7.3, let $J(t) = c \cdot \mathrm{sn}_k(t)E(t)$, then

$$g_k(w, w) = |J(b)|_{g_k}^2 = c^2 \mathrm{sn}_k^2(b) = \left|J'(0)\right|^2 \mathrm{sn}_k^2(b).$$

And let $J'(0) = a^i \left.\dfrac{\partial}{\partial x^i}\right|_p$, therefore

$$J(b) = (ba^1, \cdots, ba^n) = (w^1, \cdots, w^n).$$

Hence

$$g_k(w, w) = \sum_i \frac{\left|w^i\right|^2}{b^2}\mathrm{sn}_k^2(b) = g_c(w, w).$$

□

Prop 9.6. *Let U be a geodesic ball in $\mathbb{S}_k^n$ around p and r be the radial distance function, then*

$$\mathscr{H}_r = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r,$$

where $\pi_r : T_q\mathbb{S}_k^n \to W \subset T_q\mathbb{S}_k^n$ and W is the orthogonal complement of $\partial_r\big|_q$.
In particular,

$$\mathrm{Hess}\ r = \mathrm{sn}_k'(r)\mathrm{sn}_k(r)\hat{g},$$

$$\Delta r = (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)},$$

$$\Delta r^2 = 2 + 2(n-1)r\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}.$$

Proof. Consider the parallel vector field $E(t)$ along $\gamma : [0,b] \to M$ with $\langle E, \gamma'\rangle \equiv 0, |E| \equiv 1$.
Then $J(t) = c \cdot \mathrm{sn}_k(t)E(t)$ is the Jacobi field with $J(0) = 0$ and $J'(0) = cE(0)$.
By proposition 9.5,

$$\mathscr{H}_r(J(t)) = J'(t) = c \cdot \mathrm{sn}_k'(t)E(t), i.e.\mathscr{H}_r(E(t)) = \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}E(t).$$

And by theorem 9.3,

$$(\mathrm{Hess}\ r)(X,Y) = g(\mathscr{H}_r(X),Y) = \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}g(\pi_r(X),Y) = \mathrm{sn}_k'(r)\mathrm{sn}_k(r)\hat{g}(X,Y)$$

Moreover, in orthonormal basis $\left(E_1(t), \cdots, E_{n-1}(t), \frac{\partial}{\partial r}\right)$, $\mathrm{tr}\,\pi_r = n-1$ and so

$$\Delta r = \mathrm{tr}\,\mathscr{H}_r = (n-1)\frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)},$$

$$\Delta r^2 = 2r\Delta r + 2g(\nabla r, \nabla r) = 2r\Delta r + 2.$$

□

Prop 9.7. *Let (M,g) be a complete Riemannian manifold, $p \in M, U = M\backslash\mathrm{cut}(p)$ and r be the radial distance function on U, if*

$$\mathscr{H}_r = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r$$

holds on $U\backslash\{p\}$, then (M,g) has constant sectional curvature k.

Proof. Let $\gamma : [0,b] \to U$ be a unit-speed curve with $\gamma(0) = p, J$ is a normal Jacobi field with $J(0) = 0$, then

$$J' = \mathscr{H}_r(J) = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}J.$$

So $\frac{J}{\mathrm{sn}_k(t)}$ is a parallel vector field, let

$$\frac{J}{\mathrm{sn_k}(t)} = cE(t).$$

Similar to the proof of theorem 9.3, we have $g = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r)\hat{g} = g_k$. □

Chapter 10

Comparison theorem and applications

10.1 The Rauch comparison theorems and applications

Thm 10.1 (Rauch comparison theorem). *Let $(M,g),(\tilde{M},\tilde{g})$ be two Riemannian manifolds and $p\in M,\tilde{p}\in\tilde{M},U=M\backslash\mathrm{cut}(p),\tilde{U}=\tilde{M}\backslash\mathrm{cut}(\tilde{p}),2\leqslant\dim M\leqslant\dim\tilde{M}$.*

Consider unit-speed geodesics $\gamma:[0,l]\to M$ and $\tilde{\gamma}:[0,l]\to\tilde{M}$, suppose

(1) for any $t\in[0,l]$ and any planes $\Sigma\subset T_{\gamma(t)}M,\tilde{\Sigma}\subset T_{\tilde{\gamma}(t)}\tilde{M}$ such that $\gamma'(t)\in\Sigma,\tilde{\gamma}'(t)\in\tilde{\Sigma}$,

$$K_\Sigma(\gamma(t))\leqslant\tilde{K}_{\tilde{\Sigma}}(\tilde{\gamma}(t)).$$

(2) $\tilde{\gamma}(0)$ has no conjugate point along $\tilde{\gamma}\big|_{[0,l]}$.

Then for any two Jacobi fields $J,\tilde{J}$ along $\gamma,\tilde{\gamma}$ resp. such that

$$J(0)=c\gamma'(0),\tilde{J}(0)=c\tilde{\gamma}'(0),\left|J'(0)\right|=\left|\tilde{J}'(0)\right|,g(J'(0),\gamma'(0))=\tilde{g}(\tilde{J}'(0),\tilde{\gamma}'(0)),$$

we have $\left|\tilde{J}(t)\right|\leqslant|J(t)|$ for $t\in[0,l]$.

Proof. We first assume $J(0)=\tilde{J}(0)=0,g(J'(0),\gamma'(0))=\tilde{g}(\tilde{J}'(0),\tilde{\gamma}'(0))=0$.

Since $\tilde{\gamma}$ has no conjugate points along $\tilde{\gamma}_{[0,l]}$.

So $\frac{|J|^2}{|\tilde{J}|^2}$ is well-defined on $(0,l]$ and

$$\lim_{t\to0}\frac{|J|^2}{|J'|^2}=\lim_{t\to0}\frac{\langle J,J'\rangle}{\left\langle\tilde{J},\tilde{J}'\right\rangle}=\lim_{t\to0}\frac{\langle J,J''\rangle+|J'|^2}{\left\langle\tilde{J},\tilde{J}''\right\rangle+\left|\tilde{J}'\right|^2}=1.$$

Therefore it suffices to show that

$$\left(\frac{|J|^2}{\left|\tilde{J}\right|^2}\right)'=2\frac{\left|\tilde{J}\right|^2\langle J',J\rangle-\langle\tilde{J}',\tilde{J}\rangle|J|^2}{\left|\tilde{J}\right|^4}\geqslant0.$$

We want to show

$$\frac{\langle J',J\rangle}{|J|^2}\geqslant\frac{\left\langle\tilde{J}',\tilde{J}\right\rangle}{\left|\tilde{J}\right|^2}.$$

For fixed $s \in (0, l]$, we assume $J(s) \neq 0$ and define

$$W_s(t) = \frac{J(t)}{|J(s)|}, \tilde{W}_s(t) = \frac{\tilde{J}(t)}{\left|\tilde{J}(s)\right|}.$$

Then $|W_s(s)| = \left|\tilde{W}_s(s)\right| = 1$ and

$$\frac{\langle J', J\rangle}{|J|^2} = \frac{\langle W_s', W_s\rangle}{|W_s|^2}, \frac{\langle \tilde{J}', \tilde{J}\rangle}{\left|\tilde{J}\right|^2} = \frac{\langle \tilde{W}_s', \tilde{W}_s\rangle}{\left|\tilde{W}_s\right|^2}.$$

Moreover,

$$\begin{aligned}\left.\frac{\langle J'(t), J(t)\rangle}{|J(t)|^2}\right|_{t=s} &= \left.\langle W_s'(t), W_s(t)\rangle\right|_{t=s} = \int_0^s \left(\langle W_s'(t), W_s(t)\rangle\right)' \mathrm{d}t \\ &= \int_0^s \langle W_s', W_s'\rangle - \mathrm{R}(W_s, \gamma', \gamma', W_s)\mathrm{d}t\end{aligned}$$

$$\left.\frac{\langle \tilde{J}'(t), \tilde{J}(t)\rangle}{\left|\tilde{J}(t)\right|^2}\right|_{t=s} = \int_0^s \left\langle \tilde{W}_s', \tilde{W}_s'\right\rangle - \mathrm{R}\left(\tilde{W}_s, \tilde{\gamma}', \tilde{\gamma}', \tilde{W}_s\right)\mathrm{d}t$$

Choose parallel orthonormal frames $\{e_1(t) = \gamma'(t), e_2(t) = W_t(t), \cdots, e_n(t)\}$ along γ and $\{\tilde{e}_1(t) = \tilde{\gamma}'(t), \tilde{e}_2(t) = \tilde{W}_t(t), \cdots, \tilde{e}_m(t)\}$ along $\tilde{\gamma}$.

Let

$$W_s(t) = \sum_{i=1}^n \lambda^i(t) e_i(t)$$

and define a vector field $\tilde{V}$ along $\tilde{\gamma}$ with

$$\tilde{V}(t) = \sum_{i=1}^n \lambda^i(t)\tilde{e}_i(t).$$

Then $\tilde{V}(s) = \tilde{e}_2(s) = \tilde{W}_s(s)$ and so by corollary 7.7,

$$\begin{aligned}\left.\frac{\langle J'(t), J(t)\rangle}{|J(t)|^2}\right|_{t=s} &= \int_0^s \left\langle \tilde{V}', \tilde{V}'\right\rangle_{\tilde{g}} - \mathrm{R}(W_s, \gamma', \gamma', W_s)\mathrm{d}t \\ &\geqslant \int_0^s \left\langle \tilde{V}', \tilde{V}'\right\rangle_g - \tilde{\mathrm{R}}\left(\tilde{V}, \gamma', \gamma', \tilde{V}\right)\mathrm{d}t \\ &\geqslant \int_0^s \left\langle \tilde{W}_s', \tilde{W}_s'\right\rangle - \tilde{\mathrm{R}}\left(\tilde{W}_s, \gamma', \gamma', \tilde{W}_s\right)\mathrm{d}t\end{aligned}$$

For the general cases, let

$$J(t) = J_1(t) + \langle J, \gamma'(t)\rangle\gamma'(t), \tilde{J}(t) = \tilde{J}_1(t) + \langle \tilde{J}, \tilde{\gamma}'(t)\rangle\tilde{\gamma}'(t).$$

So

$$|J|^2 = |J_1|^2 + \left|\langle J, \gamma'(t)\rangle\right|^2, \left|\tilde{J}\right|^2 = \left|\tilde{J}_1\right|^2 + \left|\langle \tilde{J}, \tilde{\gamma}'\rangle\right|^2,$$

$$\langle J(t), \gamma'(t)\rangle = \langle J(0), \gamma'(0)\rangle + t\langle J'(0), \gamma'(0)\rangle = \langle \tilde{J}(0), \tilde{\gamma}'(0)\rangle + t\langle \tilde{J}'(0), \tilde{\gamma}'(0)\rangle = \langle \tilde{J}(t), \tilde{\gamma}'(t)\rangle.$$

Hence $|J|^2 \geqslant \left|\tilde{J}\right|^2$. □

Coro 10.1 (Jacobi field comparison)**.** *Let (M,g) be a complete Riemannian manifold $p \in M, U = M\backslash \mathrm{cut}(p)$, $\gamma : [0,b] \to U$ be a unit-speed geodesic with $\gamma(0) = p$ and J be any normal Jacobi field along γ with $J(0) = 0$.*

(1) If the sectional curvature $K_M \leqslant k$, then

$$|J(t)| \geqslant \mathrm{sn}_k(t)\big|J'(0)\big|$$

for all $t \in [0,b_1]$, where $b_1 = b$ if $k \leqslant 0$ and $b_1 = \min(\pi R, b)$ if $k = R^{-2} > 0$.

(2) If the sectional curvature $K_M \geqslant k$, then

$$|J(t)| \leqslant \mathrm{sn}_k(t)\big|J'(0)\big|.$$

Coro 10.2 (conjugate comparison)**.** *Let (M,g) be a Riemannian manifold with sectional curvature $K_M \leqslant k$, then*

(1) If $k \leqslant 0$, then M has no conjugate point.

(2) If $k = \frac{1}{R^2}$, then there is no conjugate point on any geodesic shorter than πR.

Proof. (1) See proposition 7.4.

(2) Let $\gamma : [0,b] \to M$ be a unit-speed geodesic and J be a nontrivial normal Jacobi field along γ with $J(0) = 0$.

By corollary 10.1,

$$|J(t)| \geqslant \mathrm{sn}_k(t)\big|J'(0)\big| = R\sin\left(\frac{t}{R}\right)\big|J'(0)\big|.$$

So if $t \in (0, \pi R)$, then $J(t) \neq 0$.

