Introduction to Smooth Manifolds
Exercises & Problems Solution

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Chapter 1

Smooth Manifolds

1.1 Exercises

Exer 1.1. *Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to any open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.*

Since $\hat{U}$ is open, $\exists r > 0, s.t. B_r(\varphi(p)) \subset \hat{U}$.
So $p \in \varphi^{-1}(B_r(\varphi(p)))$, $\varphi\big|_{\varphi^{-1}(B_r(\varphi(p)))} : \varphi^{-1}(B_r(\varphi(p))) \to B_r(\varphi(p))$ is homeomorphism.
And let $\psi : B_r(\varphi(p)) \xrightarrow{\cong} \mathbb{R}^n, x \mapsto \tan\left(\frac{\pi|x-\varphi(p)|}{2r}\right)\frac{x-\varphi(p)}{|x-\varphi(p)|}$.
Hence $\psi \circ \varphi\big|_{\varphi^{-1}(B_r(\varphi(p)))}$ is homeomorphism from $\varphi^{-1}(B_r(\varphi(p)))$ to $\mathbb{R}^n$.

Exer 1.6. *Show that $\mathbb{R}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.*

Because $\pi(x) = \pi\left(\frac{x}{|x|}\right) = [x]$, we can obtain that $\pi\big|_{\mathbb{S}^n}$ is surjective.
And for $x, y \in S^n$, $[x] = [y] \Leftrightarrow \exists a \in \mathbb{R}, s.t. x = ay \Leftrightarrow x = \pm y$.
So WLOG, suppose $d(x, y) < d(x, -y)$ and consider two open sets $U = (B_r(x) \cap \mathbb{S}^n) \cup (B_r(-x) \cap \mathbb{S}^n), V = (B_r(y) \cap \mathbb{S}^n) \cup (B_r(-y) \cap \mathbb{S}^n)$, where $2r < d(x, y)$.
Then U, V are neighborhood of x, y resp. in $\mathbb{S}^n$, and $U \cap V = \varnothing$
Therefore $[x] \in \pi(U), [y] \in \pi(V), \pi(U) \cap \pi(V) = \varnothing$, $i.e.\mathbb{RP}^n$ is Hausdorff.
And $\mathbb{RP}^n$ is second-countable, since $\mathbb{S}$ is second-countable and π is open map.

Exer 1.7. *Show that $\mathbb{RP}^n$ is compact.*

By proposition $A.45(a)$, $\mathbb{RP}^n$ is compact, since $\pi\big|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{RP}^n$ is surjective.

Lemma 1.13. *Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.*

(a) The collection $\left\{\bar{X} : X \in \mathcal{X}\right\}$ is also locally finite.

(b) $\overline{\bigcup_{X\in\mathcal{X}} X} = \bigcup_{X\in\mathcal{X}} \bar{X}$.

Exer 1.14. *Prove the preceding lemma.*

(a) $\forall p \in M, \exists$neighborhood $U, s.t. U$ intersects at most finitely many of the sets in $\mathcal{X}$.

We claim that if $U \cap X = \varnothing$, $U \cap \bar{X} = \varnothing$.

Then we can deduce the conclusion immediately since $\left\{\bar{X} : X \in \mathcal{X}, \bar{X} \cap U \neq \varnothing\right\} \subset \{X : X \in \mathcal{X}, X \cap U \neq \varnothing\}$.

To verify the claim, suppose $x \in U \cap \bar{X}$.

Then x must be a limit point of X, $i.e. x = \lim_{n\to\infty} x_n, x_n \in X$.

But $x_n \notin U$ since $U \cup X = \varnothing$, which means that $x \in U$ is a limit point of U^c, contradiction!

Hence $U \cap \bar{X} = \varnothing$ so the statement is true.

(b) $\bigcup_{X\in\mathcal{X}} \bar{X}$ consist of all $X \in \mathcal{X}$ and all their limit points, so it is obviously contained in $\overline{\bigcup_{X\in\mathcal{X}} X}$.

Conversely, for $p \in \overline{\bigcup_{X\in\mathcal{X}} X}$, if p is in some $X \in \mathcal{X}$, then $p \in \bigcup_{X\in\mathcal{X}} \bar{X}$ is trivial.

So we now consider the case that p is a limit point of $\bigcup_{X\in\mathcal{X}} X$.

Let U be a neighborhood of p that intersects at most finitely many of the sets in $\mathcal{X}$, and $\lim_{n\to\infty} p_i = p, \{p_n\} \subset U$.

Then by pigeonhole principle, $\exists X \in \mathcal{X}, s.t. X \cap \{p_n\}$ has infinite many elements.

Pick the subsequence of $\{p_n\}$ that contained in X, we conclude that $p \in \bar{X}$.

Hence $\overline{\bigcup_{X\in\mathcal{X}} X} = \bigcup_{X\in\mathcal{X}} \bar{X}$.

Prop 1.17. *Let M be a topological manifold.*

(a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas called the ***smooth structure determined by*** *$\mathcal{A}$.*

(b) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Exer 1.18. *Prove Proposition* 1.17(*b*).

$\Rightarrow$: Let $\mathcal{A}, \mathcal{B}$ be two smooth atlas for M that determine the same smooth structure.
So $\bar{\mathcal{A}} = \bar{\mathcal{B}} = \overline{\mathcal{A} \cup \mathcal{B}} \supset \mathcal{A} \cup \mathcal{B}$, $i.e. \mathcal{A} \cup \mathcal{B}$ is smooth.
$\Leftarrow$: Because $\mathcal{A} \cup \mathcal{B}$ is smooth and $\mathcal{A}, \mathcal{B} \subset \mathcal{A} \cup \mathcal{B}$.
So $\bar{\mathcal{A}} = \overline{\mathcal{A} \cup \mathcal{B}} = \bar{\mathcal{B}}$, $i.e. \mathcal{A}, \mathcal{B}$ determine the same smooth structure.

Prop 1.19. *Every smooth manifold has a countable basis of regular coordinate balls.*

Exer 1.20. *Prove Proposition* 1.19.

Consider the same construction as lemma 1.10, for any precompact coordinate ball V, let (U, φ) be the chart of M that contains $\bar{V}$.
So V is precompact in U, and $\hat{V} = B_r(x) \subset B_{r'}(x) \subset U$ for some $r' > r$.
Hence V must be a regular coordinate ball, $i.e.$there is a countable basis of regular coordinate balls.

Prop 1.38. *Let M be a topological n-manifold with boundary.*

(a) Int$M$ *is an open subset of M and a topological n-manifold without boundary.*

(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.

(c) M is a topological manifold if and only if $\partial M = \varnothing$.

(d) If $n = 0$, then $\partial M = \varnothing$ and M is a 0-manifold.

Exer 1.39. *Prove the preceding proposition. For this proof, you may use the theorem on topological invariance of the boundary when necessary. Which parts require it?*

(a) Int$M$ is the union of all interior chart, so it is an open subset of M.

And for every $p \in \text{Int}M$, there is a interior chart (U, φ) that contain p, and $U \subset \text{Int}M$.

So (U, φ) is a chart of Int$M$, $i.e.$Int$M$ is a topological n-manifold without boundary.

(b) By theorem 1.37, $\partial M = M - \text{Int}M$ is closed in M.

For every $p \in \partial M$, there is a boundary char (U, φ) that sends p to $\partial \mathbb{H}^n$.

So consider $V = \varphi^{-1}(\hat{U} \cap \partial \mathbb{H}^n) \subset \partial M$, the chart $\left(V, \varphi\big|_V\right)$ contain p and map V to $\partial \mathbb{H}^n \cong \mathbb{R}^{n-1}$.

Hence ∂M is a topological $(n-1)$-manifold without boundary.

(c) $\Rightarrow$: By definition, there is no boundary chart, $i.e.\partial M = \varnothing$.

$\Leftarrow$: $M = \text{Int}M$ since each point in M is either interior point or boundary point.

So by (a), M is topological manifold.

(d) Since $n = 0$, $\mathbb{R}^0 = \mathbb{H}^0$ and $\partial \mathbb{H}^0 = \varnothing$.

So $\partial M = \varnothing$, $i.e.M$ is a 0-manifold.

Prop 1.40. *Let M be a topological manifold with boundary.*

(a) M has a countable basis of precompact coordinate balls and half-balls.

(b) M is locally compact.

(c) M is paracompact.

(d) M is locally path-connected.

(e) M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.

(f) The fundamental group of M is countable.

Exer 1.41. *Prove the preceding proposition.*

(a) Similar to Lemma 1.10 and proposition 1.19, we first consider the case that there is a global smooth chart (M, φ).

If (M, φ) is interior chart, it is totally the same as Lemma 1.10.

If (M, φ) is boundary chart, consider $\mathcal{B} = \left\{B_r(x) : r, x^i \in \mathbb{Q}, B_{r'}(x) \subset \hat{U} \text{ for some } r' > r\right\} \cup \left\{B_r(x) \cap \mathbb{H}^n : r, x^i \in \mathbb{Q}, x \in \mathbb{H}^n, B_{r'}(x) \cap \mathbb{H}^n \subset \hat{U} \text{ for some } r' > r\right\}$.

Then $\mathcal{B}$ is a countable basis of $\hat{U}$, consisting of precompact balls and half-balls.

Hence $\varphi^{-1}(B)$ is a countable basis of precompact coordinate balls and half-balls.

We now let M be an arbitrary smooth manifold. By definition, each point of M is in the domain of a chart.

By proposition A.16, M can be covered by countably many charts, and each chart has a countable basis of precompact coordinate balls and half-balls.

Hence the union of all these open set forms a countable basis of precompact coordinate balls and half-balls of M.

(b) Directly from (a).

(c) We prove a stronger statement: given an open cover $\mathcal{X}$ of M and any basis $\mathcal{B}$, there exists a countable, locally finite open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$.

Let (K_j) be an exhaustion of M by compact sets, and $V_j = K_{j+1}/\mathrm{Int}K_j, W_j = \mathrm{Int}K_{j+2}/K_{j-1}$.

Then V_j is a compact set contained in the open set W_j.

For each x in V_j, there exist $X_x \in \mathcal{X}, B_x \in \mathcal{B}$ that contain x, and $B_x \subset X_x \cap W_j$

Since all B_x form a cover of V_j, so it has a finite subcover.