□

Coro 10.3 (metric comparison)**.** *Let (M,g) be a Riemannian manifold with $\dim M = n$ and (U, x^i) be a geodesic ball for g around some point $p \in M$.*
Consider the constant sectional curvature metric $g_k = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r)\hat{g}$ on $U\backslash\{p\}$.

(1) If $K_M \leqslant k$, then for any $q \in U\backslash\{p\}$ and $W \in T_qM$,

$$g(W,W) \geqslant g_k(W,W).$$

Here if $k = \frac{1}{R^2} > 0$, we need the condition $d_g(p,q) < \pi R$.

(2) if $K_M \geqslant k$, then for any $q \in U\backslash\{p\}$ and $W \in T_qM$,

$$g(W,W) \leqslant g_k(W,W).$$

Proof. (1) Since ∂_r is a unit vector w.r.t. both g_k and g.

So we only consider $w \in T_qM$ with $w \perp \partial_r$.

Consider a unit-speed geodesic $\gamma : [0,b] \to U$ with $\gamma(0) = p, \gamma(b) = q$ and Jacobi field J along γ with $J(0) = 0, J(b) = W$.

By corollary 10.1,

$$g(W,W) = |J(b)|_g^2 \geqslant \mathrm{sn}_k^2(b)\big|J'(0)\big|_{g_p}^2.$$

And since γ is also a geodesic for g_k

Hence by proposition 7.3,

$$g_k(W,W) = |J_k(b)|_{g_k}^2 = \mathrm{sn}_k^2(b)\big|J'(0)\big|_{g_k}^2 \leqslant g(W,W).$$

□

Coro 10.4. *Let $(M,g),(\tilde M,\tilde g)$ be two Riemannian manifold with $K_M \leqslant K_{\tilde M}$.*
Fix $p\in M, \tilde p\in \tilde M$ and a linear isometry $\Phi_0 : T_pM \to T_{\tilde p}\tilde M$.
Suppose $0<\delta<\min(\mathrm{inj}_p M, \mathrm{inj}_{\tilde p}\tilde M)$, then for any curve $\alpha:[0,1]\to \exp_p(B(0,\delta))$ and

$$\tilde\alpha = \exp_{\tilde p}\circ\Phi_0\circ\exp_p^{-1}\circ\alpha : [0,1]\to\tilde M,$$

we have $|\alpha'|\geqslant|\tilde\alpha'|$ and so $L(\alpha)\geqslant L(\tilde\alpha)$.

Proof. Let $c(s)=\exp_p^{-1}(\alpha(s))$ and $\tilde c(s)=\exp_{\tilde p}^{-1}(\tilde\alpha(s))$, then

$$\tilde c(s) = \Phi_0(c(s)).$$

And we have geodesic variants

$$\beta(t,s)=\exp_p(tc(s)), \tilde\beta(t,s)=\exp_{\tilde p}(t\tilde c(s))$$

For fixed $s_0\in[0,1]$, consider geodesics $\gamma_{s_0}(t)=\beta(t,s_0)$ and $\tilde\gamma_{s_0}(t)=\tilde\beta(t,s_0)$.
Then $\gamma'_{s_0}(0)=c(s_0), \tilde\gamma'_{s_0}(0)=\tilde c(s_0)$.
Consider their Jacobi fields

$$J_{s_0}(t)=\beta_*\left(\frac{\partial}{\partial s}\right)\bigg|_{s=s_0}, \tilde J(t)=\tilde\beta_*\left(\frac{\partial}{\partial s}\right)\bigg|_{s=s_0}.$$

Then

$$J_{s_0}(0)=0, J'_{s_0}(0)=c'(s_0), J_{s_0}(1)=(\mathrm{d}\exp_p)_{c(s_0)}(c'(s_0))=\alpha'(s_0)$$
$$\tilde J_{s_0}(0)=0, \tilde J_{s_0}(0)=\tilde c'(s_0), \tilde J_{s_0}(1)=\tilde\alpha'(s_0)$$

So we have $|J'_{s_0}(0)|=\left|\tilde J'_{s_0}(0)\right|$ and

$$\langle\tilde J'_{s_0}(0),\tilde\gamma'_{s_0}(0)\rangle=\langle\tilde c'(s_0),\tilde c(s_0)\rangle=\langle J'_{s_0}(0),\gamma'_{s_0}(0)\rangle.$$

Hence $|J_{s_0}(1)|\geqslant\left|\tilde J_{s_0}(1)\right|$, *i.e.*$|\alpha'(s_0)|\geqslant|\tilde\alpha'(s_0)|$. □

Coro 10.5. *Let (M,g) be a complete Riemannian manifold and suppose there exist $C_1,C_2>0$ such that $C_1\leqslant K_M\leqslant C_2$.*
Let γ be any geodesic in M and l be the distance between two consecutive conjugate points along γ, then

$$\frac{\pi}{\sqrt{C_2}}\leqslant l\leqslant\frac{\pi}{\sqrt{C_1}}.$$

In particular, $\exp_p : B\left(0,\frac{\pi}{\sqrt{C_2}}\right)\to M$ has no critical point.

Proof. Let p,q be the consecutive conjugate points along $\gamma:[0,l]\to M$ and J be the normal Jacobi field along γ and $J(0)=0$.
WLOG, we assume γ is a unit-speed geodesic.
For $\left(\tilde M,\tilde g\right)=(\mathbb{S}^n(r_2),g_{can})$, where $r_2=\sqrt{C_2^{-1}}$, $K_M\leqslant K_{\tilde M}$.
Let $\tilde\gamma:[0,\pi r_2]\to\tilde M$ be a unit-speed geodesic connecting poles and $\tilde J$ be a normal Jacobi field along $\tilde\gamma$ with $\tilde J(0)=0, \left|\tilde J'(0)\right|=|J'(0)|$.
Since $\tilde\gamma$ has no conjugate point.
So by theorem 10.1, $|J(t)|\geqslant\left|\tilde J(t)\right|$ for $t\in(0,\pi r_2)$, *i.e.*$l\geqslant\pi r_2$.
For $(\tilde M,\tilde g)=(S^n(r_1),g_{can})$, where $r_1=\sqrt{C_1^{-1}}$, $K_M\geqslant J_{\tilde M}$.

Let $\tilde{\gamma} : [0, \pi r_1] \to \tilde{M}$ be a unit-speed geodesic connecting poles and $\tilde{J}$ be a normal Jacobi field along $\tilde{\gamma}$ with $\tilde{J}(0) = 0, \left|\tilde{J}'(0)\right| = |J'(0)|$.

Suppose $l > \pi r_1$.

Then γ has no conjugate points.

So by theorem theorem 10.1 $\left|\tilde{J}(t)\right| \geqslant |J(t)|$ but $\tilde{J}(\pi r_1) = 0$.

Therefore $J(\pi r_1) = 0$, contradiction! $\square$

Lemma 10.1. *Let (M, g) be a complete Riemannian manifold and $p \in M$, then for any closed geodesic γ_0 passing p, we have*

$$L(\gamma_0) \geqslant 2d(p, \mathrm{cut}(p)).$$

Moreover, suppose $\exists q \in \mathrm{cut}(p)$ such that $d(p, q) = d(p, \mathrm{cut}(p))$.

(1) Then either q is conjugate to p along some minimizing geodesic connecting p and q, or there are exactly two minimizing geodesic $\gamma_1, \gamma_2 : [0, b] \to M$ from p to q, such that $\gamma_1'(b) = -\gamma_2'(b)$, where $b = d(p, q)$.

(2) If in addition that $\mathrm{inj}_p M = \mathrm{inj}(M)$, and q is not conjugate to p along any minimizing geodesic, then there is a closed unit-speed geodesic $\gamma : [0, 2b] \to M$ such that $\gamma(0) = \gamma(2b) = p$ and $\gamma(b) = q$ where $b = d(p, q)$.

Proof. (1) Suppose $q \notin \mathrm{conj}(p)$ and $\gamma_1, \gamma_2 : [0, b] \to M$ are two different minimizing geodesics from p to q such that $\gamma_1'(b) \neq -\gamma_2'(b)$.

WLOG, let $b = 1$.

Then there exists $w \in T_q M$, such that

$$\langle w, \gamma_1'(1) \rangle < 0, \langle w, \gamma_2'(1) \rangle < 0.$$

And since q is a regular value for $\exp_p$, $i.e.$ $\mathrm{d}\exp_p$ is nonsingular at $\gamma_1'(0), \gamma_2'(0)$.

So for small s, there exists smooth curves $v_i(s) \in T_p M$, such that

$$v_i(0) = \gamma_i'(0), \exp_p(v_i(s)) = \exp_q(sw).$$

Consider the variations

$$\alpha_i(t, s) = \exp_p(t v_i(s)).$$

Then similar to corollary 7.1, we have

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} L(\alpha_i(\bullet, s)) &= \left\langle (\alpha_i)_* \left(\frac{\partial}{\partial s}\right)\Big|_{(1,0)}, \gamma_i'(1) \right\rangle \\ &= \left\langle \left(\mathrm{d}\exp_q\right)_0 (w), \gamma_i'(1) \right\rangle \\ &= \left\langle w, \gamma_i'(1) \right\rangle < 0 \end{aligned}$$

Therefore for small s, $L(\alpha_i(\bullet, s)) < L(\gamma_i) = d(p, q)$ and $v_1(s) \neq v_2(s)$.

Hence $\exp_p$ is not bijective in $\Sigma(p)$, contradiction!

(2) Since $\mathrm{inj}_p M = \mathrm{inj}(M) = \inf\limits_{q \in M} \mathrm{inj}_q M$.

So $d(p, \mathrm{cut}(p)) \leqslant d(q, \mathrm{cut}(q))$, and there are two minimizing geodesics from p to q, say $\gamma_1, \gamma_2 : [0, b] \to M, s.t. \gamma_1'(b) = -\gamma_2'(b)$.

On the other hand $p \in \mathrm{cut}(q), i.e. d(q, \mathrm{cut}(q)) \leqslant d(q, p) = d(p, \mathrm{cut}(p))$.

Therefore $d(q,p) = d(q, \mathrm{cut}(q))$, $i.e. \gamma_1'(0) = -\gamma_2'(0)$.

Consider $\gamma(t) = \gamma_1 \cdot \gamma_2^{-1}$, then

$$\gamma(0) = \gamma(2b) = p, \gamma(b) = q, \gamma'(0) = \gamma_1'(0) = -\gamma_2'(0), \gamma'(b) = \gamma_1'(b) = -\gamma_2'(b).$$

Hence γ is a closed unit-speed geodesic.

□

Thm 10.2 (Klingenberg's estimate for injective radius)**.** *Let (M,g) be a compact Riemannian manifold with $K_M \leqslant C$ where $C > 0$, we define*

$$l(M,g) = \inf\{L(\gamma) | \gamma \textit{ is a smooth closed geodesic in } M\}$$

Then $\mathrm{inj}(M) \geqslant \frac{\pi}{\sqrt{C}}$ *or* $\mathrm{inj}(M) = \frac{l(M,g)}{2}$.

Proof. Consider $p, q \in M$ such that $d(p,q) = d(p, \mathrm{cut}(p)) = \mathrm{inj}(M)$.

If p, q are conjugate to each other, then by corollary 10.5, $d(p,q) \geqslant \frac{\pi}{\sqrt{C}}$.

If q is not a conjugate point to p, then by lemma 10.1, there is a closed geodesic $\gamma : [0, 2b] \to M$ with $\gamma(0) = p, \gamma(b) = q$.

So $l(M,g) \leqslant 2b = 2\mathrm{inj}(M)$.

And since every closed geodesic contains a cut point.

Hence $\mathrm{inj}(M) = \frac{l(M,g)}{2}$. □

Exam 10.1. *In $(\mathbb{S}^1 \times \mathbb{S}^1, g_{can})$, $K_M = 0 \leqslant C$ for any $C > 0$.*

So $\mathrm{inj}(M) = \frac{2\pi}{2} = \pi$.

10.2 Hessian comparison and Laplacian comparison theomem

Thm 10.3. *Let (M,g) and $\left(\tilde{M}, \tilde{g}\right)$ be two Riemannian manifolds with* $\dim M = \dim \tilde{M}$*, $p \in M, \tilde{p} \in \tilde{M}$ and $U = M \backslash \mathrm{cut}(p), \tilde{U} = \tilde{M} \backslash \mathrm{cut}(\tilde{p})$.*

Suppose $\gamma : [0,b] \to U$ and $\tilde{\gamma} : [0,b] \to \tilde{U}$ are two unit-speed geodesics with $\gamma(0) = p, \gamma(b) = q$ and $\tilde{\gamma}(0) = \tilde{p}, \tilde{\gamma}(b) = \tilde{q}$.