The union of all finite subcover as j range over $\mathbb{Z}^+$ is a countable open cover of M, which is a refinement of $\mathcal{X}$.

And the cover is locally finite, since $W_j \cap W_{j'} = \varnothing$ for $|j - j'| > 2$.

(d) Directly from (a).

(e) By proposition $A.43(a)$, each component is open in M, thus they form an open cover of M.

Since M is second-countable, this cover must have a countable subcover.

But the components are all disjoint, which means that there must have only countable different components.

And they are connected topological manifold with boundary, since they are open subset.

(f) Using the conclusion of $(a)-(e)$, the prove of this statement is totally the same as proposition 1.16, which is pretty complicated and I don't want to repeat it here.

Exer 1.42. *Show that every smooth manifold with boundary has a countable basis consisting of regular coordinate balls and half-balls.*

Consider the same construction as proposition 1.40, for any precompact coordinate ball V, let (U, φ) be the chart of M that contains $\bar{V}$.

So V is precompact in U, and $\hat{V} = B_r(x) \subset B_{r'}(x) \subset U$ for some $r' > r$.

Hence V must be a regular coordinate ball.

And for any precompact coordinate half-ball V, let (U, φ) be the boundary chart of M that contains $\bar{V}$.

So V is precompact in U, and $\hat{V} = B_r(0) \cap \mathbb{H}^n \subset B_{r'}(0) \cap \mathbb{H}^n \subset U$ for some $r' > r$.

Hence V must be a regular coordinate half-ball.

Therefore there is a countable basis consisting of regular coordinate balls and half-balls.

Exer 1.43. *Show that the smooth manifold chart lemma (Lemma 1.35) holds with "$\mathbb{R}^n$" replaced by "$\mathbb{R}^n$ or $\mathbb{H}^n$" and "smooth manifold" replaced by "smooth manifold with boundary".*

Totally the same as lemma 1.35 since there is no particularity of $\mathbb{R}^n$ been used while proving.

Exer 1.44. *Suppose M is a smooth n-manifold with boundary and U is an open subset of M. Prove the following statements:*

*(a) U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V, φ) for M such that $V \subset U$ defines a smooth structure on U. With this topology and smooth structure, U is called an **open submanifold with boundary**.*

*(b) If $U \subset \mathrm{Int}M$, then U is actually a smooth manifold (without boundary); in this case we call it an **open submanifold of** M.*

(c) $\mathrm{Int}M$ is an open submanifold of M (without boundary)

(a) Consider the atlas in the statement, denote it as $\mathcal{A}$.

For every point $p \in U$, since there is a chart (V, φ) of M that contains p, the chart $\left(U \cap V, \varphi|_{U \cap V}\right) \in \mathcal{A}$ also contains p.

And we can easily check that every two charts in $\mathcal{A}$ are compatible.

Hence $\mathcal{A}$ is a well-define smooth atlas and U is a smooth manifold with boundary.

(b) By proposition 1.40, Int$M$ is a topological manifold without boundary, and all charts in the smooth atlas in (a) are interior charts.

So Int$M$ is a smooth manifold without boundary.

By example 1.26, U is an open submanifold of Int$M$.

(c) Directly by (b).

1.2 Problems

Prob 1.1. *Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the line with two origins).*

Denote two different origins as a, b resp.
For $x \neq 0$, $x \in (x - \varepsilon, x + \varepsilon)$ is an open set in M, where $|\varepsilon| < |x|$.
And $(-\varepsilon, 0) \cup (0, \varepsilon) \cup \{a\}, (-\varepsilon, 0) \cup (0, \varepsilon) \cup \{b\}$ are both open and homeomorphic to $(-\varepsilon, \varepsilon)$.
Hence M is locally Euclidean.
The set $\{(x, y) : x, y \in \mathbb{Q}, xy > 0\} \cup \{(x, 0) \cup (0, y) \cup \{p\} : x, y \in \mathbb{Q}, x < 0 < y, p \in \{a, b\}\}$ is a countable basis of M, so M is also second-countable.
But every neighborhood of a must contain $(-\varepsilon, 0) \cup (0, \varepsilon) \cup \{a\}$ for sufficiently small ε, which means that any two neighborhood of a, b must have nonempty intersection.
Therefore M is not Hausdorff.

Remark 1.1. This is a famous counterexample of the statement "quotient space of Hausdorff space is Hausdorff".

Prob 1.2. *Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally euclidean and Hausdorff, but not second-countable.*

$X = \bigsqcup_{\alpha \in \mathscr{A}} \mathbb{R}_\alpha$ is obviously locally Euclidean and Hausdorff, we now prove that it is not second-countable.
Consider a cover $\mathcal{B} = \{\mathbb{R}_\alpha | \alpha \in \mathscr{A}\}$.
Then by proposition $A.16$, if X is second-countable, $\mathcal{B}$ must have a countable subcover.
But countable many sets in $\mathcal{B}$ can only cover countable many X_i, contraction!
Hence X is not second-countable.

Remark 1.2. In generally, any disjoint union of uncountable many topological spaces is not second-countable.

Prob 1.3. *A topological space is said to be σ**-compact** if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.*

$\Rightarrow$: By lemma 1.10, topological manifold is obviously σ-compact.
$\Leftarrow$: Since X is locally Euclidean, for every point p, there is a chart (U, φ) containing p.
So X can be covered by countably many charts, since X is σ-compact.

And because every charts are homeomorphic to open subset of $\mathbb{R}^n$, which are all second-countable, we can obtain that X is also second-countable.

Hence X is a topological manifold.

Prob 1.4. *Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.*

(a) Assuming that each set in $\mathcal{U}$ intersects only finite many others, show that $\mathcal{U}$ is locally finite.

(b) Give an example to show that the converse to (a) may be false.

(c) Now assume that the sets in $\mathcal{U}$ are precompact in M, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

(a) Suppose there is a point p in M such that any of its neighborhood intersects with infinitely many of the sets in $\mathcal{U}$.

Then p must be contained in some $U \in \mathcal{U}$, since $\mathcal{U}$ is an open cover of M.

So U intersects with infinitely many of the sets in $\mathcal{U}$, contradiction!

(b) let $M = \mathbb{R}, \mathcal{U} = \{\mathbb{R}\} \cup \{(n, n+1) : n \in \mathbb{Z}\}$, then it is clear that $\mathbb{R}$ intersects with infinitely many different sets in $\mathcal{U}$, but $\mathcal{U}$ is locally finite.

(c) For every open set $U \in \mathcal{U}$, $\bar{U}$ is compact.

Since every point $p \in \bar{U}$ has a neighborhood that intersects with finitely many of sets in $\mathcal{U}$.

Consider all these neighborhood as a cover of $\bar{U}$, it must have a finite subcover.

So there must have only finitely many of the sets in $\mathcal{U}$ that intersects with $\bar{U}$, otherwise one of the neighborhood in the finite subcover will intersects with infinitely many of sets in $\mathcal{U}$.

Prob 1.5. *Suppose M is a locally Euclidean Hausdorff space. Show that M is second-countable if and only if it is paracompact and has countably many connected components.*

$\Rightarrow$: By proposition 1.11 and theorem 1.15, M is paracompact and has countably many connected components.

$\Leftarrow$: We first prove that M is second-countable when it is connected, once this is done, there is also a countable basis for countably many connected components cases.

For every point $p \in M$, there is a precompact chart (U, φ) containing p.

All these charts form a open cover of M, so it has a locally finite refinement denoted by $\mathcal{U}$.

By problem 1.4(c), every set in $\mathcal{U}$ intersects only finitely many others.

Consider a set sequence $\mathcal{V}_n$ in $\mathcal{U}$, inductively define by letting $\mathcal{V}_0$ be an arbitrary set in $\mathcal{U}$, and $\mathcal{V}_n$ be the sets in $\mathcal{U}$ that intersect sets in $\bigcup_{i=0}^{n-1} \mathcal{V}_i$ but not contained in it.

We claim that the collection $\bigcup_{i=0}^{\infty} \mathcal{V}_i$ cover M.

Since M is locally Euclidean and connected, it is locally path-connected, *i.e.*M is path-connected.

So for each point $x \in M$, there must have a path $\gamma : [0,1] \to M$ that connects x and a point in $\mathcal{V}_0$ together.

And $\gamma([0,1])$ is compact, it can be covered by finite many sets in $\mathcal{U}$.

Divided $[0,1]$ into several parts $0 = a_0 < a_1 < \cdots < a_k = 1$, such that $f([a_{i-1}, a_i]) \subset U_i$ for some $U_i \in \mathcal{U}$.

Then it is easy to see that $U_i \subset \bigcup_{j=0}^{i} \mathcal{V}_j$, which means that x is covered by $\bigcup_{i=0}^{k} \mathcal{V}_i$.

Hence the collection $\bigcup_{i=0}^{\infty} \mathcal{V}_i$ cover M, and it is countable since every $\mathcal{V}_i$ are countable.

Therefore M is second-countable, since every elements in $\mathcal{V}_i$ are charts.

Prob 1.6. *Let M be a nonempty topological manifold of dimension $n \geqslant 1$. If M has a smooth structure, show that it has uncountably many distinct ones.*

Let $f_s : \mathbb{B}^n \to \mathbb{B}^n, x \to |x|^{s-1}x$, which is obviously a homeomorphism, we first prove that $f_s(x)$ is a diffeomorphism iff $s = 1$.

When $s = 1$, $f_s(x) = x$ is obviously a diffeomorphism.

Then we consider the case that $s < 1$, we have $\left.\frac{\partial f_{s,1}}{\partial x_1}\right|_{x=0} = |x|^{s-1} + (s-1)x_1^2|x|^{s-3}$. If we approx 0 along the x_1-axis, we find that $\left.\frac{\partial f_{s,1}}{\partial x_1}\right|_{x=0} = (s-1)|x_1|^{s-1} \to \infty$, is not well-defined.

So $f_s(x)$ is not a diffeomorphism when $s < 1$. And for $s > 1$, $f_s^{-1}(x) = |x|^{\frac{1}{s}-1}x$ is not diffeomorphism.

Hence, for a given smooth manifold M with a smooth structure $\mathcal{A}$, we consider an arbitrary point p and a chart $(U, \varphi) \in \mathcal{A}$ that contains p.

We first make (U, φ) to be the only chart that covers p, let $\mathcal{A}' = \{(V\backslash\{p\})|(V, \psi) \in \mathcal{A}\} \cup \{(U, \varphi)\}$, which is obviously a smooth chart.

Then since $\hat{U} \subset \mathbb{R}^n$ is open, there exist an r such that $B(\hat{p}, r) \subset \hat{U}$.

So let $\mathcal{A}'' = \{(V\backslash\{p\}, \psi)|(V, \psi) \in \mathcal{A}\} \cup \{(W, \varphi)\}$, where $W = \varphi^{-1}(B(\hat{p}, r)) \subset U$.

Now we try to modify φ, define $\phi = \frac{\varphi - \hat{p}}{r}$, then ϕ maps W to the unit ball.

Therefore we define $\mathcal{A}_s''' = \{(V\backslash\{p\}, \psi)|(V, \psi) \in \mathcal{A}\} \cup \{(W, f_s \circ \phi)\}$ and prove that it is smooth.

Since every two charts that not containing p are smoothly compatible, we now focus on the chart $(V\backslash\{p\}, \psi)$ and $(W, f_s \circ \phi)$.

Then $(F_s \circ \phi) \circ \psi^{-1} : \psi(W \cup V\backslash\{p\}) \to \mathbb{B}^n\backslash\{0\}$ is a diffeomorphism.

In the end, we prove that any two $\mathcal{A}_s'''$ and $\mathcal{A}_t'''$ are not compatible.

Consider charts $(W, f_s \circ \phi)$ and $(W, f_t \circ \phi)$, the transition map is $f_s \circ \phi \circ \phi^{-1} \circ f_t^{-1} = f_{\frac{s}{t}}$ is not diffeomorphism, *i.e.*these two charts are not smooth compatible.

Hence it has uncountably many distinct ones.

Prob 1.7. *Let N denote the north pole $(0, \cdots, 0, 1) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, and let S denote the south pole $(0, \cdots, 0, -1)$. Define the* ***stereographic projection*** *$\sigma : \mathbb{S}^n\backslash\{N\} \to \mathbb{R}^n$ by $\sigma\left(x^1, \cdots, x^{n+1}\right) = \frac{(x^1, \cdots, x^n)}{1 - x^{n+1}}$. Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}\backslash\{S\}$.*

(a) For any $x \in \mathbb{S}\backslash\{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\tilde{\sigma}$ is called ***stereographic projection from the south pole****.)*

(b) Show that σ is bijective, and $\sigma^{-1}(u^1, \cdots, u^n) = \frac{(2u^1, \cdots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}$.

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n\backslash\{N\}, \sigma)$ and $(\mathbb{S}^n\backslash\{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\tilde{\sigma}$ are called ***stereographic coordinates****.)*

(d) Show that this smooth structure is the same as the one defined in Example 1.31.

(a) We only need to prove that $(\sigma(x), 0)$, x and N are collinear.

This is because $\frac{x^i - (\sigma(x))^i}{x^{n+1}} = \frac{x^{n+1}x^i}{x^{n+1}(x^{n+1} - 1)} = \frac{x^i}{x^{n+1} - 1} = \frac{x^i - N^i}{x^{n+1} - N^{n+1}}$

(b) By (a), since a line can only have two intersect with the sphere.

So σ is injective.

Consider an arbitrary point $u = (u^1, \cdots, u^n)$, we need to prove that the line connecting u and N has another intersect with the sphere.

Then the point x on the line satisfy that $x^i = (1 - x^{n+1})u^i$ for $i = 1, \cdots, n$

Since $x \in \mathbb{S}\backslash\{N\}$, we can obtain that $(1 - x^{n+1})^2|u|^2 + (x^{n+1})^2 = 1, i.e. x^{n+1} = \frac{|u|^2-1}{|u|^2+1}$.

Hence σ is bijective and $\sigma^{-1}(u^1, \cdots, u^n) = \frac{(2u^1, \cdots, 2u^n, |u|^2-1)}{|u|^2+1}$.

(c) $\tilde{\sigma} \circ \sigma^{-1}(u^1, \cdots, u^n) = \tilde{\sigma}\left(\frac{(2u^1, \cdots, 2u^n, |u|^2-1)}{|u|^2+1}\right) = \frac{(u^1, \cdots, u^n)}{|u|^2}$ is diffeomorphism, whose inverse is itself.

So these two charts are smooth compatible, $i.e.$the atlas consisting of these two charts defines a smooth structure on $\mathbb{S}^n$.

(d) For $i \neq n+1$, $\varphi_i^{\pm} \circ \sigma^{-1} = \frac{\left(2u^1, \cdots, \hat{u^i}, \cdots, 2u^n, |u|^2-1\right)}{|u|^2+1}$ and $\sigma \circ \left(\varphi_i^{\pm}\right)^{-1} = \frac{\left(u^1, \cdots, \sqrt{1-|u|^2}, u^{n-1}\right)}{1-u^n}$ are diffeomorphism.

For $i = n+1$, $\varphi_i^{\pm} \circ \sigma^{-1} = \frac{2}{|u|^2+1}u$ and $\sigma \circ \left(\varphi_i^{\pm}\right)^{-1} = \frac{1}{1-\sqrt{1-|u|^2}}u$ are diffeomorphism.

So $(\mathbb{S}\backslash\{N\}, \sigma)$ is smooth compatible with $\left(U_i^{\pm}, \varphi_i^{\pm}\right)$, so is $(\mathbb{S}\backslash\{S\}, \tilde{\sigma})$.

Hence this smooth structure is the same as the one defined in Example 1.31.

Prob 1.8. *By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. And angle function on a subset $U \subset \mathbb{S}^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exist an angle function θ on an open subset $U \subset \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.*

If there exist such angle function θ, and suppose $U = \mathbb{S}^1$.
Since $\mathbb{S}^1$ is connected and compact, let $\theta(\mathbb{S}^1) = [a, b]$.
And θ is injective because $e^{iz} \circ \theta = \mathrm{Id}$ is injective.
So $\theta : \mathbb{S}^1 \to [a, b]$ is homeomorphic, but $\mathbb{S}^1$ is not simply connected, contradiction!
If U is proper subset of $\mathbb{S}^1$, then every component of U is simply connected.
Let $\theta(z) = -i\ln(z)$ where ln is a well-define branch of logarithm function.
Then θ is a well-define continuous function satisfying that $e^{i\theta(z)} = z$.
And since $\sigma \circ \theta^{-1} = \frac{\cos x}{1-\sin x} = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right), \tilde{\sigma} \circ \theta^{-1} = \tan\left(-\frac{x}{2} + \frac{\pi}{4}\right)$ are diffeomorphism.
Hence U is smooth compatible with $\mathbb{S}^1\backslash\{N\}$ and $\mathbb{S}^1\backslash\{S\}$, $i.e.(U, \theta)$ is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

Prob 1.9. ***Complex projective n-space***, *denoted by $\mathbb{CP}^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We use the correspondence $(x^1+iy^1, \cdots, x^{n+1}+iy^{n+1}) \leftrightarrow (x^1, y^1, \cdots, x^{n+1}, y^{n+1})$ to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.)*

For each $i = 1, \cdots, n+1$, let $\tilde{U}_i \subset \mathbb{C}^{n+1}\backslash\{0\}$ be the set where $z^i \neq 0$ and let $U_i = \pi\left(\tilde{U}_i\right)$.
Since $\tilde{U}_i$ is a saturated open subset, U_i is open and $\pi\big|_{\tilde{U}_i} : \tilde{U}_i \to U_i$ is a quotient map.
Define a map $\varphi_i : U_i \to \mathbb{C}^n$ by $\varphi_i[z^1, \cdots, z^{n+1}] = \left(\frac{z^1}{z^i}, \cdots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots, \frac{z^{n+1}}{z^i}\right)$.
This map is well define because its value is unchanged by multiplying z by a nonzero constant.
Because $\varphi_i \circ \pi$ is continuous, φ_i is continuous by the universal property of quotient maps.
In fact, φ_i is a homeomorphism because it has a continuous inverse given by $\varphi_i^{-1}(u^1, \cdots, u^n) = [u^1, \cdots, u^{i-1}, 1, u^{i+1}, \cdots, u^n]$ as you can check.
Because the sets $U_1, \cdots, U_{n+1}$ cover $\mathbb{CP}^n$, this shows that $\mathbb{CP}^n$ is locally Euclidean.
And let $f : \mathbb{CP}^n \times \mathbb{CP}^n \to \mathbb{C}, (z, w) \to \sum\limits_{i<j} |z_i w_j - w_i z_j|$

Then f is obviously continuous, and $f^{-1}(0) = \{(z,z)|z \in \mathbb{CP}^n\}$.

So $\{(z,z)|z \in \mathbb{CP}^n\}$ is closed, *i.e.*$\mathbb{CP}^n$ is a Hausdorff space and a topological manifold.

More precisely, because $\varphi_i \circ \varphi_j^{-1} = \left(\frac{u^1}{u^i}, \cdots, \frac{u^{i-1}}{u^i}, \frac{u^{i+1}}{u^i}, \cdots, \frac{u^{j-1}}{u^i}, \frac{1}{u^i}, \frac{u^j}{u^i}, \cdots, \frac{u^n}{u^i}\right)$ is a diffeomorphism, all these charts are compatible to each other.

Hence $\mathbb{CP}^n$ is a smooth manifold.

Prob 1.10. *Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subset \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \cdots, e_k)$ and $(e_{k+1}, \cdots, e_n)$, respectively, where e_i is the ith standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $S \subset \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\begin{bmatrix} I_k \\ B \end{bmatrix}$, where I_k denotes the $k \times k$ identity matrix.*

Since $e_i + \varphi(S)e_i \in S$ and are linear independent, since $\varphi(S)e_i \in Q$.

So S is spanned by $\{e_i + \varphi(S)e_i | i = 1, \cdots, k\}$, *i.e.*it is spanned by the columns of the matrix $\begin{bmatrix} I_k \\ B \end{bmatrix}$ and B is obviously unique.