If for any $t \in [0,b]$ and any planes $\Sigma \subset T_{\gamma(t)}M$ and $\tilde{\Sigma} \subset T_{\tilde{\gamma}(t)}\tilde{M}$ with $\gamma'(t) \in \Sigma, \tilde{\gamma}'(t) \in \tilde{\Sigma}$, the corresponding sectional curvatures satisfy

$$K_\Sigma(\gamma(t)) \geqslant \tilde{K}_{\tilde{\Sigma}}(\tilde{\gamma}(t)),$$

Then for any $X \in T_qM, \tilde{X} \in T_{\tilde{q}}\tilde{M}$ with $|X| = \left|\tilde{X}\right| = 1$ and $X \perp \gamma(b)', \tilde{X} \perp \tilde{\gamma}'(b)$,

$$\mathrm{Hess}\ r(X,X) \leqslant \mathrm{Hess}\ \tilde{r}\left(\tilde{X}, \tilde{X}\right).$$

In particular, for $t \in (0,b]$,

$$(\Delta r)(\gamma(t)) \leqslant \left(\tilde{\Delta}\tilde{r}\right)(\tilde{\gamma}(t)).$$

Moreover, if the identity holds for all $t \in (0,b]$, then $K_\Sigma(\gamma(t)) = \tilde{K}_{\tilde{\Sigma}}(\tilde{\gamma}(t))$.

Proof. Let $(e_1(t), \cdots, e_{n-1}(t), e_n(t) = \gamma'(t))$ be a parallel orthonormal frame along γ and $(\tilde{e}_1(t), \cdots, \tilde{e}_{n-1}(t), \tilde{e}_n(t) = \tilde{\gamma}'(t))$ be a parallel orthonormal frame along $\tilde{\gamma}$.

We assume at point $\gamma(b), \tilde{\gamma}(b)$, $\langle X, e_i(b)\rangle = \langle \tilde{X}, \tilde{e}_i(b)\rangle$ for $i = 1, \cdots, n-1$.

Let J be the Jacobi field along γ with $J(0) = 0, J(b) = X$ and $\tilde{J}$ be the Jacobi field along $\tilde{\gamma}$ with $\tilde{J}(0) = 0, \tilde{J}(b) = \tilde{X}$.

Then $J(t) \perp \gamma'(t), \tilde{J}(t) \perp \tilde{\gamma}'(t)$ for every $t \in [0, b]$.

If we write $\tilde{J}(t) = \sum_{i=1}^{n-1} \lambda^i(t)\tilde{e}_i(t)$, then

$$X = \sum_{i=1}^{n-1} \langle X, e_i(b) \rangle e_i(b) = \sum_{i=1}^{n-1} \left\langle \tilde{X}, \tilde{e}_i(b) \right\rangle e_i(b) = \sum_{i=1}^{n} \lambda^i(b) e_i(b).$$

Set $Z(t) = \sum_{i=1}^{n-1} \lambda^i(t)e_i(t)$ along γ.

Then $Z(0) = 0, Z(b) = J(b) = Z$ and

$$\left|\tilde{J}'(t)\right| = \left|\sum_{i=1}^{n-1} (\lambda^i)'\tilde{e}_i(t)\right| = \left|\sum_{i=1}^{n-1} (\lambda^i)' e_i(t)\right| = \left|Z'(t)\right|$$

So by proposition 9.5 and corollary 7.7,

$$\begin{aligned}
(\text{Hess } r)_q(X, X) =& I_\gamma(J, J) \leqslant I_\gamma(Z, Z) \\
=& \int_0^b (\left|Z'(t)\right|^2 - \mathrm{R}(Z(t), \gamma'(t), \gamma'(t), Z(t)))\mathrm{d}t \\
=& \int_0^b (\left|\tilde{J}'(t)\right|^2 - \mathrm{R}(Z(t), \gamma'(t), \gamma'(t), Z(t)))\mathrm{d}t \\
\leqslant& \int_0^b (\left|\tilde{J}'(t)\right|^2 - \tilde{\mathrm{R}}(\tilde{J}(t), \tilde{\gamma}'(t), \tilde{\gamma}'(t), \tilde{J}(t)))\mathrm{d}t \\
=& I_{\tilde{\gamma}}\left(\tilde{J}, \tilde{J}\right) = (\text{Hess } \tilde{\gamma})\left(\tilde{X}, \tilde{X}\right)
\end{aligned}$$

□

Coro 10.6. *Let (M, g) be a complete Riemannian manifold, $p \in M, U = M\backslash\mathrm{cut}(p)$ and r is the radial distance function.*

(1) If $K_M \leqslant k$, then on $U_0\backslash\{p\}$,

$$\mathscr{H}_r \geqslant \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r, \Delta r \geqslant (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)},$$

where $U_0 = U$ if $k \leqslant 0$ and $U_0 = U \cap B_{\pi R}(0)$ if $k = R^{-2} > 0$.

Moreover, if the identity holds on $U_0\backslash\{p\}$, then g has constant sectional curvature k on $U_0\backslash\{p\}$.

(2) If $K_M \geqslant k$, then on $U\backslash\{p\}$,

$$\mathscr{H}_r \leqslant \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r, \Delta r \leqslant (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}.$$

Moreover, if the identity holds on $U\backslash\{p\}$, then g has constant sectional curvature k on $U\backslash\{p\}$.

Proof. The result follows from proposition 9.6 and theorem 10.3. □

Thm 10.4 (Laplacian comparison theorem)**.** *Let (M, g) be a complete Riemannian manifold, $p \in M, U = M \backslash \mathrm{cut}(p)$ and r is the radial distance function.*

If there exists some constant $k \in \mathbb{R}$ such that

$$\mathrm{Ric}(g) \geqslant (n-1)kg,$$

then the following inequality holds on $U \backslash \{p\}$

$$\Delta r \leqslant (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}.$$

Moreover, if the identity holds on $U \backslash \{p\}$, then (M, g) has constant sectional curvature k.

Proof. Let $q \in U \backslash \{p\}$, $\gamma : [0, b] \to U$ be a unit-speed minimal geodesic with $\gamma(0) = p, \gamma(b) = q$ and $(e_1(t), \cdots, e_{n-1}(t), e_n(t) = \gamma'(t))$ be a parallel orthonormal frame along γ.

By proposition 9.5,

$$\Delta r(q) = \sum_{i=1}^{n-1} (\mathrm{Hess}\ r)(e_i(b), e_i(b)) = \sum_{i=1}^{n} I_\gamma(J_i, J_i),$$

where J_i are Jacobi fields on γ such that $J_i(0) = 0, J_i(b) = e_i(b)$.

Let $\tilde{M} = \mathbb{S}_k^n, \tilde{p} \in \tilde{M}, \tilde{U} = \mathbb{S}_k^n \backslash \mathrm{cut}(\tilde{p})$, $\tilde{r}$ be the radial distance function on $\tilde{U}$, $\tilde{\gamma} : [0, b] \to \tilde{U}$ be a unit-speed geodesic with $\tilde{\gamma}(0) = \tilde{p}, \tilde{\gamma}(b) = \tilde{q}$ and $(\tilde{e}_1(t), \cdots, \tilde{e}_n(t) = \tilde{\gamma}'(t))$ be a parallel orthonormal frame along $\tilde{\gamma}$.

Similarly, we define Jacobi fields $\tilde{J}_i$ along $\tilde{\gamma}$ such that $\tilde{J}_i(0) = 0, \tilde{J}_i(b) = e_i(b)$, then

$$\tilde{\Delta}\tilde{r}(\tilde{q}) = \sum_{i=1}^{n-1} (\mathrm{Hess}\ \tilde{\gamma})(\tilde{e}_i(b), \tilde{e}_i(b)) = \sum_{i=1}^{n-1} \tilde{I}\left(\tilde{J}_i, \tilde{J}_i\right)$$

And by proposition 9.6, $\tilde{J}_i(t) = f(t)\tilde{e}_i(t)$ where $f(t) = \frac{\mathrm{sn}_k(t)}{\mathrm{sn}_k(b)}$, so

$$\tilde{\Delta}\tilde{r}(\tilde{q}) = \sum_{i=1}^{n-1} \int_0^b \left(\left|\tilde{J}_i'(t)\right|^2 - \tilde{\mathrm{R}}\left(\tilde{J}_i(t), \tilde{\gamma}'(t), \tilde{\gamma}'(t), \tilde{J}_i(t)\right)\right) \mathrm{d}t = (n-1)\int_0^b \left(\left|f'(t)\right| - kf^2(t)\right) \mathrm{d}t$$

We set $Z_i = f(t)e_i(t)$, by corollary 7.7,

$$\begin{aligned}
\Delta r =& \sum_{i=1}^{n-1} I_r(J_i, J_i) \leqslant \sum_{i=1}^{n-1} I_r(Z_i, Z_i) \\
=& \sum_{i=1}^{n-1} \int_0^b \left(\left|Z_i'(t)\right|^2 - R(Z_i(t), \gamma'(t), \gamma'(t), Z_i(t))\right) \mathrm{d}t \\
=& \sum_{i=1}^{n-1} \int_0^b \left(\left|f'(t)\right|^2 - |f(t)|^2 R(e_i(t), \gamma'(t), \gamma'(t), e_i(t))\right) \\
=& \int_0^b (n-1)\left|f'(t)\right|^2 \mathrm{d}t - \int_0^b f^2 \mathrm{Ric}(\gamma'(t), \gamma'(t)) \\
\leqslant& (n-1)\int_0^b (\left|f'(t)\right|^2 - k|f(t)|^2)\mathrm{d}t = \tilde{\Delta}\tilde{r}(\tilde{q})
\end{aligned}$$

Moreover, if $\Delta r = (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$, then

$$\partial_r(\Delta r) + \frac{(\Delta r)^2}{n-1} + (n-1)k = 0.$$

On the other hand, by Bochner formula,

$$0 = \frac{1}{2}\Delta|\nabla r|^2 = |\text{Hess } r|^2 + \text{Ric}(\nabla r, \nabla r) + \partial_r \Delta r \geqslant \partial_r(\Delta r) + |\text{Hess } r|^2 + (n-1)k.$$

So we have

$$\frac{(\Delta r)^2}{n-1} \geqslant |\text{Hess } r|^2.$$

And by Cauchy-Scharwz inequality,

$$(\Delta r)^2 \leqslant (n-1)|\text{Hess } r|^2.$$

Since

$$g(g, \mathrm{d}r \otimes \mathrm{d}r) = |\mathrm{d}r|^2 = 1, g(\text{Hess } r, \mathrm{d}r \otimes \mathrm{d}r) = \text{Hess } r(\partial_r, \partial_r) = 0.$$

Therefore we obtain

$$\left|\text{Hess } r - \frac{\text{sn}_k'(r)}{\text{sn}_k(r)}(g - \mathrm{d}r \otimes \mathrm{d}r)\right|^2 = |\text{Hess } r|^2 + (n-1)\frac{\text{sn}_k'(r)}{\text{sn}_k(r)} - 2\frac{\text{sn}_k'(r)}{\text{sn}_k(r)}\Delta r = 0.$$

Hence

$$\text{Hess } r = \frac{\text{sn}_k'(r)}{\text{sn}_k(r)}(g - \mathrm{d}r \otimes \mathrm{d}r) = \text{sn}_k'(r)\text{sn}_k(r)\hat{g}.$$

□

Coro 10.7. *Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature, $p \in M, U = M \backslash \text{cut}(p)$ and r is the radial distance function, then on $U \backslash \{p\}$,*

$$\Delta r \leqslant \frac{n-1}{r}, \Delta r^2 \leqslant 2n.$$

Proof. directly follows from the Laplacian comparison theorem. □

Lemma 10.2 (Ricatti comparison principle)**.** *If $f : (0, b) \to \mathbb{R}$ is a smooth function satisfying $f(t) = \frac{1}{t} + O(1)$ and for some $k > 0$ with $b \leqslant \frac{\pi}{\sqrt{k}}$,*

$$f' + f^2 + k \leqslant 0,$$

then for any $t \in (0, b)$,

$$f(t) \leqslant \frac{\text{sn}_k'(t)}{\text{sn}_k(t)}.$$

Proof. Let $f_k(t) = \frac{\text{sn}_k'(t)}{\text{sn}_k(t)}$.

Then $f_k(t) = \frac{1}{t} + O(1)$ and

$$f_k' + f_k^2 + k = 0.$$

Consider a smooth function $F : (0, b) \to \mathbb{R}$ such that $F(t) = 2\log(t) + O(1)$ and

$$F' = f + f_k.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(e^F(f - f_k)\big) = e^F(f' - f_k' + f^2 - f_k^2) \leqslant 0.$$

So $e^R(f - f_k)$ is decreasing and

$$\lim_{t \to 0} e^{F(t)}(f(t) - f_k(t)) = 0.$$

Hence $f(t) \leqslant f_k(t)$ for $t \in (0, b)$. □

Lemma 10.3. *Let (M, g) be a Riemannian manifold, (U, x^i) be a geodesic ball chart around a point $p \in M$ with normal coordinates $\{x^i\}$ and r be the radial distance function, then on $U \backslash \{p\}$,*

$$\Delta r = \partial_r \log\left(r^{n-1}\sqrt{\det g}\right).$$

Moreover, for unit-speed geodesic $\gamma : [0, b] \to U$ with $\gamma(0) = p$, let $f(t) = (\Delta r)(\gamma(t))$, then

$$f(t) = \frac{n-1}{t} + O(1).$$

Proof. In normal coordinates $(x^1, \cdots, x^n)$, by corollary 4.4 and corollary 9.1,

$$\begin{aligned}\Delta r =&\frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det(g)}\frac{\partial r}{\partial x^j}\right) = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det(g)}\frac{x^j}{r}\right)\\ =&\frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(\frac{x^i}{r}\sqrt{\det g}\right) = \sum_i \frac{\partial}{\partial x^i}\left(\frac{x^i}{r}\right) + \frac{1}{\sqrt{\det(g)}}\sum_i \frac{x^i}{r}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g)}\right)\\ =&\frac{n-1}{r} + \frac{1}{\sqrt{\det(g)}}\partial_r\sqrt{\det(g)} = \partial_r \log\left(r^{n-1}\sqrt{\det g}\right)\end{aligned}$$

Let $\gamma'(0) = v$, then

$$r(\gamma(t)) = \left|\exp_p^{-1}(\gamma(t))\right| = |tv| = t,$$
$$\partial_r\big|_{\gamma(t)} = \tfrac{x^i}{r}\tfrac{\partial}{\partial x^i}\big|_{\gamma(t)} = \tfrac{tv^i}{t}\tfrac{\partial}{\partial x^i}\big|_{\gamma(t)} = \gamma'(t).$$

So

$$f(t) = \frac{n-1}{r(\gamma(t))} + \frac{\mathrm{d}}{\mathrm{d}t}\log\sqrt{\det(g)} = \frac{n-1}{t} + \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}$$

□

Alternative proof of Laplacian comparison theorem. By Bochner formula,

$$0 = \frac{1}{2}\Delta|\nabla r| = |\text{Hess } r|^2 + \text{Ric}(\nabla r, \nabla r) + \partial_r\Delta r \geqslant \partial_r\Delta r + |\text{Hess } r|^2 + (n-1)k.$$

And by Cauchy-Scharwz inequality,

$$(\Delta r)^2 \leqslant (n-1)|\text{Hess } r|^2.$$

So

$$\partial_r(\Delta r) + \frac{(\Delta r)^2}{n-1} + (n-1)k \leqslant 0.$$

For any unit-speed geodesic $\gamma : [0, b] \to U$ with $\gamma(0) = p$, let

$$f(t) = \frac{(\Delta r)(\gamma(t))}{n-1}.$$

Then by lemma 10.3 and Ricatti comparison principle,

$$f(t) \leqslant \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}, i.e. \Delta r \leqslant (n-1)\frac{\mathrm{sn}_k'(\mathrm{r})}{\mathrm{sn}_k(r)}.$$

□

10.3 Volume comparison theorems and applications

Def 10.1. The metric ball of radius δ is defined as

$$B(p,\delta)=\{q\in M|d(p,q)<\delta\}.$$

Lemma 10.4. *Let (M,g) be a complete and connected manifold, $p\in M$, then for any $\delta>0$,*

$$\exp_p(B(0,\delta)\cap\Sigma(p))\subset B(p,\delta)\subset\exp_p(B(0,\delta)\cap\Sigma(p))\cup\mathrm{cut}(p).$$

In particular, under the trivialization map

$$\mathbb{R}^+\times\mathbb{S}^{n-1}\xrightarrow{\Phi}\mathbb{R}^n\backslash\{0\}\cong T_pM\backslash\{0\}\to B(p,\delta)\backslash\{p\},$$

where $\Phi(\rho,\omega)=\rho\omega$, we have

$$\mathrm{Vol}_g(B(p,\delta))=\int_{\mathbb{S}^{n-1}}\int_0^\delta\chi_{\Sigma(p)}\sqrt{\det(g)}\exp_p(\Phi(\rho,\omega))\rho^{n-1}\mathrm{d}\rho\mathrm{dvol}_{\mathbb{S}^{n-1}}$$

Proof. Since $\mathrm{cut}(p)$ is of measure zero, so

$$\begin{aligned}\mathrm{Vol}_g(B(p,\delta))=&\mathrm{Vol}_g(\exp_p(B(0,\delta)\cap\Sigma(p)))\\=&\int_{B(0,\delta)\cap\Sigma(p)}\exp_p^*(\mathrm{dvol}_g)\\=&\int_{B(0,\delta)}\chi_{\Sigma(p)}\exp_p^*(\mathrm{dvol}_g)\\=&\int_{\mathbb{S}^{n-1}}\int_0^\delta\chi_{\Sigma(p)}\sqrt{\det(g)}\exp_p(\Phi(\rho,\omega))\rho^{n-1}\mathrm{d}\rho\mathrm{dvol}_{\mathbb{S}^{n-1}}\end{aligned}$$

$\square$

Coro 10.8. *Let $p\in S^n_k$.*

(1) If $k\leqslant 0$, then for any $\delta>0$,

$$\mathrm{Vol}_{g_k}(B(p,\delta))=\int_{\mathbb{S}^{n-1}}\int_0^\delta\mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\mathrm{dvol}_{\mathbb{S}^{n-1}}.$$

(2) If $k=R^{-2}>0$, then for any $\delta>0$,

$$\mathrm{Vol}_{g_k}(B(p,\delta))=\int_{\mathbb{S}^{n-1}}\int_0^\delta\chi_{B(0,\pi R)}\cdot\mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\mathrm{dvol}_{\mathbb{S}^{n-1}}.$$

Proof. directly follows from lemma 10.4. $\square$

Lemma 10.5. *Let (M,g) be a complete Riemannian manifold, (U,x^i) be a geodesic ball chart of radius b with normal coordinates $\{x^i\}$.*

For each fixed $\omega\in\mathbb{S}^{n-1}$, the volume density ratio is defined as

$$\lambda(\rho,\omega)=\frac{\rho^{n-1}\sqrt{\det(g)}\circ\exp_p(\Phi(\rho,\omega))}{\mathrm{sn}_k^{n-1}(\rho)}.$$

(1) If $K_M\leqslant k$, then $\lambda(\rho,\omega)$ is increasing in $\rho\in(0,b_0)$, where

$$b_0=\begin{cases}b & k\leqslant 0\\ \min\{b,\pi R\} & k=\frac{1}{R^2}\end{cases}$$

Moreover, $\lim\limits_{\rho\to0}\lambda(\rho,\omega)=1$.

(2) If $K_M \geqslant k$ or $\mathrm{Ric}(g) \geqslant (n-1)kg$, then $\lambda(\rho, \omega)$ is decreasing in $\rho \in (0, b)$ and $\lim_{\rho \to 0} \lambda(\rho, \omega) = 1$.

Proof. By lemma 10.3 and corollary 10.6,

$$\partial_r \log\left(r^{n-1}\sqrt{\det(g)}\right) = \Delta r \geqslant (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} = \partial_r(\log(\mathrm{sn}_k^{n-1}(r))).$$

So along each radial geodesic $\gamma = \exp_p(\Phi(\bullet, \omega))$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log(\lambda(t, \omega))) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(\frac{r^{n-1}\sqrt{\det(g)}}{\mathrm{sn}_k^{n-1}(r)}\right) \circ \gamma(t)\right) \geqslant 0.$$

Hence $\lambda(\rho, \omega)$ is increasing in $\rho \in (0, b_0)$.
And when $\rho \to 0$,

$$\frac{\rho}{\mathrm{sn}_k(\rho)} \to 1, \sqrt{\det(g)} \to 0, i.e. \lambda(\rho, \omega) \to 1.$$

The proof of (2) is similar. □

Lemma 10.6. *Let $f : [0, +\infty) \to [0, +\infty)$ and $g : [0, +\infty) \to (0, +\infty)$ be two integrable functions, if*

$$\lambda(t) = \frac{f(t)}{g(t)} : [0, +\infty) \to [0, +\infty)$$

is nonincreasing, show that

$$F(t) = \frac{\int_0^t f(\tau)\mathrm{d}\tau}{\int_0^t g(\tau)\mathrm{d}\tau} : [0, +\infty) \to [0, +\infty)$$

is nonincreasing.
Moreover, if there exists $0 < t_1 < t_2$ such that

$$F(t_1) = F(t_2),$$

show that $\lambda(t) \equiv \lambda(t_1)$ for all $t \in [0, t_2]$.

Proof. Since for $0 \leqslant \tau \leqslant t$,

$$\frac{f(\tau)}{g(\tau)} \geqslant \frac{f(t)}{g(t)}.$$

So

$$F(t) = \frac{\int_0^t f(\tau)\mathrm{d}\tau}{\int_0^t g(\tau)\mathrm{d}\tau} \geqslant \frac{\int_0^t \lambda(t)g(\tau)\mathrm{d}\tau}{\int_0^t g(\tau)\mathrm{d}\tau} = \lambda(t).$$

And for $t \leqslant \tau \leqslant s$,

$$\frac{f(\tau)}{g(\tau)} \leqslant \frac{f(t)}{g(t)}.$$

Therefore

$$F(s) = \frac{\int_0^s f(\tau)\mathrm{d}\tau}{\int_0^s g(\tau)\mathrm{d}\tau} \leqslant \frac{\int_0^t f(\tau)\mathrm{d}\tau + \lambda(t)\int_t^s g(\tau)\mathrm{d}\tau}{\int_0^t g(\tau)\mathrm{d}\tau + \int_t^s g(\tau)\mathrm{d}\tau} \leqslant \frac{\int_0^t f(\tau)\mathrm{d}\tau + F(t)\int_t^s g(\tau)\mathrm{d}\tau}{\int_0^t g(\tau)\mathrm{d}\tau + \int_t^s g(\tau)\mathrm{d}\tau} = F(t).$$

Hence $F(t)$ is nonincreasing.
And assume there exists $0 < t_1 < t_2$ such that

$$F(t_1) = F(t_2),$$

then $f(\tau) = \lambda(t_1)g(\tau), i.e. \lambda(t) = \lambda(t_1)$ for every $\tau \in [0, t_2]$. □

Thm 10.5 (Bishop-Gromov)**.** *Let (M,g) be a complete Riemannian manifold with*

$$\mathrm{Ric}(g) \geqslant (n-1)kg$$

for some $k \in \mathbb{R}$.
Suppose $p \in M, B(p,\delta)$ is a metric ball and g_k is metric with $\sec \equiv k$ on $B(p,\delta)\backslash\{p\}$.
Then the volume ratio

$$\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))}$$

is nonincreasing for $\delta \in \mathbb{R}^+$ and

$$\lim_{\delta\to 0}\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} = 1.$$

In particular,

$$\mathrm{Vol}_g(B(p,\delta)) \leqslant \mathrm{Vol}_{g_k}(B(p,\delta)),$$

Moreover, if there exist $0 < \delta_1 < \delta_2$ such that

$$\frac{\mathrm{Vol}_g(B(p,\delta_1))}{\mathrm{Vol}_{g_k}(B(p,\delta_1))} = \frac{\mathrm{Vol}_g(B(p,\delta_2))}{\mathrm{Vol}_{g_k}(B(p,\delta_2))},$$

then

$$\mathrm{Vol}_g(B(p,\delta)) = \mathrm{Vol}_{g_k}(B(p,\delta))$$

for any $\delta \in [0,\delta_2]$ and g has constant sectional curvature k on $B(p,\delta_2)$.