Prob 1.11. *Let $M = \bar{\mathbb{B}}^n$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$.*

Let $U = \mathbb{B}^n, U_i^+ = \left\{(x^1, \cdots, x^n) \in \bar{\mathbb{B}}^n \middle| x^i > 0\right\}, U_i^- = \left\{(x^1, \cdots, x^n) \in \bar{\mathbb{B}}^n \middle| x^i < 0\right\}$ and define:

$$\varphi_i^\pm : U_i^\pm \to \mathbb{B}_i^{n,\pm}, (x^1, \cdots, x^n) \to \left(x^1, \cdots, x^i \mp \sqrt{1 - (x^1)^2 - \cdots - \widehat{(x^i)^2} - \cdots - (x^n)^2}, \cdots, x^n\right),$$

where $\mathbb{B}_i^{n,+} = \left\{(x^1, \cdots, x^n) \in \mathbb{B}^n \middle| x^i \geqslant 0\right\}, \mathbb{B}_i^{n,-} = \left\{(x^1, \cdots, x^n) \in \mathbb{B}^n \middle| x^i \leqslant 0\right\}$.

It is easy to check that $\varphi_i^\pm$ is diffeomorphism, so the transition maps are all diffeomorphism.

Hence $\bar{\mathbb{B}}^n$ is a smooth manifold, with charts $(U, \mathrm{Id}), \left(U_i^\pm, \varphi_i^\pm\right)$.

Prop 1.45. *Suppose $M_1, \cdots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then $M_1 \times \cdots \times M_k \times N$ is a smooth manifold with boundary, and $\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$.*

Prob 1.12. *Prove Proposition 1.45(a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).*

By example 1.34, $M_1 \times \cdots M_k$ is a smooth manifold, denoted by M.

Consider the charts $(U, \varphi), (V, \psi)$ for M, N resp., and let $(U \times V, \varphi \times \psi)$ be a chart for $M \times N$.

And For any two such charts $(U_1 \times V_1, \varphi_1 \times \psi_1), (U_2 \times V_2, \varphi_2 \times \psi_2)$, note that $(\varphi_2 \times \psi_2) \circ (\varphi_1 \times \psi_1)^{-1} = (\varphi_2 \circ \varphi_1^{-1}) \circ (\psi_2 \circ \psi_1^{-1})$ is diffeomorphism.

Hence $M \times N$ is a smooth manifold with boundary.

And let $(x,y) \in \partial(M \times N), (U \times V, \varphi \times \psi)$ be a boundary chart that contains (x,y).

Since $\varphi(U)$ is open, (V, ψ) must be a boundary chart.

Moreover, $\psi(y)$ must be contained in $\partial\mathbb{H}^n$, otherwise there is an interior chart whose image is a neighborhood of $\psi(y)$.

Therefore $(x,y) \in M \times \partial N$.

Conversely, if $(x,y) \in M \times \partial N$, there is a boundary chart (V, ψ) that contains y such that $\psi(y) \in \partial\mathbb{H}^n$.

So for a chart (U, φ) that contains x, $(U \times V, \varphi \times \psi)$ contains (x,y) and $(\varphi \times \psi)(x,y) = (\varphi(x), \psi(y)) \in \partial\mathbb{H}^{m+n}$.

Hence $\partial(M \times N) = M \times \partial N$.

Chapter 2

Smooth Maps

2.1 Exercises

Exer 2.1. *Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.*

For each point $p \in M$, consider smooth charts $(U, \varphi), (V, \psi)$ whose domain contain p, and $f \circ \varphi^{-1}, g \circ \psi^{-1}$ is smooth.

So $(U \cap V, \varphi)$ is a smooth chart, and in $\varphi(U \cap V)$, $g \circ \varphi^{-1} = g \circ \psi^{-1} \circ (\psi \circ \varphi^{-1})$ is smooth.

Therefore $(f + g) \circ \varphi^{-1}, (fg) \circ \varphi^{-1}, (cf) \circ \varphi^{-1}$ are all smooth in $\varphi(U \cap V)$.

Hence $f + g, fg, cf \in C^\infty(M)$, *i.e.*it is a commutative and associative algebra over $\mathbb{R}$.

Exer 2.2. *Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f : U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$.*

For each point $p \in U$, $f \circ \mathrm{Id}^{-1} = f$ in U, since (U, Id) is the global smooth chart.

So f is smooth in the sense just defined iff it is smooth in the sense of ordinary calculus.

Exer 2.3. *Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, φ) for M.*

For each point $p \in U$, there exists a chart (V, ψ) containing p and $f \circ \psi^{-1}$ is smooth.

So $(U \cap V, \varphi)$ is a smooth chart, and in $\varphi(V \cap U)$, $f \circ \varphi^{-1} = f \circ \psi^{-1} \circ (\psi \circ \varphi^{-1})$ is smooth.

Hence $f \circ \varphi^{-1}$ is smooth at every point $\hat{p} \in \hat{U}$, *i.e.*$f \circ \varphi^{-1}$ is smooth.

Prop 2.5 (Equivalent Characterizations of smoothness)**.** *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is smooth if and only if either of the following conditions is satisfied:*

(a) For every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Prop 2.6 (Smoothness Is Local)**.** *Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map.*

(a) If every point $p \in M$ has a neighborhood U such that the restriction $F\big|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Exer 2.7. *Prove the preceding two propositions.*

(a) If F satisfies the condition, notice that $(U \cap F^{-1}(V), \varphi)$ is a smooth chart since $U \cap F^{-1}(V)$ is open.

So F is smooth by definition.

If F is smooth, by definition, for every $p \in M$, there exist smooth chart (U, φ) containing p and (V, ψ) containing $F(p)$, such that $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1}$ is smooth.

So $U \cap F^{-1}(V) = U$ is open, $i.e.F$ satisfies the condition.

(b) If F satisfies the condition, then every point $p \in M$ is contained in some $(U_\alpha, \varphi_\alpha)$, and $F(p)$ is contained in some (V_β, ψ_β), and $U_\alpha \cap F^{-1}(V_\beta)$ open.

So by (a), F is smooth.

If F is smooth, then there exists smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N resp., such that for each α there exists a β such that $F(U_\alpha) \subset V_\beta$ and $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth.

We now prove that these atlases satisfies the condition.

For each α, β, let β' satisfying that $F(U_\alpha) \subset V_{\beta'}$ and $\psi_{\beta'} \circ F \circ \varphi_\alpha^{-1}$ is smooth.

So $\psi_\beta \circ F \circ \varphi_\alpha^{-1} = (\psi_\beta \circ \psi_{\beta'}^{-1}) \circ (\psi_{\beta'} \circ F \circ \varphi_\alpha^{-1})$ is smooth from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Remark 2.1. The conditions that $U \cap F^{-1}(V)$ is open in proposition 2.5(a) is necessary, problem 2.1 is a counterexample of the situation $U \cap F^{-1}(V)$ is closed.

(a) There exist smooth charts (V, φ) containing p and (W, ψ) containing $F(p)$ such that $V \subset U, F(V) \subset W$ and $\psi \circ F \circ \varphi^{-1}$ is smooth, since $F\big|_U$ is smooth.

So by definition, F is smooth.

(b) For each open set U and point $p \in U$, there exist smooth charts (V, φ) containing p and (W, ψ) containing $F(p)$ such that $F(V) \subset W$ and $\psi \circ F \circ \varphi^{-1}$ is smooth.

And since $(U \cap V, \varphi)$ is a smooth chart and $F(U \cap V) \subset W$.

Hence by definition, $F\big|_U$ is smooth

Exer 2.9. *Suppose $F : M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of F with respect to every pair of smooth charts for M and N is smooth.*

For each pair of smooth charts $(U, \varphi), (V, \psi)$ and $p \in U$, there exist smooth charts (U', φ') containing p and (V', ψ') containing $F(p)$ such that $F(U') \subset V'$ and $\psi' \circ F \circ (\varphi')^{-1}$ is smooth.

So $\psi \circ F \circ \varphi^{-1} = (\psi \circ (\psi')^{-1}) \circ (\psi' \circ F \circ (\varphi')^{-1}) \circ (\varphi' \circ \varphi^{-1})$ is smooth from $\varphi(U \cap U' \cap F^{-1}(V))$ to $\psi(F(U) \cap V' \cap V)$.

Hence $\psi \circ F \circ \varphi^{-1}$ is smooth at each $\hat{p} \in \hat{U}$,$i.e.\hat{F}$ is smooth.

Prop 2.10. *Let M, N, and P be smooth manifolds with or without boundary.*

(a) Every constant map $c : M \to N$ is smooth.

(b) The identity map of M is smooth.

(c) If $U \subset M$ is an open submanifold with or without boundary, then the inclusion map $U \hookrightarrow M$ is smooth.

(d) If $F : M \to N$ and $G : N \to P$ are smooth, then so is $G \circ F : M \to P$.

Exer 2.11. *Prove parts (a) – (c) of the preceding proposition.*

(a) Let (V, ψ) be a chart that contains $c(M)$.

Then for each point $p \in M$ and a chart (U, φ) containing p, $\psi \circ c\varphi^{-1}$ is smooth since it is a constant map whose image is $\psi(c(M))$.

Hence c is smooth.

(b) For each point $p \in M$ and a chart (U, φ) containing $p = \mathrm{Id}(p)$, $\varphi \circ \mathrm{Id} \circ \varphi^{-1} = \mathrm{Id}$ is smooth.

So Id is smooth.

(c) For each point $p \in U$ and a chart (V, φ) containing $p = i(p)$, $\varphi \circ i \circ \varphi^{-1} = \mathrm{Id}$ is smooth.

So i is smooth.

Prop 2.15 (Properties of Diffeomorphism)**.**

(a) Every composition of diffeomorphisms is a diffeomorphism

(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.

(c) Every diffeomorphism is a homeomorphism and an open map.

(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.

(e) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

Exer 2.16. *Prove the preceding proposition.*

(a) Let M, N, P be the smooth manifolds with or without boundary, $F : M \to N, G : N \to P$ are diffeomorphisms.

Then $G \circ F$ is a bijection and by proposition 2.10, $G \circ F$ and $F^{-1} \circ G^{-1}$ are smooth.

Hence the composition $G \circ F$ is a diffeomorphism.

(b) Let $M_1, \cdots, M_n, N_1, \cdots, N_n$ be the smooth manifolds with or without boundary, and $F_i : M_i \to N_i$ are diffeomorphisms, $M = M_1 \times \cdots \times M_n$, $N = N_1 \times \cdots \times N_n$ and $F{=}F_1 \times \cdots F_n$.

Then for each $p \in M$, there are smooth charts (U_i, φ_i) containing p_i and (V_i, ψ_i) containing $F_i(p_i)$, such that $F(U_i) \subset V_i$ and $\psi_i \circ F_i \circ \varphi_i^{-1}$ is smooth.

So $(U = U_1 \times \cdots \times U_n, \varphi = \varphi_1 \times \cdots \times \varphi_n)$ and $(V = V_1 \times \cdots \times V_n, \psi = \psi_1 \times \cdots \times \psi_n)$ are smooth charts of M and N resp., and $F(U) \subset V$, $\psi \circ F \circ \varphi^{-1} = (\psi_1 \circ F \circ \varphi_1^{-1}) \times \cdots \times (\psi_n \circ F \circ \varphi_n^{-1})$ is smooth.

Therefore F is smooth, so is F^{-1}.

And since F is bijective, we conclude that F is diffeomorphism.

(c) By proposition 2.4, F and F^{-1} are continuous.

Hence F is a homeomorphism since F is bijection.

(d) Let M, N be the smooth manifolds with or without boundary, $F : M \to N$ is a diffeomorphism, U is a open submanifold of M and $V = F(U)$.

By proposition 2.6(b), $F\big|_U$ is smooth and $F^{-1}(V)$ is smooth since $U = F^{-1}(V)$.

Therefore $F\big|_U$ is smooth.

(e) By proposition 2.10(b), identity map is smooth and bijective, $i.e. M \approx M$.

If $M \approx N$, let $F : M \to N$ be the diffeomorphism, then F^{-1} is a diffeomorphism, $i.e. N \approx M$.

If $M \approx N, N \approx P$, then by (a), we can obtain that $M \approx P$.

Hence "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

Thm 2.18 (Diffeomorphism Invariance of the Boundary)**.** *Suppose M and N are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from* $\text{Int}M$ *to* $\text{Int}N$*.*

Exer 2.19. *Use Theorem* 1.46 *to prove the preceding theorem.*

For each $p \in \partial M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1}$ is smooth.

By proposition 2.15(d), $\psi \circ F\big|_U \circ \varphi^{-1}$ is diffeomorphism from $\hat{U}$ to $\widehat{F(U)}$.

So $\widehat{F(U)}$ is also a subset of $\mathbb{H}^n$ and $\widehat{F(p)} \in \partial \mathbb{H}^n, i.e. F(p) \in \partial N$.

Similarly, we have $\partial M \subset F^{-1}(\partial N), i.e. F(\partial M) = \partial N$.

And $F\big|_{\text{Int}M}$ is a diffeomorphism to $\text{Int}N$, since $F(\text{Int}M) = \text{Int}N$.

Thm 2.23 (Existence of Partitions of Unity)**.** *Suppose M is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

Exer 2.24. *Show how the preceding proof needs to be modified for the case in which M has nonempty boundary.*

By exercise 1.42, each set X_α has a basis $\mathcal{B}_\alpha$ of regular coordinate balls and half-balls.

So we can obtain a basis $\mathcal{B} = \bigcup\limits_{\alpha \in A} \mathcal{B}_\alpha$ of M and by proposition 1.40(c), $\mathcal{X}$ has a countable, locally finite refinement $\{B_i\} \cup \{C_i\}$ consisting of elements of $\mathcal{B}$, where $\{B_i\}$ are regular coordinate balls while $\{C_i\}$ are half-balls.

By lemma 1.13(a), the cover $\left\{\bar{B}_i\right\} \cup \left\{\bar{C}_i\right\}$ is also locally finite.

We can define the smooth function $f_i : M \to \mathbb{R}$ such that $\text{supp} f_i = \bar{B}_i$ in the same way as theorem 2.23, so it is sufficient to define this function to $\bar{C}_i$.

Since there exist some $C_i' \subset X_\alpha$ such that $\bar{C}_i \subset C_i'$ and a smooth coordinate map $\varphi_i : C_i' \to \mathbb{R}^n$ such that $\varphi_i\left(\bar{C}_i\right) = \bar{B}_{r_i}(0) \cap \mathbb{H}^n, \varphi_i(B_i') = B_{r_i'}(0) \cap \mathbb{H}^n$ for some $r_i < r_i'$.

Define $g_i = H_i \circ \varphi_i$ on C_i', and zero else-where, where $H_i : \mathbb{R}^n \to \mathbb{R}$ is a smooth function that is positive in $B_{r_i}(0)$ and zero else-where, as in lemma 2.22.

So g_i is a smooth function such that $\text{supp} g_i = \bar{C}_i$.

And the remaining part is totally the same to the proof for smooth manifold without boundary.

Lemma 2.26. *Suppose M is a smooth manifold with or without boundary, $A \subset M$ is a closed subset, and $f : A \to \mathbb{R}^k$ is a smooth function. For any open subset U containing A, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}\Big|_A = f$ and* $\text{supp}\tilde{f} \subset U$*.*

Exer 2.27. *Give a counterexample to show that the conclusion of the extension lemma can be false if A is not closed.*

Let $M = \mathbb{R}, A = U = (0,1)$ and $f = 1$.
Suppose such $\tilde{f}$ exists, then supp$\tilde{f}$ is a closed set contained in $U = A$.
This is impossible since $A \subset$ supp$\tilde{f}$.

2.2 Problems

Prob 2.1. *Define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1 & x \geqslant 0, \\ 0 & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi(U \cap f^{-1}(V))$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

For $x < 0$, consider a sufficiently small neighborhood U such that $U \subset (-\infty, 0)$, then $\psi \circ f \circ \varphi^{-1} = f\big|_U \equiv 0$ is a constant function, which is definitely smooth.
Similarly, for $x > 0$, consider U such that $U \subset (0, \infty)$, then $\psi \circ f \circ \varphi^{-1} \equiv 1$ is smooth.
For $x = 0$, let $U = (-\varepsilon, \varepsilon), V = (1 - \varepsilon, 1 + \varepsilon)$ where $\varepsilon < 1$
Then $U \cap f^{-1}(V) = [0, \varepsilon)$, *i.e.*$\psi \circ f \circ \varphi^{-1} \equiv 1$ is smooth.
But for chart (V, ψ) such that $f(U) \subset V$, V must contains $0, 1$ since $U \cap (-\infty, 0) \neq \varnothing, 0 \in U$.
Then $\psi \circ f \circ \varphi^{-1} = f\big|_U$ is not smooth since it has no derivative at 0.

Prop 2.12. *Suppose $M_1, \cdots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \cdots, M_k$ has nonempty boundary. For each i, let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ denote the projection onto the M_i factor. A map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F : N \to M_i$ is smooth.*

Prob 2.2. *Prove Proposition* 2.12*(smoothness of maps into product manifolds).*

If F is smooth, then $F_i = \pi_i \circ F$ is smooth since π_i is smooth by Example 2.13.
If each of F_i is smooth, then for each point $p \in N$, there exist smooth charts (U_i, φ_i) containing p and (V_i, ψ_i) containing $F_i(p)$, such that $F_i(U_i) \subset V_i$ and $\psi_i \circ F_i \circ \varphi_i^{-1}$ is smooth.
So let $U = \bigcap\limits_{i=1}^{k} U_i, V = \prod\limits_{i=1}^{k} V_i, \psi = \prod\limits_{i=1}^{k} \psi_i$, we have $F(U) \subset V$, and for each i, $\pi_i \circ \psi \circ F \circ \varphi_1^{-1} = \psi_i \circ F_i \circ \varphi_i^{-1} \circ (\varphi_i \circ \varphi_1^{-1})$ is smooth, *i.e.*$\psi \circ F \circ \varphi_1^{-1}$ is smooth.
Hence F is smooth.

Prob 2.3. *For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.*

*(a) $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ is the n-**th power map** for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.*

*(b) $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is the **antipodal map** $\alpha(x) = -x$.*

(c) $F : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, where we think of $\mathbb{S}^3$ as the subset $\{(w, z) : |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$.

(a) Notice that p_n is continuous, it remains to prove that stereographic coordinates satisfies the condition of proposition $2.5(b)$.

For convenience, we let $N = 1, S = -1$.

For $U = \mathbb{S}^1 \backslash \{N\}, V = \mathbb{S}^1 \backslash \{N\}$, $(\sigma \circ p_n \circ \sigma^{-1})(x) = \tan\big(n \arctan x - \frac{n-1}{2}\pi\big)$ is smooth.

For $U = \mathbb{S}^1 \backslash \{N\}, V = \mathbb{S}^1 \backslash \{S\}$, $(\tilde{\sigma} \circ p_n \circ \sigma^{-1})(x) = \tan\big(\frac{n}{2}\pi - n \arctan x\big)$ is smooth.

For $U = \mathbb{S}^1\backslash\{S\}, V = \mathbb{S}^1\backslash\{N\}$, $(\sigma \circ p_n \circ \tilde{\sigma}^{-1})(x) = \tan\left(\frac{\pi}{2} - n \arctan x\right)$ is smooth.

For $U = \mathbb{S}^1\backslash\{S\}, V = \mathbb{S}^1\backslash\{S\}$, $(\tilde{\sigma} \circ p_n \circ \tilde{\sigma}^{-1})(x) = \tan(n \arctan x)$ is smooth.

Hence by proposition 2.5(b), p_n is smooth.

(b) For point $p \neq N$, smooth chart $(\mathbb{S}^n\backslash\{N\}, \sigma)$ contains p and $(\mathbb{S}^n\backslash\{S\}, \tilde{\sigma})$ contains $\alpha(p) = -p$, and we have $\alpha(\mathbb{S}^n\backslash\{N\}) = \mathbb{S}^n\backslash\{S\}, \tilde{\sigma} \circ \alpha \circ \sigma^{-1}(u) = -u$ is smooth.