Proof. If $k \leqslant 0$, by lemma 10.4 and corollary 10.8,

$$\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} = \frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} \frac{\int_0^\delta \chi_{\Sigma(p)}\sqrt{\det(g)}\circ\exp_p(\Phi(\rho,\omega))\rho^{n-1}\mathrm{d}\rho}{\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho}\mathrm{dvol}_{\mathbb{S}^{n-1}}.$$

By lemma 10.5 and lemma 10.6, it is nonincreasing.
If $k = R^{-2} > 0$, then

$$\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} = \frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} \frac{\int_0^\delta \chi_{\Sigma(p)}\sqrt{\det(g)}\circ\exp_p(\Phi(\rho,\omega))\rho^{n-1}\mathrm{d}\rho}{\int_0^\delta \chi_{B(0,\pi R)}\cdot\mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho}\mathrm{dvol}_{\mathbb{S}^{n-1}}.$$

By Myers' theorem, $\mathrm{diam}(M,g) \leqslant \pi R$, *i.e.* $\Sigma(p) \subset B(0,\pi R)$.
So the volume ratio is nonincreasing.
Assume there exist $0 < \delta_1 < \delta_2$ such that

$$\frac{\mathrm{Vol}_g(B(p,\delta_1))}{\mathrm{Vol}_{g_k}(B(p,\delta_1))} = \frac{\mathrm{Vol}_g(B(p,\delta_2))}{\mathrm{Vol}_{g_k}(B(p,\delta_2))}.$$

If $k \leqslant 0$, by lemma 10.6, for any $\rho \in (0,\delta_2)$,

$$\frac{\chi_{\Sigma(p)}\sqrt{\det(g)}\circ\exp_p(\Phi(\rho,\omega))\rho^{n-1}}{\mathrm{sn}_k^{n-1}(\rho)} \equiv 1.$$

So $B(0,\delta_2) \subset \Sigma(p)$.
By lemma 10.3 and Laplacian comparison theorem,

$$\Delta r = (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$

and so g has constant sectional curvature k on $B(p, \delta_2)$.
Suppose $k = R^{-2} > 0$, if $\delta_2 \leqslant \pi R$, then for $\rho \in (0, \delta_2)$,

$$\frac{\chi_{\Sigma(p)} \sqrt{\det(g)} \circ \exp_p(\Phi(\rho, \omega))\rho^{n-1}}{\chi_{B(0,\pi R)} \cdot \mathrm{sn}_k^{n-1}(\rho)} \equiv 1$$

So $B(0, \delta_2) \subset \Sigma(p)$ and so g has constant sectional curvature k on $B(p, \delta_2)$.
If $\delta_2 > \pi R$, then

$$\mathrm{Vol}_g(B(p, \delta_2)) = \mathrm{Vol}_g(B(p, \pi R)), \mathrm{Vol}_{g_k}(B(p, \delta_2)) = \mathrm{Vol}_{g_k}(B(p, \pi R)).$$

Hence we obtain the same conclusion. □

Thm 10.6 (Günther)**.** *Let (M, g) be a complete Riemannian manifold with*

$$\sec(g) \leqslant k$$

for some $k \in \mathbb{R}$.
Suppose $p \in M, B(p, \delta)$ is a metric ball and g_k is metric with $\sec \equiv k$ on $B(p, \delta)\backslash\{p\}$.
Then the volume ratio

$$\frac{\mathrm{Vol}_g(B(p, \delta))}{\mathrm{Vol}_{g_k}(B(p, \delta))}$$

is nondecreasing for $\delta \in \mathbb{R}^+$ and

$$\lim_{\delta \to 0} \frac{\mathrm{Vol}_g(B(p, \delta))}{\mathrm{Vol}_{g_k}(B(p, \delta))} = 1.$$

In particular,

$$\mathrm{Vol}_g(B(p, \delta)) \geqslant \mathrm{Vol}_{g_k}(B(p, \delta)),$$

Moreover, if there exist $0 < \delta_1 < \delta_2$ such that

$$\frac{\mathrm{Vol}_g(B(p, \delta_1))}{\mathrm{Vol}_{g_k}(B(p, \delta_1))} = \frac{\mathrm{Vol}_g(B(p, \delta_2))}{\mathrm{Vol}_{g_k}(B(p, \delta_2))},$$

then

$$\mathrm{Vol}_g(B(p, \delta)) = \mathrm{Vol}_{g_k}(B(p, \delta))$$

for any $\delta \in [0, \delta_2]$ and g has constant sectional curvature k on $B(p, \delta_2)$.

Proof. Similar to theorem 10.5. □

Coro 10.9. *Let (M, g) be a complete Riemannian manifold, $p \in M$, $B(p, \delta)$ be the metric ball centered at p with radius δ and g_k be the metric with constant sectional curvature k on $B(p, \delta)\backslash\{p\}$.*
If $\mathrm{Ric}(g) \geqslant (n-1)kg$, then for any $0 \leqslant \delta_1 < \min(\delta_2, \delta_3) \leqslant \max(\delta_2, \delta_3) < \delta_4$,

$$\frac{\mathrm{Vol}_g(B(p, r_4)) - \mathrm{Vol}_g(B(p, r_3))}{\mathrm{Vol}_g(B(p, r_2)) - \mathrm{Vol}_g(B(p, r_1))} \leqslant \frac{\mathrm{Vol}_{g_k}(B(p, r_4)) - \mathrm{Vol}_{g_k}(B(p, r_3))}{\mathrm{Vol}_{g_k}(B(p, r_2)) - \mathrm{Vol}_{g_k}(B(p, r_1))}$$

Proof. Let

$$f(r) = \frac{\mathrm{Vol}_g(B(p, r))}{\mathrm{Vol}_{g_k}(B(p, r))}, h(r) = \mathrm{Vol}_{g_k}(B(p, r)).$$

By Bishop-Gromov theorem, $f(r)$ is nonincreasing and $\lim\limits_{r \to 0} f(r) = 1$.

If $r_3 \geqslant r_2$, then

$$\frac{f(r_4)h(r_4) - f(r_3)h(r_3)}{h(r_4) - h(r_3)} \leqslant f(r_3) \leqslant \frac{f(r_2)h(r_2) - f(r_1)h(r_1)}{h(r_2) - h(r_1)}.$$

If $r_3 < r_2$, then since

$$\frac{f(r_4)h(r_4) - f(r_2)h(r_2)}{h(r_4) - h(r_2)} \leqslant \frac{f(r_2)h(r_2) - f(r_3)h(r_3)}{h(r_2) - h(r_3)} \leqslant \frac{f(r_3)h(r_3) - f(r_1)h(r_1)}{h(r_3) - h(r_1)},$$

we have

$$\frac{f(r_4)h(r_4) - f(r_3)h(r_3)}{h(r_4) - h(r_3)} \leqslant \frac{f(r_2)h(r_2) - f(r_1)h(r_1)}{h(r_2) - h(r_1)}.$$

Hence we obtain the desired formula. □

Prop 10.1 (Gromov). *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric}(g) \geqslant (n-1)kg$ for some constant $k > 0$, then*

$$\mathrm{Vol}_g(M) \leqslant \mathrm{Vol}_{g_k}\left(\mathbb{S}^n\left(\frac{1}{\sqrt{k}}\right)\right).$$

If the identity holds, then (M, g) is isometric to $\left(\mathbb{S}^n\left(\frac{1}{\sqrt{k}}\right), g_{can}\right)$.

Proof. Let $k = R^{-2}$.

Then by Myers' theorem, $\mathrm{diam}(M, g) \leqslant \pi R$.

So for any $p \in M, \Sigma(p) \subset B(0, \pi R)$ and therefore

$$\mathrm{Vol}_g(B(p, \pi R)) = \mathrm{Vol}_g(M),$$

where $B(p, \pi R)$ is a metric ball in M.

And since

$$\mathrm{Vol}_{g_k}(\mathbb{S}^n(R)) = \mathrm{Vol}_{g_k}(B(p, \pi R)).$$

Hence by Bishop-Gromov theorem,

$$\mathrm{Vol}_g(M) \leqslant \mathrm{Vol}_{g_k}(\mathbb{S}^m(R)).$$

Moreover, if the identity holds, then g has constant sectional curvature on $B(p, \pi R)$.

Since $\overline{B(p, \pi R)} = M$.

So (M, g) has constant sectional curvature k.

Suppose $\pi : \mathbb{S}^{n-1}(R) \to M$ is the universal covering, then

$$\mathrm{Vol}_g(M) = |\pi_1(M)| \cdot \mathrm{Vol}_{g_k}(\mathbb{S}^n(R)).$$

Hence $|\pi_1(M)| = 1, i.e.(M, g)$ is isometric to $(\mathbb{S}^n(R), g_{can})$. □

There is a generalized quantitive rigidity theorem.

Thm 10.7 (Cheeger-Colding). *For any integer $n \geqslant 2$, there exists $\delta(n) \in (0, 1)$ such that if (M, g) is a compact Riemannian manifold with $\mathrm{Ric}(g) \geqslant (n-1)g$ and*

$$\mathrm{Vol}_g(M) \geqslant (1 - \delta(n))\mathrm{Vol}(\mathbb{S}^n, g_{can}),$$

then M is diffeomorphic to $\mathbb{S}^n$.

Proof. This is difficult and need some analytic tools so we are not going to prove it. □

Thm 10.8 (Myers-Cheng). *Let (M,g) be a complete Riemannian manifold with* $\mathrm{Ric}(g) \geqslant (n-1)kg$ *for some constant* $k = R^{-2} > 0$.

If $\mathrm{diam}(M,g) = \pi R$, *then* (M,g) *is isometric to* $(\mathbb{S}^n(R), g_{can})$.

Proof. There exist points $p, q \in M$ such that $d(p,q) = \pi R$.

Then for any $\delta \in (0, \pi R)$, $B(p,\delta) \cap B(q, \pi R - \delta) = \varnothing$.

And since for any $x, y \in M$, $\mathrm{Vol}_g(B(x,\pi R)) = \mathrm{Vol}_g(M), \mathrm{Vol}_{g_k}(B(x,\pi R)) = \mathrm{Vol}_{g_k}(\mathbb{S}^n(R))$, and $\mathrm{Vol}_{g_k}(B(p,\delta)) + \mathrm{Vol}_{g_k}(B(q,\pi R - \delta)) = \mathrm{vol}_{g_k}(\mathbb{S}^n(R))$.

So by Bishop-Gromov theorem,

$$\begin{aligned}
\mathrm{Vol}_g(M) \geqslant& \mathrm{Vol}_g(B(p,\delta)) + \mathrm{Vol}_g(B(q,\pi R - \delta)) \\
\geqslant& \mathrm{Vol}_{g_k}(B(p,\delta)) \cdot \frac{\mathrm{Vol}_g(B(p,\pi R))}{\mathrm{Vol}_{g_k}(B(p,\pi R))} + \mathrm{Vol}_{g_k}(B(q,\pi R - \delta)) \cdot \frac{\mathrm{Vol}_g(B(q,\pi R))}{\mathrm{Vol}_{g_k}(B(q,\pi R))} \\
=& \mathrm{Vol}_g(M)
\end{aligned}$$

Therefore for any $\delta \in (0, \pi R)$,

$$\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} = \frac{\mathrm{Vol}_g(B(p,\pi R))}{\mathrm{Vol}_{g_k}(B(p,\pi R))} = \frac{\mathrm{Vol}_g(M)}{\mathrm{Vol}_{g_k}(\mathbb{S}^n(R))}.$$

Let $\delta \to 0$, we deduce $\mathrm{Vol}_g(M) = \mathrm{Vol}_{g_k}(\mathbb{S}^n(R))$.

By proposition 10.1, (M,g) is isometric to $(\mathbb{S}^n(R)), g_{can})$. □

Thm 10.9. *Let (M,g) be a compact orientable Riemannian manifold with* $\mathrm{Ric}(g) \geqslant \lambda g, \lambda > 0$.

(1) (Lichnerowicz)The first nonzero eigenvalue λ_1 *of* $\Delta = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ *satisfies*

$$\lambda_1 \geqslant \frac{n}{n-1}\lambda.$$

(2) (Obata)If $\lambda_1 = \frac{n}{n-1}\lambda$, *then* (M,g) *is isometric to* $\left(\mathbb{S}^n\left(\sqrt{\frac{n-1}{\lambda}}\right), g_{can}\right)$.

Proof. (1) Suppose f is a nonzero eigenfunction,*i.e.* $\Delta_g f = -\Delta f = -\lambda_1 f$.

Then by Bochner formula, Cauchy-Scharwz inequality and divergence theorem,

$$\begin{aligned}
0 = \int_M \frac{1}{2}\Delta_g |\nabla f|^2 =& \int_M |\mathrm{Hess}\ f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla \Delta_g f, \nabla f) \\
\geqslant& \int_M \frac{(\Delta_g f)^2}{n} + (\lambda - \lambda_1)|\nabla f|^2 \\
=& -\int_M \frac{\lambda_1}{n} f \Delta_g f + (\lambda - \lambda_1)|\nabla f|^2 \\
=& \int \frac{\lambda_1}{n}|\nabla f|^2 + (\lambda - \lambda_1)|\nabla f|^2
\end{aligned}$$

Hence $\lambda_1 \geqslant \frac{n}{n-1}\lambda$.