And for $p = N$, smooth chart $(\mathbb{S}^n\backslash\{S\}, \tilde{\sigma})$ contains p and $(\mathbb{S}^n\backslash\{N\}, \sigma)$ contains $\alpha(p) = S$, and $\alpha(\mathbb{S}^n\backslash\{S\}) = \mathbb{S}^n\backslash\{N\}, \sigma \circ \alpha \circ \tilde{\sigma}^{-1}(u) = -u$ is smooth.

Hence α is smooth.

(c) Notice that $F(x, y, z, w) = (2xz + 2yw, 2yz - 2xw, x^2 + y^2 - z^2 - w^2)$ is continuous.

So similar to (a), we check that the stereographic coordinates satisfies the condition of proposition 2.5(b).

For $U = \mathbb{S}^3\backslash\{N\}, V = \mathbb{S}^2\backslash\{N\}, \sigma \circ F \circ \sigma^{-1}(x) = \frac{(4x^1x^3 + 2x^2(|x|^2-1), 4x^2x^3 - 2x_1(|x|^2-1))}{(|x|^2-1)^2 + 4(x^3)^2}$ is smooth.

For $U = \mathbb{S}^3\backslash\{N\}, V = \mathbb{S}^2\backslash\{S\}$, $\tilde{\sigma} \circ F \circ \sigma^{-1}(x) = \frac{(2x^1x^3 + x^2(|x|^2-1), 2x^2x^3 - x^1(|x|^2-1))}{2(x^1)^2 + 2(x^2)^2}$ is smooth.

For $U = \mathbb{S}^3\backslash\{S\}, V = \mathbb{S}^2\backslash\{N\}, \sigma \circ F \circ \tilde{\sigma}^{-1}(x) = \frac{(4x^1x^3 - 2x^2(|x|^2-1), 4x^2x^3 + 2x^1(|x|^2-1))}{(|x|^2-1)^2 + 4(x^3)^2}$ is smooth.

For $U = \mathbb{S}^3\backslash\{S\}, V = \mathbb{S}^2\backslash\{S\}$, $\tilde{\sigma} \circ F \circ \tilde{\sigma}^{-1}(x) = \frac{(2x^1x^3 - x^2(|x|^2-1), 2x^2x^3 + x^1(|x|^2-1))}{2(x^1)^2 + 2(x^2)^2}$ is smooth.

Hence F is smooth.

Prob 2.4. *Show that the inclusion map $\bar{\mathbb{B}}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\bar{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary.*

For $p \in \mathbb{B}^n$, the smooth chart(U, Id) contains p, and $i \circ \mathrm{Id} = \mathrm{Id}\big|_{\mathbb{B}^n}$ is smooth.
For $p \in \partial\mathbb{B}^n$, there exist a smooth chart (U_i^+, φ_i^+) containing p, WLOG.
So $i \circ \varphi^{-1}(x) = \left(x^1, \cdots, x^i + \sqrt{1 - (x^1)^2 - \cdots - \widehat{(x^i)^2} - \cdots - (x^n)^2}, \cdots, x^n\right)$ is smooth.
Hence the inclusion map i is smooth.

Prob 2.5. *Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\tilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.*

(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$.

(b) Show that f is smooth as a map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

(a) Since $(\psi \circ f \circ \mathrm{Id}^{-1})(x) = f(x)^3$ is smooth.

So f is smooth as a map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$.

(b) f is smooth as a map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$ is equivalent to say that $(f \circ \psi^{-1})(x) = f\left(x^{\frac{1}{3}}\right)$ is smooth.

Let $g(x) = f\left(x^{\frac{1}{3}}\right)$.

Notice that $f^{(n)}(x) = \sum_{k_1 + \cdots + nk_n = n} \frac{n!}{k_1! \cdots k_n!} g^{(k_1 + \cdots + k_n)}\left(x^3\right) \prod_{i=1}^{n} \left(\frac{\left(x^3\right)^{(i)}}{i!}\right)^{k_i}$.

So if g is smooth, $f^{(n)}(0)$ can only be nonzero when $n = 3k_3$.

And if $f^{(n)}(0) = 0$ for all $3 \nmid n$, by Taylor's theorem, we have $f(x) = f(0) + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(3n)}(0)}{(3n)!}x^{3n} + x^{3n+3}R_n(x)$ for some smooth R_n.

Therefore $g(x) = g(0) + \frac{f^{(3)}(0)}{3!}x + \cdots + \frac{f^{(3n)}(0)}{(3n)!}x^n + x^{n+1}R_n\left(x^{\frac{1}{3}}\right)$, then it remains to prove that $R_n \in C^n(\mathbb{R})$.

By induction, $\left(x^{n+1}R_n\left(x^{\frac{1}{3}}\right)\right)' = x^n\left((n+1)R_n\left(x^{\frac{1}{3}}\right) + \frac{x^{\frac{1}{3}}}{3}R_n'\left(x^{\frac{1}{3}}\right)\right) \in C^{n-1}(\mathbb{R})$.

Hence g is smooth.

Prob 2.6. *Let $P : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}^{k+1}\backslash\{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}, P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R}\backslash\{0\}$ and $x \in \mathbb{R}^{n+1}\backslash\{0\}$. (Such a function is said to be homogeneous of degree d.) Show that the map $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well defined and smooth.*

If $[x] = [y]$, then $x = \lambda y, \tilde{P}([x]) = [P(x)] = [\lambda^d P(y)] = [P(y)] = \tilde{P}([y])$.
So $\tilde{P}$ is well-defined.
And for any two chart $(U_i, \varphi_i), (V_j, \varphi_j)$ for $\mathbb{RP}^n$ and $\mathbb{RP}^k$ resp., we have $\varphi_i \circ \tilde{P} \circ \varphi_i^{-1}(u) = \left(\frac{P_1(x)}{P_j(x)}, \cdots, \frac{P_{j-1}(x)}{P_j(x)}, \frac{P_{j+1}(x)}{P_j(x)}, \cdots, \frac{P_{k+1}(x)}{P_j(x)}\right)$, where $x = (u^1, \cdots, u^{i-1}, 1, u^i, \cdots, u^n)$
Since P is smooth and $P_j \neq 0$.
Hence $\tilde{P}$ is smooth.

Prob 2.7. *Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geqslant 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.*

We first prove that if $f_1, \cdots, f_k \in C^\infty(M)$ have nonempty disjoint supports, then they are linearly independent.
Suppose they are linearly dependent, $c_1f_1 + \cdots + c_kf_k \equiv 0$.
For every point $p \in \text{supp}f_1$, p is not contained in any other $\text{supp}f_i$ since they are disjoint.
So $c_1f_1(p) = 0$, *i.e.*$f_1 \equiv 0$, contradiction!
consider an arbitrary chart (U, φ), let $B_1, B_2, \cdots, B_k$ be k disjoint balls in $\hat{U}$.
For each B_i, let $f_i : M \to \mathbb{R}$ maps U to $H_i \circ \varphi$ and be zero else-where, where H_i is a smooth function such that $\text{supp}(H_i) \subset B_i$ by lemma 2.22.
Hence $C^\infty(M)$ is infinite-dimensional since k can be arbitrarily big.

Prob 2.8. *Define $F : \mathbb{R}^n \to \mathbb{RP}^n$ by $F(x^1, \cdots, x^n) = [x^1, \cdots, x^n, 1]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. Do the same for $G : \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G(z^1, \cdots, z^n) = [z^1, \cdots, z^n, 1]$. (see Problem 1.9).*

Since $F = \varphi_{n+1}^{-1}$ is the coordinate map.
So it is a diffeomorphism from $\mathbb{R}^n$ to U_{n+1}.
And every open set in $\mathbb{RP}^n$ intersect with U_{n+1}, *i.e.*U_{n+1} is dense.
Similarly, for chart (U_{n+1}, φ_{n+1}) in $\mathbb{CP}^n$, $G = \varphi_{n+1}^{-1}$ is also the coordinate map.
So it is also a diffeomorphism from $\mathbb{C}^n$ to U_{n+1}, which is dense in $\mathbb{CP}^n$.

Prob 2.9. *Given a polynomial p in one variable with complex coefficients, not identically zero, show that there is a unique smooth map $\tilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1-dimensional complex projective space and $G : \mathbb{C} \to \mathbb{CP}^1$ is the map of Problem 2.8:*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{CP}^1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ \mathbb{C} & \xrightarrow[G]{} & \mathbb{CP}^1 \end{array}$$

Since $\mathbb{CP}^1 \backslash U_2 = \{[1,0]\}$ is a one point set.
So we define $\tilde{p}([1,0]) = [1,0]$, and $\tilde{p}(x) = G(p(G^{-1}(x)))$ for $x \in U_2$.
For point $x \in U_2$, $\tilde{p}(U_2) \subset U_2$ and $\varphi_2 \circ \tilde{p} \circ \varphi_2^{-1}(z) = p(z)$ is smooth.
There is only finite many point in $\mathbb{CP}^1$ that is mapped to $[0,1]$ by $\tilde{p}$ since p is a polynomial.
Therefore there exist a small neighborhood U of $[1,0]$ contained in U_1 such that $\tilde{p}(U) \subset U_1$.
We then have $\varphi_1 \circ \tilde{p} \circ \varphi_1^{-1}(x) = \begin{cases} \frac{1}{P(x^{-1})} & x \in U_1, \\ 0 & x = 0. \end{cases}$, this is smooth because $\lim_{x \to 0} \frac{1}{P(x^{-1})} = 0$.
Hence $\tilde{p}$ is smooth and unique.

Prob 2.10. *For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. Given a continuous map $F : M \to N$, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$.*

(a) Show that F^* is a linear map.

(b) Suppose M and N are smooth manifolds. Show that $F : M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.

(c) Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Show that it is diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

Remark 2.2. This result shows that in a certain sense, the entire smooth structure of M is encoded in the subset $C^\infty(M) \subset C(M)$. In fact, some authors define a smooth structure on a topological manifold M to be a subalgebra of $C(M)$ with certain properties; see, e.g., Nestruev's *Smooth Manifolds and Observables.*

(a) $F^*(f+g) = (f+g)\circ F = f\circ F + g\circ F = F^*(f) + F^*(g), F^*(cf) = (cf)\circ F = c(f\circ F) = cF^*(f)$.
Hence F^* is a linear map.

(b) If F is smooth, then $F^*(f) = f \circ F \in C^\infty(M)$ for any $f \in C^\infty(N)$.
If $F^*(C^\infty(N)) \subset C^\infty(M)$, then for any charts $(U, \varphi), (V, \psi)$ for M and N resp., consider an arbitrary smooth function $f : N \to \mathbb{R}$.
We have $f \circ \psi^{-1}$ is smooth from $\tilde{V}$ to $\mathbb{R}$, and $f \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\mathbb{R}$.
let $f = \pi_i \circ \psi$, where $\pi_i(x) = x^i$ is smooth.
So $\pi_i \circ (\psi \circ F \circ \varphi)$, which equals the i-th component of image of $\psi \circ F \circ \varphi$, is smooth
Hence $\psi \circ F \circ \varphi$ is smooth, $i.e. F$ is smooth.

(c) If F is diffeomorphism, then $f = g \Leftrightarrow f \circ F \circ F^{-1} = g \circ F \circ F^{-1} \Leftrightarrow f \circ F = g \circ F$.
And by (b), we can obtain that F^* is restricts to an isomorphism.
If F^* is restricts to an isomorphism, then by (b), F and F^{-1} are both smooth, $i.e. F$ is a diffeomorphism.

Prob 2.11. *Suppose V is a real vector space of dimension $n \geqslant 1$. Define the* ***projectivization of*** *V, denoted by $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi : V \backslash \{0\} \to \mathbb{P}(V)$ that sends x to its span. (Thus $\mathbb{P}(\mathbb{R}^n) = \mathbb{RP}^{n-1}$.) Show that $\mathbb{P}(V)$ is a topological $(n-1)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \cdots, E_n)$ for V, the map $E : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E[v^1, \cdots, v^n] = [v^i E_i]$(where brackets denote equivalence classes) is a diffeomorphism.*

For an arbitrary basis $(B_1, \cdots, B_n)$, we can define a homeomorphism from $\mathbb{R}^n$ to V, by mapping $(x^1, \cdots, x^n)$ to $x_1B_1 + \cdots + x_nB_n$.

So there is a quotient map $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{P}(V)$ induced by π.

By the universal property of quotient map $q : \mathbb{R}^n\backslash\{0\} \to \mathbb{RP}^{n-1}$, we have a continuous map $\tilde{f} : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ such that $\tilde{f}([x]) = \tilde{f}([y]) \Leftrightarrow \pi^{-1}\left(\tilde{f}([x])\right) = \pi^{-1}\left(\tilde{f}([y])\right) \Leftrightarrow \operatorname{span} x = \operatorname{span} y \Leftrightarrow [x] = [y]$, *i.e.*$\tilde{f}$ is a homeomorphism.

Therefore $\mathbb{P}(V)$ is homeomorphic to $\mathbb{RP}^{n-1}$, which is a topological $(n-1)$-manifold.

Define the smooth structure the same as $\mathbb{RP}^{n-1}$, such that $\left(\tilde{f}(U_i), \varphi_i \circ \tilde{f}^{-1}\right)$ are the charts.

For any two charts, the transition map is $\varphi_j \circ \tilde{f}^{-1} \circ \tilde{f} \circ \varphi_i^{-1} = \varphi_j \circ \varphi_i^{-1}$, which means that they are compatible.

And for any other basis $(E_1, \cdots, E_n)$, we first prove that $\tilde{f}^{-1} \circ E$ and $E^{-1} \circ \tilde{f}$ are smooth.

This is because $\tilde{f}^{-1} \circ E[v^1, \cdots, v^n] = [x^1, \cdots, x^n]$, where $x_1B_1 + \cdots x_nB_n = v_1E_1 + \cdots v_nE_n$, is an invertible linear map.

So $\varphi_j \circ \tilde{f}^{-1} \circ E \circ \varphi_i$ and $\varphi_i \circ E^{-1} \circ \tilde{f} \circ \varphi_j^{-1}$, which are the coordinate representation of E and R^{-1} resp., are smooth.

Hence E is a diffeomorphism since it is homeomorphism.

Prob 2.12. *State and prove an analogue of Problem* 2.11 *for complex vector spaces.*

Statement: Suppose V is a complex vector space of dimension $n \geqslant 1$. Define the **projectivization of** V, denoted by $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi : V\backslash\{0\} \to \mathbb{P}(V)$ that sends x to its span. (Thus $\mathbb{P}(\mathbb{C}^n) = \mathbb{CP}^{n-1}$.) Show that $\mathbb{P}(V)$ is a topological $(2n-2)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \cdots, E_n)$ for V, the map $E : \mathbb{CP}^{n-1} \to \mathbb{P}(V)$ defined by $E[v^1, \cdots, v^n] = [v^iE_i]$(where brackets denote equivalence classes) is a diffeomorphism.

The proof is totally the same as the real case and I don't want to repeat it again.

Prob 2.13. *Suppose M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a partition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.*

Consider a partition of unity (ψ_α) of open cover $\mathcal{X} = (X_\alpha)$, let $U_\alpha = \psi_\alpha^{-1}((0, +\infty))$.

Then each $U_\alpha \subset \operatorname{supp}\psi_\alpha \subset X_\alpha$ and (U_α) is locally finite since $(\operatorname{supp}\psi_\alpha)$ is locally finite.

Moreover, since $\sum \psi_\alpha(x) = 1$, we conclude that (U_α) cover M.

Hence (U_α) is a open, locally finite refinement of $\mathcal{X}$, *i.e.*M is paracompact.

Prob 2.14. *Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leqslant f(x) \leqslant 1$ for all $x \in M$, $f^{-1}(0) = A$ and $f^{-1}(1) = B$.*

By theorem 2.29, there exist function g, h such that $g^{-1}(0) = A, h^{-1}(0) = B$, and $g, h \geqslant 0$.

Let $f = \frac{g}{g+h}$, then $0 \leqslant f(x) \leqslant 1$ and $f^{-1}(0) = A, f^{-1}(1) = B$.

Chapter 3

Tangent Vectors

3.1 Exercises

Lemma 3.4 (Properties of Tangent Vectors on Manifolds)**.** *Suppose M is a smooth manifold with or without boundary, $p \in M, v \in T_pM$, and $f, g \in C^\infty(M)$.*

(a) If f is a constant function, then $vf = 0$.

(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Exer 3.5. *Prove Lemma* 3.4*.*

1. WLOG, assume $f \equiv 1$.
   Then $v(f) = v(f \cdot f) = f(p)v(f) + f(p)v(f) = 2v(f)$, *i.e.*$vf = 0$.
2. $v(fg) = f(p)v(g) + g(p)v(f) = 0$.

Prop 3.6 (Properties of Differentials)**.** *Let M, N and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.*

(a) $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ *is linear.*

(b) $\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p : T_pM \to T_{G\circ F(p)}P$.

(c) $\mathrm{d}(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.

(d) If F is a diffeomorphism, then $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(\mathrm{d}F_p)^{-1} = \mathrm{d}\left(F^{-1}\right)_{F(p)}$.

Exer 3.7. *Prove Proposition* 3.6*.*

(a) $\mathrm{d}F_p(v + w)(f) = (v + w)(f \circ F) = v(f \circ F) + w(f \circ F) = \mathrm{d}F_p(v)(f) + \mathrm{d}F_p(w)(f)$.
$\mathrm{d}F_p(cv)(f) = (cv)(f \circ F) = cv(f \circ F) = c\mathrm{d}F_p(v)(f)$.
Hence $\mathrm{d}F_p$ is linear.

(b) $\mathrm{d}(G \circ F)_p(v)(f) = v(f \circ G \circ F) = \mathrm{d}F_p(v)(f \circ G) = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p(v)(f)$.

(c) $\mathrm{d}(\mathrm{Id}_M)_p(v)(f) = v(f \circ \mathrm{Id}) = v(f)$, *i.e.* $\mathrm{d}(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$.

(d) Since $F \circ F^{-1} = \mathrm{Id}$.
So by $(b)(c)$, $\mathrm{d}\left(F^{-1}\right)_{F(p)} \circ \mathrm{d}F_p = \mathrm{Id}_{T_pM}$
Hence $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(\mathrm{d}F_p)^{-1} = \mathrm{d}\left(F^{-1}\right)_{F(p)}$.

Exer 3.17. *Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where $\tilde{x} = x, \tilde{y} = y + x^3$. Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that $\frac{\partial}{\partial x}\big|_p \neq \frac{\partial}{\partial \tilde{x}}\big|_p$, even though the coordinate functions x and $\tilde{x}$ are identically equal.*

$$\frac{\partial}{\partial x}\Big|_p = \frac{\partial \tilde{x}}{\partial x}(\hat{p})\,\frac{\partial}{\partial \tilde{x}}\Big|_p + \frac{\partial \tilde{y}}{\partial x}(\hat{p})\,\frac{\partial}{\partial \tilde{y}}\Big|_p = \frac{\partial}{\partial \tilde{x}}\Big|_p + 3\,\frac{\partial}{\partial \tilde{y}}\Big|_p \neq \frac{\partial}{\partial \tilde{x}}\Big|_p$$

Exer 3.19. *Suppose M is a smooth manifold with boundary. Show that TM has a natural topology and smooth structure making it into a smooth manifold with boundary, such that if $(U, (x^i))$ is any smooth boundary chart for M, then rearranging the coordinates in the natural chart $(\pi^{-1}(U), (x^i, v^i))$ for TM yields a boundary chart $(\pi^{-1}, (v^i, x^i))$.*

For boundary chart (U, φ), $\pi^{-1}(U) \subset TM$ is the set of all tangent vectors to M at all points of U.

Let $(x^1, \cdots, x^n)$ denote the coordinate functions of φ, and define $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by $\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\big|_p\right) = (v^1, \cdots, v^n, x^1(p), \cdots, x^n(p))$.

Its image set is $\mathbb{R}^n \times \varphi(U)$, which is an open subset of $\mathbb{H}^{2n}$. It is a bijection onto its image, because its inverse can be written explicitly as $\tilde{\varphi}^{-1}(v^1, \cdots, v^n, x^1, \cdots, x^n) = v^i \frac{\partial}{\partial x^i}\big|_{\varphi^{-1}(x)}$.