(2) If $\lambda_1 = \frac{n}{n-1}\lambda$, then

$$\frac{1}{2}\Delta_g |\nabla f|^2 \geqslant -\frac{\lambda_1 f}{n}\Delta_g f + (\lambda - \lambda_1)|\nabla f|^2 = -\frac{\lambda}{n-1}\left(f\Delta_g f + |\nabla f|^2\right).$$

And since $\frac{1}{2}\Delta_g f^2 = f\Delta_g f + |\nabla f|^2$, we obtain

$$\frac{1}{2}\Delta_g\left(|\nabla f|^2 + \frac{\lambda}{n-1}f^2\right) \geqslant 0,$$

which implies $|\nabla f|^2 + \frac{\lambda}{n-1} f^2$ is constant.

WLOG, we assume $\sup f^2 = 1$, since $\nabla f = 0$ at the extremal point, we deduce

$$|\nabla f|^2 + \frac{\lambda}{n-1} f^2 = \frac{\lambda}{n-1}, \sup f = 1, \inf f = -1.$$

Let $f(p) = 1, f(q) = -1$, γ be a unit-speed minimal geodesic from p to q and $u(t) = f(\gamma(t))$. Then outside the measure zero set $V = \{x \in M | f^2(x) = 1\}$, one has

$$\frac{|u'(t)|}{\sqrt{\frac{\lambda}{n-1}(1-u^2)}} \leqslant \frac{|\nabla f|}{\sqrt{\frac{\lambda}{n-1}(1-f^2)}} = 1.$$

So integral over $[0, l]$, we have

$$\int_0^l \frac{|u'(t)|}{\sqrt{\frac{\lambda}{n-1}(1-u^2)}} \mathrm{d}t = \sqrt{\frac{n-1}{\lambda}}\pi \leqslant l \leqslant d.$$

Hence by theorem 10.8, $(M, g) \cong \left(\mathbb{S}^n\left(\sqrt{\frac{n-1}{\lambda}}\right), g_{can}\right)$.

$\square$

Thm 10.10 (Bishop-Yau). *Let (M^n, g) be a complete non-compact Riemannian manifold* $\mathrm{Ric}(g) \geqslant 0$, *then the volume growth of (M, g) satisfies*

$$c_n \mathrm{Vol}(B(p,1)) r \leqslant \mathrm{Vol}_g(B(p,r)) \leqslant \mathrm{Vol}(B(p,r)) = \frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{n} r^n$$

for some constant $c_n = c(n, r) > 0$ depending only on n and large r.

Proof. Let $x \in \partial B(p, 1+r)$, then

$$B(p,1) \subset B(x, 2+r) \backslash B(x,r), B(x,r) \subset B(p, 1+2r).$$

So by corollary 10.9,

$$\begin{aligned}
\mathrm{Vol}_g(B(p,1)) \leqslant & \mathrm{Vol}_g(B(x,2+r)) - \mathrm{Vol}_g(B(x,r)) \\
\leqslant & \frac{\mathrm{Vol}_0(B(x,2+r)) - \mathrm{Vol}_0(B(x,r))}{\mathrm{Vol}_0(B(x,r))} \mathrm{Vol}_g(B(x,r)) \\
\leqslant & \frac{(r+2)^n - r^n}{r^n} \mathrm{Vol}(B(x,r)) \\
\leqslant & \frac{\tilde{C}_n}{r} \mathrm{Vol}(B(p, 1+2r))
\end{aligned}$$

$\square$

Prop 10.2. *Let (M^n, g) be a complete Riemannian manifold with* $\mathrm{Ric}(g) \geqslant 0$, *if*

$$\lim_{r \to \infty} \frac{\mathrm{Vol}_g(B(p,r))}{r^n} \geqslant \frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{n},$$

then (M, g) is isometric to $(\mathbb{R}^n, g_{can})$.

Proof. By Bishop-Gromov theorem, the volume ratio

$$\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))}$$

is nonincreasing and

$$\lim_{r\to 0}\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))}=1.$$

And since

$$\mathrm{Vol}_{g_0}(B(p,r))=\int_0^r \mathrm{Vol}_{g_0}(\partial B(p,x))\mathrm{d}x=\mathrm{Vol}(\mathbb{S}^{n-1})\int_0^r x^{n-1}\mathrm{d}x=\frac{r^n}{n}\mathrm{Vol}(\mathbb{S}^{n-1}).$$

So we obtain

$$\lim_{r\to\infty}\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))}\geqslant 1.$$

Therefore for every $r>0$,

$$\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))}\equiv 1.$$

Hence (M,g) has constant sectional curvature 0 and similar to proposition 10.1, we deduce that $|\pi_1(M)|=1, i.e.(M,g)$ is isometric to $(\mathbb{R}^n, g_{can})$. □

Prop 10.3. *Let (M,g) be a Cartan-Hadamard manifold with $\mathrm{Ric}(g)\leqslant -kg$ for some $k>0$, then for any $q\in M$,*

$$\mathrm{Vol}(B(p,r))\geqslant c_n e^{\sqrt{k}r}$$

Proof. I don't know how to prove this):, maybe you can work it out and teach me! □

10.4 Cheeger-Gromoll's splitting theorem

Def 10.2. A geodesic ray is a unit-speed geodesic $\gamma:[0,+\infty)\to M$ such that for any $s,t>0$,

$$d(\gamma(s),\gamma(t))=|s-t|.$$

Lemma 10.7. *Let (M,g) be a complete Riemannian manifold, TFAE*

(1) M is noncompact

(2) For any $p\in M$, there exists a geodesic ray $\gamma:[0,+\infty)\to M, \gamma(0)=p$.

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2): Let $\{p_i\}$ be the points such that $d(p,p_i)=i$ and $\gamma_i=\exp_p(tv_i)$ be the unit-speed minimal geodesic from p to p_i.

By passing to a subsequence, we assume $v_i\to v$.

We claim that $\gamma(t)=\exp_p(tv)$ is a unit speed geodesic ray.

Indeed, for any $s,t\geqslant 0$ and $k>\max\{s,t\}$, we have

$$d(\gamma_k(s),\gamma_k(t))=|s-t|.$$

So by continuity of $\exp_p$,

$$d(\gamma(s),\gamma(t))=d(\exp_p(sv),\exp_p(tv))=\lim_{k\to+\infty}d(\exp_p(sv_k),\exp_p(tv_k))=|s-t|.$$

□

Def 10.3. Let $\gamma : [0, +\infty) \to M$ be a geodesic ray starting from $p \in M$, we define

$$b_\gamma^t(x) = d(x, \gamma(t)) - t = d(x, \gamma(t)) - d(\gamma(0), \gamma(t)).$$

Prop 10.4. *Given $t \in [0, +\infty)$, the function $b_\gamma^t : M \to \mathbb{R}$ has the following properties:*

(1) For any fixed $x \in M, b_\gamma^+$ is nonincreasing in t.

(2) For any $x \in M$ and $t \geqslant 0, \left|b_\gamma^t(x)\right| \leqslant d(x, \gamma(0))$.

(3) For any $x, y \in M$ and $t \geqslant 0, \left|b_\gamma^t(x) - b_\gamma^t(y)\right| \leqslant d(x, y)$

Proof. (1) for $t > s \geqslant 0$,

$$\begin{aligned} b_\gamma^t(x) - b_\gamma^s(x) =& d(x, \gamma(t)) - t - d(x, \gamma(s)) + s \\ \leqslant& d(\gamma(t), \gamma(s)) + s - t \\ =& |t - s| + s - t = 0 \end{aligned}$$

(2) and (3) follows from the triangle inequality.

□

Def 10.4. The Busemann function w.r.t. a geodesic ray $r : [0, +\infty)$ is defined as

$$b_\gamma(x) = \lim_{t \to +\infty} b_\gamma^t(x).$$

Prop 10.5. *The Busemann function $b_\gamma : M \to \mathbb{R}$ is Lipschitz continuous with* $\operatorname{Lip}(b_\gamma) \leqslant 1$.

Proof. Follows by Ascoli-Arezla theorem. □

Def 10.5. Let $\gamma : [0, +\infty) \to M$ be a geodesic ray and $p \in M \backslash \operatorname{Im} \gamma$.

According to the proof of lemma 10.7, the unit-speed minimal geodesics from p to $c(t)$ is converges to a geodesic ray $\tilde{\gamma} : [0, +\infty) \to M$ with $\tilde{\gamma}(0) = p$ by passing to a subsequence.

Such geodesic ray $\tilde{\gamma}$ is called the asymptote for γ from p.

Lemma 10.8. *Let γ be a geodesic ray and $\tilde{\gamma}$ is the asymptote for γ from $p \in M$, then*

(1) $b_\gamma(\tilde{\gamma}(t)) = b_\gamma(p) - t$.

(2) $b_\gamma(x) \leqslant b_\gamma(p) + b_{\tilde{\gamma}}(x)$.

Proof. (1) Let γ_i be the unit-speed minimal geodesics from p to $c(t_i)$ that converge to $\tilde{\gamma}$, then

$$d(p, c(t_i)) - t_i = d(p, \sigma_i(s)) + d(\sigma_i(s), c(t_i)).$$

So when $i \to \infty$,

$$\begin{aligned} b_\gamma(p) =& \lim \left(d(p, \gamma(t_i)) - t_i\right) \\ =& \lim \left(d\left(p, \tilde{\gamma}(s)\right) + d\left(\tilde{\gamma}(s), \gamma(t_i)\right) - t_i\right) \\ =& s + b_\gamma\left(\tilde{\gamma}(s)\right) \end{aligned}$$

(2) when $s \to \infty$,

$$\begin{aligned} b_\gamma(x) =& \lim \left(d(x, \gamma(s)) - s\right) \\ \leqslant& \lim \left(d\left(x, \tilde{\gamma}(t)\right) + d\left(\tilde{\gamma}(t), \gamma(s)\right) - s\right) \\ =& \lim \left(d\left(x, \tilde{\gamma}(t)\right) - t\right) + b_\gamma(p) \\ =& b_{\tilde{\gamma}}(x) + b_\gamma(p) \end{aligned}$$

□

Def 10.6. Let (M, g) be a Riemannian manifold, $f \in C^0(M)$ and f_q is a C^2 function defined in a neighborhood of U of $q \in M$.

(1) f_q is called a upper barrier function of f at q if

$$f_q(q) = f(q), f_q(x) \geqslant f(x), x \in U.$$

(2) We say

$$\Delta f(q) \leqslant c$$

in the barrier sense if for any $\varepsilon > 0$, there exists a upper barrier function $f_{q,\varepsilon}$ of f at q such that

$$\Delta f_{q,\varepsilon}(q) \leqslant c + \varepsilon.$$

Prop 10.6. *Let (M, g) be a complete noncompact Riemannian manifold with* $\mathrm{Ric}(g) \geqslant 0$ *and γ be a geodesic ray starting from $p \in M$, then*

$$\Delta b_\gamma \leqslant 0$$

in the barrier sense.

Proof. By lemma 10.8(2), we only need to prove that $\Delta b_\gamma(x) \leqslant 0$ for $x = \gamma(0)$.

By Laplacian comparison theorem,

$$\Delta b_\gamma^t(x) = \Delta d(x, \gamma(t)) \leqslant \frac{n-1}{d(x, \gamma(t))}.$$

And by proposition 10.4, b_γ^t is a upper barrier function of b_γ.

So for fix x and large t, we deduce $\Delta b_\gamma \leqslant 0$ in the barrier sense. □

Def 10.7. A geodesic line is a unit-speed geodesic $\gamma : (-\infty, +\infty) \to M$ such that for any $s, t \in \mathbb{R}$,

$$d(\gamma(s), \gamma(t)) = |s - t|.$$

Lemma 10.9. *Let (M, g) be a connected, complete, noncompact Riemannian manifold.*

If M contains a compact subset K such that $M \backslash K$ has at least two unbounded components, then M contains a geodesic line passing through K.

Proof. Since $M \backslash K$ has at least two unbounded components.

So there are two unbounded sequences of points $\{p_i\}$ and $\{q_i\}$ in different components such that any curve from p_i to q_i passes through K.

Let $\gamma_i : [-a_i, b_i] \to M$ be the unit-speed minimal geodesics from p_i to q_i with $\gamma_i(0) \in K$.

Then by passing to a subsequence, γ_i converges to a geodesic line γ_∞. □

Lemma 10.10. *Let (M, g) be a connected Riemannian manifold, $f \in C^0(M)$ such that $\Delta f(x) \leqslant 0$ in the barrier sense for every $x \in M$.*

If f has a local minimum at $p \in M$, then f is constant on a neighborhood of p.

In particular, f has global minimum if and only if f is constant.

Proof. Suppose p is a local minimum of f and f is not constant on any neighborhood of p.

Then there exists $\varepsilon > 0$ and $x \in \partial B_\varepsilon(p)$ such that $f(p) > f(x)$.