The remaining part it totally the same as proposition 3.18.

Exer 3.27. *Show that any (covariant or contravariant) functor from C to D takes isomorphisms in C to isomorphisms in D.*

Let $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ be a isomorphism, $g \in \mathrm{Hom}_{\mathsf{C}}(Y, X)$ such that $f \circ g = \mathrm{Id}_Y, g \circ f = \mathrm{Id}_X$.
Then $\mathcal{F}(f) \circ \mathcal{F}(g) = \mathcal{F}(f \circ g) = \mathrm{Id}_{\mathcal{Y}} = \mathrm{Id}_{\mathcal{F}(Y)}$, $\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f) = \mathrm{Id}_{\mathcal{X}} = \mathrm{Id}_{\mathcal{F}(X)}$.
Hence covariant functor takes isomorphisms to isomorphisms, so is contravariant functor.

3.2 Problems

Prob 3.1. *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.*

If $\mathrm{d}F_p$ is zero for each $p \in M$, then for smooth charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$, we have $\mathrm{d}F_p\left(\frac{\partial}{\partial x^i}\big|_p\right) = \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\,\frac{\partial}{\partial y^j}\Big|_{F(p)}$.

So $\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) = 0$ for each i, j and $p \in U$, *i.e.*$\hat{F}$ is constant on each component of $\hat{U}$.

Hence F is constant on each component of M.

If F is constant on each component of M, Then for each smooth charts (U, φ) for M and (V, ψ) for N, we have $\frac{\partial \hat{F}^j}{\partial x^i} \equiv 0$.

So $\mathrm{d}F_p\left(\frac{\partial}{\partial x^i}\big|_p\right) = 0$ for each $p \in U \cap F^{-1}(V)$.

Hence $\mathrm{d}F_p \equiv 0$ for each $p \in M$.

Prop 3.14 (The Tangent Space to a Product Manifold)**.** *Let $M_1, \cdots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \cdots, p_k) \in M_1 \times \cdots \times M_k$, the map $\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$ defined by $\alpha(v) = (d(\pi_1)_p(v), \cdots, d(\pi_k)_p(v))$ is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.*

Prob 3.2. *Prove Proposition* 3.14 *(the tangent space to a product manifold).*

For smooth charts (U_i, φ_i) containing p_i, $\alpha\left(\frac{\partial}{\partial x^{i,j}}\Big|_p\right) = \left(\mathrm{d}(\pi_1)_p\left(\frac{\partial}{\partial x^{i,j}}\Big|_p\right), \cdots, \mathrm{d}(\pi_k)_p\left(\frac{\partial}{\partial x^{i,j}}\Big|_p\right)\right)$.
Since $\mathrm{d}(\pi_{i'})_p\left(\frac{\partial}{\partial x^{i,j}}\Big|_p\right) = \frac{\partial \widehat{\pi_{i'}}^{j'}}{\partial x^{i,j}} \frac{\partial}{\partial x^{i',j'}}\Big|_{\pi_{i'}(p)} = \delta_i^{i'} \frac{\partial}{\partial x^{i,j}}\Big|_p$, where δ is the Kronecker symbol.
So $\alpha\left(\frac{\partial}{\partial x^{i,j}}\Big|_p\right) = \left(0, \cdots, \frac{\partial}{\partial x^{i,j}}\Big|_p, \cdots, 0\right)$ are linear independent.
And by proposition 3.15, domain and codomain of α has the same dimensions.
Hence α is an isomorphism.

Prob 3.3. *Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.*

By proposition 3.14, we can define $F : T(M \times N) \to TM \times TN, u_p \oplus v_q \mapsto (u_p, v_q)$.
Let $(U, \varphi), (V, \psi)$ be smooth charts for M, N resp.
Then $\left(\pi_M^{-1}(U), \tilde{\varphi}\right), \left(\pi_N^{-1}(V), \tilde{\psi}\right), \left(\pi_{M\times N}^{-1}(U \times V), \widetilde{\varphi \times \psi}\right)$ are the smooth charts for TM, TN, $T(M \times N)$ resp.
So $\left(\tilde{\varphi} \times \tilde{\psi}\right) \circ F \circ \widetilde{\varphi \times \psi}^{-1}(x, y, u, v) = \left(\tilde{\varphi} \times \tilde{\psi}\right)\left(u^i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)}, v^i \frac{\partial}{\partial y^i}\Big|_{\psi^{-1}(y)}\right) = (x, u, y, v)$.
Hence F and F^{-1} are smooth, *i.e.*F is a diffeomorphism.

Prob 3.4. *Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.*

Define $F : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}, v\frac{\partial}{\partial x_\varphi}\Big|_p \mapsto (p, v)$, where $\varphi : \mathbb{S}^1 \backslash \{-p\} \to \mathbb{R}, e^{i\theta} \mapsto \theta$.
Consider smooth charts $\left(\pi^{-1}(\mathbb{S}^1 \backslash \{p\}), \tilde{\varphi}\right), ((\mathbb{S}^1 \backslash \{p\}) \times \mathbb{R}, \varphi \times \mathrm{Id})$ for $T\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{R}$ resp.
Notice that for $q \in \mathbb{S}^1 \backslash \{-p\}$, $\frac{\partial}{\partial x_\varphi}\Big|_q = \frac{\partial}{\partial x_\psi}\Big|_q$ for $\psi : \mathbb{S}^1 \backslash \{-q\} \to \mathbb{R}, e^{i\theta} \to \theta$.
We have $(\varphi \times \mathrm{Id}) \circ F \circ \tilde{\varphi}^{-1}(\theta, v) = (\theta, v)$.
Hence F and F^{-1} are smooth, *i.e.*F is a diffeomorphism.

Prob 3.5. *Let $\mathbb{S}^1 \subset \mathbb{R}^2$ be the unit circle, and let $K \subset \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) : \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no diffeomorphism with the same property.*

Consider an arc γ of $\mathbb{S}^1$, such that $F \circ \gamma(0) = (1, 1), F \circ \gamma(t) = (1, y(t))$ for $t > 0$ and $F \circ \gamma(t) = (x(t), 1)$ for $t < 0$.
Then $(F \circ \gamma)' = \begin{cases} x'(t) \frac{\partial}{\partial x}\Big|_{(x(t),1)} & t < 0, \\ y'(t) \frac{\partial}{\partial y}\Big|_{(1,y(t))} & t > 0. \end{cases}$
So $\mathrm{d}F(\gamma'(0)) = (F \circ \gamma)'(0) = 0$, but $\gamma'(0) \neq 0$, contradiction!

Prob 3.6. *Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z : \mathbb{R} \to \mathbb{S}^3$ by $\gamma_z(t) = (e^{it}z^1, e^{it}z^2)$. Show that γ_z is a smooth curve whose velocity is never zero.*

For smooth chart (U_1^+, φ_1^+), $\varphi_1^+ \circ \gamma_z(t) = \left(a^1 \sin t + b^1 \cos t, a^2 \cos t - b^2 \sin t, a^2 \sin t + b^2 \cos t\right)$ is smooth and the same for other charts.
So γ_z is a smooth curve.
And $\gamma_z'(t_0) = \mathrm{d}\gamma_z\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0}\right) = \frac{\mathrm{d}\left(\varphi_1^+ \circ \gamma_z\right)^j}{\mathrm{d}t} \frac{\partial}{\partial x^j}\Big|_{\gamma_z(t)} = (a^1 \cos t - b^1 \sin t) \frac{\partial}{\partial x^1}\Big|_{\gamma_z(t)} + (-a^2 \sin t - b^2 \cos t) \frac{\partial}{\partial x^2}\Big|_{\gamma_z(t)} + (a^2 \cos t - b^2 \sin t) \frac{\partial}{\partial x^3}\Big|_{\gamma_z(t)}$ is nonzero since $a^1 \cos t - b^1 \sin t > 0$.
Hence γ_z is a smooth curve whose velocity is never zero.

Prob 3.7. *Let M be a smooth manifold with or without boundary and p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\Phi : \mathcal{D}_pM \to T_pM$ by $(\Phi v)f = v([f]_p)$. Show that Φ is an isomorphism.*

It is easy to check that Φ is well-defined and linear.
And for a smooth chart (U,φ), (x^i,U) are not equivalent to each other in $C_p^\infty(M)$.
So $\dim \mathcal{D}_pM \geqslant \dim T_pM$, *i.e.*it is sufficient to prove that Φ is injective.

For a closed ball $\bar{B}$ centered at p contained in U, there exist $\tilde{f}:M\to\mathbb{R}$ such that $\tilde{f}\Big|_{\bar{B}} = f\big|_{\bar{B}}$ and supp$\tilde{f}\subset U$ by lemma 2.26.

Therefore if $\Phi v\equiv 0$, then $v([f]_p) = v\left(\left[\tilde{f}\right]_p\right) = \Phi v\left(\tilde{f}\right) = 0, i.e. v\equiv 0$.

Hence Φ is a isomorphism.

Prob 3.8. *Let M be a smooth manifold with or without boundary and $p\in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1\sim\gamma_2$ if $(f\circ\gamma_1)'(0) = (f\circ\gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi:\mathcal{V}_pM\to T_pM$ defined by $\Psi[\gamma]=\gamma'(0)$ is well defined and bijective.*

Since $\gamma_1\sim\gamma_2 \Leftrightarrow \mathrm{d}f(\gamma_1'(0)) = \mathrm{d}f(\gamma_2'(0)) \Leftrightarrow \mathrm{d}f(\gamma_1'(0)-\gamma_2'(0)) = 0 \Leftrightarrow \gamma_1'(0)=\gamma_2'(0)$.
We can obtain that Ψ is well-defined and injective.
And for smooth chart (U,φ) and $v = v^i\,\frac{\partial}{\partial x^i}\big|_p$, defined $\gamma(t)=\varphi^{-1}(v^1t,\cdots,v^nt)$
So $\Psi[\gamma]=\gamma'(0)=\mathrm{d}\gamma\left(\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\right) = \frac{\mathrm{d}(\varphi\circ\gamma)^i}{\mathrm{d}t}\,\frac{\partial}{\partial x^i}\big|_p = v^i\,\frac{\partial}{\partial x^i}\big|_p = v$
Hence Ψ is well-defined and bijective.