Let $(x^1, \cdots, x^n)$ be an normal coordinate containing $\bar{B}_\varepsilon(p)$ and $x = (\varepsilon, 0, \cdots, 0)$, consider

$$\varphi = x^1 - C_1((x^2)^2 + \cdots + (x^n)^2), \psi = e^{C_2\varphi} - 1.$$

So for sufficiently large C_1, we have $\varphi(y) < 0$ for every $y \in \partial B_\varepsilon(p)$ with $f(y) = f(p)$ and

$$\Delta\psi = (C_2^2|\nabla\varphi|^2 + C_2\Delta\phi)e^{C_2\varphi} > 0$$

for sufficiently large C_2.
Therefore there exists $\lambda > 0$ such that $(f - \lambda\psi)(y) > f(p)$ for every $y \in \partial B_\varepsilon(p)$.
Let q be the minimum of $f - \lambda\psi$ inside $B_\varepsilon(p)$.
Consider the upper barrier function $f_{q,\delta}$ of f at q for some small δ.
Then $f_{q,\delta} - \lambda\psi$ is also a upper barrier function of $f - \lambda\psi$ at q.
So for sufficiently small δ, we have

$$\Delta(f_{q,\delta} - \lambda\psi) < \delta - \lambda\Delta\psi < 0.$$

But q is also a local minimum of $f_{q,\delta} + \lambda\psi$, which deduce

$$\Delta(f_{q,\delta} + \lambda\psi)(q) \geqslant 0,$$

contradiction!
Hence f is constant on a neighborhood of p. □

Prop 10.7. *Let (M, g) be a complete, noncompact Riemannian manifold* $\mathrm{Ric}(g) \geqslant 0$.
If M has a geodesic line, then $b_{\gamma_+}, b_{\gamma_-} : M \to \mathbb{R}$ are smooth harmonic function with

$$\left|\nabla b_{\gamma_\pm}\right| = 1, \mathrm{Hess}\ b_{\gamma_\pm} = 0,$$

where $\gamma_\pm = \gamma(\pm t) : [0, +\infty) \to M$.

Proof. Let $b(x) = b_{\gamma_+}(x) + b_{\gamma_-}(x)$, then

$$b(x) = \lim_{t\to+\infty} d(x, \gamma_+(t)) + d(x, \gamma_-(t)) - 2t = \lim_{t\to+\infty} d(x, \gamma(s)) + d(x, \gamma(-s)) - 2x \geqslant 0.$$

And since $b(\gamma(t)) = 0$.
By proposition 10.6 and lemma 10.10, $b(x) \equiv 0$.
So by proposition 10.6 again, we obtain that $b_{\gamma_+} = -b_{\gamma_-}$ are harmonic and smooth.
Let $f = b_{\gamma_+}$, the Bochner formula shows

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}\ f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla\Delta f, \nabla f) \geqslant |\mathrm{Hess}\ f|^2 \geqslant 0.$$

Therefore $|\nabla f|^2$ is superharmonic.
On the other hand, by proposition 10.5, $\mathrm{Lip}(f) \leqslant 1$ and so $|\nabla f| \leqslant 1$, for $x = \gamma_+(t)$

$$f(x) = \lim_{s\to+\infty} d(\gamma_+(t), \gamma_+(s)) - s = \lim_{s\to+\infty} |t - s| - s = -t$$

$$|\nabla f|(x) = |\nabla f|\left|\gamma_+'(x)\right| \geqslant \left|\langle\nabla f, \gamma_+'(t)\rangle\right| = 1.$$

Hence $|\nabla f| \equiv 1, \mathrm{Hess}\ f \equiv 0$. □

Prop 10.8. *Let (M, g) be a complete noncompact Riemannian manifold, suppose $f \in C^\infty(M, \mathbb{R})$ satisfies*

$$|\nabla f| = 1, \mathrm{Hess}\ f = 0.$$

We set $N = f^{-1}(0)$ and $h = g\big|_N$, then:

(1) (N, h) is totally geodesic in (M, g)

(2) the map

$$F : (\mathbb{R} \times N, \mathrm{d}t \otimes \mathrm{d}t \oplus h) \to (M, g), F(p,t) = \exp_p(t\nabla_p f)$$

is an isometry.

Proof. Let B be the second fundamental form of the map $\iota : (N,h) \to (M,g)$.
Since for any $X \in \Gamma(N,TN)$,

$$\langle \iota^*\nabla f, \iota_* X\rangle_{\hat{g}} = \langle \nabla f, X\rangle_g = X(f) = 0.$$

So for any $X, Y \in \Gamma(N,TN)$,

$$\begin{aligned}\langle B(X,Y), \iota^*\nabla f\rangle_{\hat{g}} =& \left\langle \hat{\nabla}_X(\iota_* Y) - \iota_*\left(\nabla^h_X Y\right), \iota^*\nabla f\right\rangle_{\hat{g}} \\ =& X\left\langle \iota_* Y, \iota^*\nabla f\right\rangle_{\hat{g}} - \left\langle \iota_* Y, \hat{\nabla}_X\left(\iota^*\nabla f\right)\right\rangle_{\hat{g}} \\ =& -\left\langle Y, \nabla_X(\nabla f)\right\rangle_g = -(\text{Hess } f)(X,Y) = 0\end{aligned}$$

Let $\iota^*TM = T^\perp N \oplus \iota_*(TN)$ be the orthonormal decomposition.
Then $T^\perp N = \mathrm{span}_{\mathbb{R}}\{\iota^*\nabla f\}$.
Therefore $B = 0 \in \Gamma(N, T^*N \otimes T^*N \otimes T^\perp N)$, *i.e.*$(N,h)$ is a totally geodesic in (M,g).
Let $X = \nabla f$ and $\gamma_p(t) = \exp_p(tX_p)$.
Since $\nabla X = 0$.
So $E_p(t) = X_{\gamma_p(t)}$ and $\gamma_p'(t)$ are two parallel vector fields along γ_p with the same initial value $E_p(0) = \gamma_p'(0) = X_p$.
Therefore $\gamma_p'(t) = X \circ \gamma_p(t)$, *i.e.*$\gamma_p$ is an integral curve of X.
And since $|X| = 1$ is a complete vector field.
Thus F is well-defined diffeomorphism.
It remains to prove that F is an isometry.
For $v \in T_pN$, let J be the Jacobi field along γ_p with $J(0) = 0$ and $J'(0) = v$.
By the radial curvature equation corollary 6.1,

$$\mathrm{R}(\bullet, \nabla f, \nabla f, \bullet) = \text{Hess}\left(\frac{1}{2}|\nabla f|^2\right)(\bullet,\bullet) - \left(\nabla_{\nabla f}\text{Hess } f\right)(\bullet,\bullet) - \text{Hess } f(\nabla_\bullet \nabla f, \bullet) = 0$$

So the Jacobi equation is

$$J''(t) + \mathrm{R}(J, \gamma_p')\gamma_p' = J''(t) = 0$$

It implies that $J'(t)$ is a parallel vector field and so $|J'(1)| = |J'(0)| = |v|$
By the uniqueness of Jacobi field, we deduce

$$J(t) = tJ'(t).$$

On the other hand, J is given by the geodesic variation $\alpha(t,s) = \exp_p(t(X_p + sv))$ and

$$J(1) = (\mathrm{d}\exp_p)_{X_p}(v) = (\mathrm{d}F)_p v.$$

Therefore we have

$$|(\mathrm{d}F)_p v| = |J(1)| = \left|J'(1)\right| = |v|.$$

Hence F is an isometry. □

Thm 10.11 (Cheeger-Gromoll's splitting theorem)**.** *Let (M,g) be a complete Riemannian manifold of dimension n with $\mathrm{Ric}(g) \geqslant 0$, if there is a geodesic line in M, then (M,g) is isometric to $(\mathbb{R} \times N, g_{\mathbb{R}} \oplus g_N)$ where $\mathrm{Ric}(g_N) \geqslant 0$.*

Proof. It follows by proposition 10.7 and proposition 10.8. ☐

Coro 10.10. *Let (M, g) be a compact Riemannian manifold with* $\operatorname{Ric}(g) \geqslant 0$*, then*

(1) there exist a nonnegative k and a compact Riemannian manifold (N, g_N) with $\operatorname{Ric}(g_N) \geqslant 0$ such that N does not contain a geodesic line and $(M, g) \cong \left(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N\right)$.

(2) $\operatorname{Iso}(M, g) = \operatorname{Iso}(\mathbb{R}^k) \times \operatorname{Iso}(N, g_N)$.

Proof. (1) follows from Cheeger-Gromoll's splitting theorem

(2) Let $F : \mathbb{R}^k \times N \to \mathbb{R}^k \times N$ be an isometry.

If $\gamma(t) = (\gamma_1(t), \gamma_2(t)) : \mathbb{R} \to \tilde{M}$ is a geodesic line, then γ_1 and γ_2 are geodesic in $(\mathbb{R}^k, g_{\mathbb{R}^k})$ and (N, g_N) resp.

And since (N, g_N) has no geodesic line.

So $\gamma_2(t) \equiv p_0 \in N$.

Therefore we can find $F_2(p) \in N$ such that

$$F : \mathbb{R}^k \times \{p\} \to \mathbb{R}^k \times \{F_2(p)\}$$

since F maps geodesic line to geodesic line.

This implies that $F(v, p) = (F_1(v, p), F_2(p))$ and for any $(v, p) \in \mathbb{R}^k \times N$, the tangent map $(\mathrm{d}F)_{(v,p)}$ is an isometry that preserves $T_v\mathbb{R}^k$.

Thus $(\mathrm{d}F)_{(v,p)}$ also preserves T_pN, *i.e.* $(\mathrm{d}F_1)_{v,p}\big|_{T_pN} = 0$.

Hence $F_1(v, p) = F_1(v)$ and so $\operatorname{Iso}(M, g) = \operatorname{Iso}(\mathbb{R}^k) \times \operatorname{Iso}(N, g_N)$

☐

Def 10.8. A subgroup B_n of $\operatorname{Iso}(\mathbb{R}^n) = O(n) \rtimes \mathbb{R}^n$ is called a Bieberbach group it acts freely on $\mathbb{R}^n$ and $\mathbb{R}^n/B_n$ is a compact manifold.

Thm 10.12 (Bieberbach)**.** *(1) Every Bieberbach group B_n is torsion free and contains $\mathbb{Z}^n$ as finite index subgroup.*

(2) Every compact quotient of $\mathbb{R}^n$ by a discrete group G of isometries of $\mathbb{R}^n$ is finitely covered by a flat torus $\mathbb{R}^n/\Gamma$.

Proof. The proof need some technology from geometric group theory, go see the paper of Bieberbach written in 1911 if you want. ☐

Thm 10.13 (Structure theorem for manifold with $\operatorname{Ric} \geqslant 0$)**.** *Let (M, g) be a compact Riemannian manifold with $\operatorname{Ric}(g) \geqslant 0$ and $\pi : \left(\tilde{M}, \tilde{g}\right) \to (M, g)$ be its universal covering with the pullback metric.*

(1) There exists some $k \geqslant 0$ and a compact Riemannian manifold (N, g_N) with $\operatorname{Ric}(g_N) \geqslant 0$ such that $\left(\tilde{M}, \tilde{g}\right)$ is isometric to $(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N)$.

(2) The isometry group splits

$$\operatorname{Iso}\left(\tilde{M}, \tilde{g}\right) \cong \operatorname{Iso}(\mathbb{R}^k, g_{\mathbb{R}^k}) \times \operatorname{Iso}(N, g_N).$$

(3) There exists a finite normal subgroup G of $\operatorname{Iso}(N, h)$, a Bieberbach group B_k and an exact sequence

$$0 \to G \to \pi_1(M) \to B_k \to 0.$$

Proof. By corollary 10.10, $\left(\tilde{M}, \tilde{g}\right)$ is isometric to $\left(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N\right)$, where N does not contain a geodesic line and $\mathrm{Ric}(g_N) \geqslant 0$, we only need to show that N is compact.

Suppose N is noncompact, by lemma 10.7, fix some $x_0 \in N$, there exists a geodesic ray $\gamma : [0, +\infty) \to N$ starting from x_0.

Since M is compact.

Then there exists a compact subset $\tilde{K} \subset \tilde{M}$ such that

$$\mathrm{Aut}_\pi\left(\tilde{M}\right) \cdot \tilde{K} = \tilde{M}.$$

Let $\pi_{\mathbb{R}^k}, \pi_N$ be the projections of $\tilde{M}$ to $\mathbb{R}^k$ and N resp.

Since $\mathrm{Aut}_\pi\left(\tilde{M}\right)$ is a subgroup of $\mathrm{Iso}\left(\tilde{M}, \tilde{g}\right) \cong \mathrm{Iso}(\mathbb{R}^k, g_{\mathbb{R}^k}) \times \mathrm{Iso}(N, h)$, so we have

$$\mathrm{Aut}_\pi\left(\tilde{M}\right) \cdot \pi_{\mathbb{R}^k}\left(\tilde{K}\right) = \mathbb{R}^k, \mathrm{Aut}_\pi\left(\tilde{M}\right) \cdot \pi_N\left(\tilde{K}\right) = N.$$

Moreover, there exists a sequence $\{\beta_m\}$ in $\mathrm{Aut}_\pi\left(\tilde{M}\right)$, such that

$$\beta_m(\gamma(m)) \in \pi_N\left(\tilde{K}\right).$$

By passing to a subsequence, we assume

$$\lim \beta_m(\gamma(m)) = p, \lim(\mathrm{d}\beta_m)(\gamma'(m)) = v \in T_pM.$$

Let $\gamma_m : [-m, +\infty) \to N$ be the geodesic rays defined by

$$\gamma_m(t) = \beta_m(\gamma(m+t)).$$

Then γ_m converges to the geodesic line

$$\tilde{\gamma} : \mathbb{R} \to N, \sigma(t) = \exp_p(tv),$$

contradiction!

For (3), consider the projection

$$\mathrm{Aut}_\pi\left(\tilde{M}\right) \to \mathrm{Iso}\left(\mathbb{R}^k\right)$$

and let B_k, G be its image and kernel resp.

By construction, B_k, G acts freely and properly on $\mathbb{R}^k$ and N resp.

Hence B_k is a Bieberbach group and G is finite since N is compact. □

Coro 10.11. *Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric}(g) \geqslant 0$ and $(\tilde{M}, \tilde{g})$ be its universal cover.*

(1) If $\tilde{M}$ is contractible, then $\left(\tilde{M}, \tilde{g}\right)$ is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$ and (M, g) is flat

(2) If $\left(\tilde{M}, \tilde{g}\right)$ does not contain a geodesic line, then $|\pi_1(M)|$ is finite.

(3) If $|\pi_1(M)|$ is finite, then $\tilde{M}$ is compact and $b_1(M) = 0$.

Proof. (1) Since $\tilde{M}$ splits as $\mathbb{R}^k \times N$ with compact N and $\tilde{M}$ is contractible.

So N is contractible, *i.e.* N is a point.

(2) $k = 0$ and $\tilde{M} = N$ is compact.

So $\pi_1(M)$ is finite.

(3) There is a natural surjection

$$h : \pi_1(M) \to H_1(M, \mathbb{Z}) = \mathbb{Z}^{b_1} \times T,$$

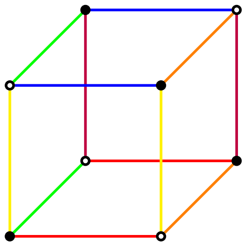
where T is a finite abelian group.

Hence $b_1(M) = 0$, and $\tilde{M} \to M$ is finite covering, *i.e.* $\tilde{M}$ is compact.

□

Prob 10.1. *Does there exist a compact Riemannian manifold* (M, g) *with* $\mathrm{Ric}(g) \geqslant 0, b_1(M) = 0$ *and* $|\pi_1(M)| = +\infty$.

Proof. Consider the Hantzsche-Wendt manifold $M = T^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.



So the representation of $\pi_1(M)$ is

$$\langle ab^{-1}, ac^{-1}, ad^{-1}, ae^{-1}, af^{-1} | ad^{-1}ac^{-1}, bd^{-1}bc^{-1}, ae^{-1}be^{-1}, af^{-1}bf^{-1}, cf^{-1}ce^{-1}, df^{-1}de^{-1} \rangle,$$

where a, b, c, d, e, f are red, blue, green, orange, yellow, purple line resp. , then

$$H_1(M) = \langle ac^{-1}, be^{-1} \rangle \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

Hence M is flat, $b_1(M) = 0$ but $|\pi_1(M)| = +\infty$. □

Remark 10.1. Hantzsche-Wendt manifold is the only closed flat 3-manifold with $b_1(M) = 0$.

Coro 10.12. *Let* (M, g) *be a compact Riemannian manifold with* $\mathrm{Ric}(g) \geqslant 0$, *if there exists* $p \in M$ *such that* $\mathrm{Ric}(g)(p) > 0$, *then* $|\pi_1(M)|$ *is finite and* $b_1(M) = 0$.

Proof. Let $\left(\tilde{M}, \tilde{g}\right)$ be the universal covering of (M, g).

If $\left(\tilde{M}, \tilde{g}\right)$ splits as $\left(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N\right)$ with $k > 0$, then $\mathrm{Ric}\left(g_{\mathbb{R}^k}\right) = 0$, *i.e.*$\mathrm{Ric}(g) = 0$, which is impossible.

Hence $\tilde{M} = N$ is compact, *i.e.* $|\pi_1(M)|$ is finite and $b_1(M) = 0$. □

Coro 10.13. *Let* (M, g) *be a compact Riemannian manifold with* $\mathrm{Ric}(g) \geqslant 0$, *then*

$$b_1(M) \leqslant \dim M.$$

If $b_1(M) = \dim M$, *then* M *is flat.*

Proof. By theorem 6.4, $b_1(M) \leqslant \dim M$.

If $b_1(M) = \dim M$, then there are n linearly independent parallel 1-forms.

So for any $p \in M$, there exists a parallel local frame on a neighborhood of p.

Hence M is flat. □

Coro 10.14. $\mathbb{S}^3 \times \mathbb{S}^1$ *has no Ricci flat metric.*

Proof. Suppose $\mathbb{S}^3 \times \mathbb{S}^1$ has a Ricci-flat metric.

Since the universal covering $\tilde{M}$ of $\mathbb{S}^3 \times \mathbb{S}^1$ is homeomorphic to $\mathbb{S}^3 \times \mathbb{R}$.

So $\left(\tilde{M}, \tilde{g}\right)$ must split as $(N \times \mathbb{R}, h \oplus g_{\mathbb{R}})$, where N is simply connected and $\mathrm{Ric}(h) = 0$.

And by theorem 1.4, (N, h) is isometric to $\left(\mathbb{R}^3, g_{\mathbb{R}^3}\right)$, contradiction! □

Appendix A

Convention of tensor calculus

The reason why I write this appendix is that many people asked me about the rules and notations of tensor calculus and sometimes I feel confused by these notations too. So I think should write a "axiom system" of tensor calculus that is true most of the time. It may be not very rigorous(actually, it is a little bit ridiculous) but it is helpful if you are not familiar with these calculus I think.

By the way, If you still feel confused while reading this, feel free to contact me, I will try my best to answer your question and so that I can know where is unclear and need to improve.

A.1 General tensor calculus

Def A.1. Index is a letter that can take a value in a given finite set(mostly the integer $1 \sim n$).

Def A.2. A symbol with some upper and lower indices are called tensor:

$$A^{ij\cdots}_{kl\cdots}.$$

The tensor with p upper indices and q lower indices is said to be of type (p, q).

The total number of indices is called the degree of the tensor, e.g. a type-(p, q) tensor has degree $p + q$.

If the definition of indices has m elements totally, then we say the tensor is m-dimensional.

Def A.3. A term is the multiplication of some tensors, where these tensors may share the same indices.

Def A.4. Some terms connected by '+' is called a tensor expression.

Two expressions connected by '=' is called a tensor equation.

Axiom A.1. *The indices are replaceable in the tensor expression.*

Exam A.1. *An example of a correct change is:*

$$A^i B^k_j C_{kl} + D^i_j E_l \to A^s B^t_j C_{tl} + D^s_j E_l,$$

whereas an erroneous change is:

$$A^i B^k_j C_{kl} + D^i_j E_l \not\to A^s B^k_j C_{tl} + D^i_j E_l$$

since 's' does not fully replace 'i' and 't' does not fully replace 'k'.

Axiom A.2. *In one term, the same index symbol can only appear once as an upper or lower index respectively.*

Exam A.2.

$$A^j_{ik}B^{jk}_{ij}$$

is incorrect since there are two 'i' in the lower index and two 'j' in the upper index.

Def A.5. The index that appears as upper and lower index both is called dummy(or summation) index.

The index that only appears once in the term is called free index.

Axiom A.3. *A term with p free upper indices and q free lower indices can be seen as a type-(p, q) tensor.*

Exam A.3.

$$A^k_i B^i C^j_k$$

Can be regarded as a $(1, 0)$-tensor D^j.

Axiom A.4. *The free indices in a tensor expression always appear in the same (upper or lower) position throughout every term, and in a tensor equation the free indices are the same on each side.*

Exam A.4.

$$A^i B^k_j C_{kl} + D^i_l E_j = T^i_{jl}$$

is legal, and as for an erroneous expression:

$$A^i B^k_j C_{kl} + D^k_{ij} E^l.$$

While j is a free lower index in both terms, the position of 'i','l' are wrong and 'k' is dummy in the first term but free in the second term.

So far, all the things that we defined are very abstract. Thus, we now try to give you the specific meaning of tensor.

Axiom A.5. *If we take a specific value for every index of a tensor, the result is in a given $\mathbb{R}$-module.*

Axiom A.6. *Free indices can freely choose a value in its domain and the dummy indices should be summed over its domain.*

Exam A.5.

$$A_i B^i \equiv \sum_i A_i B^i.$$

The summation may occur more than once within a term:

$$A^k_i B^i C^j_k \equiv \sum_i \sum_k A^k_i B^i C^j_k.$$

Axiom A.7. *A tensor equation of dimension m with n free indices represents m^n real-value equations: each free index takes on every value of the definition.*

Exam A.6. *Let the definition of indices be $\{0, 1, 2, 3\}$, then the tensor equation*

$$A^i B^k_j C_{kl} + D^i_j E_l = T^i_{jl}$$

Then since there are three free indices (i, j, l), there are $4^3 = 64$ real-value equations. In particular, here are three of them:

$$A^0 B^0_1 C_{00} + A^0 B^1_1 C_{10} + A^0 B^2_1 C_{20} + A^0 B^3_1 C_{30} + D^0_1 E_0 = T^0_{10},$$

$$A^1 B^0_0 C_{00} + A^1 B^1_0 C_{10} + A^1 B^2_0 C_{20} + A^1 B^3_0 C_{30} + D^1_0 E_0 = T^1_{00},$$

$$A^1 B^0_2 C_{02} + A^1 B^1_2 C_{12} + A^1 B^2_2 C_{22} + A^1 B^3_2 C_{32} + D^1_2 E_2 = T^1_{22}.$$

Def A.6. Kronecker delta is a type-$(1,1)$ tensor:

$$\delta_i^j := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Prop A.1.

$$\delta_i^j A^i = A^j$$

$$\delta_i^j B_j = B_i$$

Proof.

$$\delta_i^j A^i = \sum_i \delta_i^j A^i = A^j.$$

$$\delta_i^j B_j = \sum_j \delta_i^j B_j = B_i.$$

□

Def A.7. For a $(2,0)$-tensor A^{ij}, its inverse A_{ij} is a $(0,2)$-tensor such that

$$A_{ij}A^{jk} = \delta_i^k.$$

A.2 Ricci calculus

Def A.8. Given an n-dimensional manifold M, we assume the the definition of indices is $\{1, \cdots, n\}$.

For a element $T \in \Gamma(M, (\otimes^p TM) \otimes (\otimes^q T^*M))$, we can define a (p,q)-tensor:

$$T^{i_1 \cdots i_p}_{j_1 \cdots j_q} = T\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_p}}, \mathrm{d}x^{j_1}, \cdots, \mathrm{d}x^{j_q}\right).$$

Prop A.2. *For some elements $T_1, \cdots, T_n \in \Gamma(M, (\otimes^\bullet TM) \otimes (\otimes^\bullet T^*M))$, the product of all these tensors corresponds to the product of $T_1, \cdots, T_n$ as section of vector bundle.*

Exam A.7.

$$A^i_j B_k = A\left(\frac{\partial}{\partial x^i}, \mathrm{d}x^j\right) B\left(\mathrm{d}x^k\right) = (A \otimes B)\left(\frac{\partial}{\partial x^i}, \mathrm{d}x^j, \mathrm{d}x^k\right).$$

Prop A.3. *For covariant derivative, we have the similar property:*

$$\nabla_i T^{i_1 \cdots i_p}_{j_1 \cdots j_q} = (\nabla T)\left(\mathrm{d}x^i, \frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_p}}, \mathrm{d}x^{j_1}, \cdots, \mathrm{d}x^{j_q}\right)$$

Remark A.1. When you are not sure what a tensor expression mean, just "throw" every index to the back for every term and be careful that the position(upper or lower) must reverse in this process.