Algebraic Topology
Lecture Notes

Note taker: Jacky567

Qiuzhen College, Tsinghua University
2025 Spring

Contents

Chapter 1

Singular homology

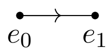
1.1 Singular homology

Def 1.1.1. The standard n-simplex is the convex hull of the standard basis $\{e_0, \cdots, e_n\} \in \mathbb{R}^{n+1}$:

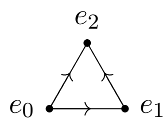
$$\Delta^n = \left\{ \sum_{i=0}^{n} t_i e_i \middle| t_i \geqslant 0, \sum_{i=0}^{n} t_i = 1 \right\}.$$

Exam 1.1.1. *Δ^0 is a point:*

Δ^1 is line:



Δ^2 is triangle:



and Δ^3 is a tetrahedron.

Def 1.1.2. For $0 \leqslant i \leqslant n$, we have the face inclusion map

$$\mathrm{d}^i : \Delta^{n-1} \to \Delta^n, (t_0, \cdots, t_{n-1}) \mapsto (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}).$$

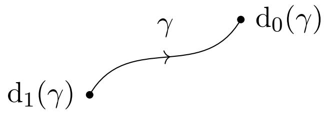
Def 1.1.3. A singular n-simplex in X is a continuous map $\sigma : \Delta^n \to X$ and

$$\sin_n(X) = \{\text{singular } n\text{-simplices in } X\}.$$

For $0 \leqslant i \leqslant n$, we have a map

$$\mathrm{d}_i : \sin_n(X) \to \sin_{n-1}(X), \sigma \mapsto \sigma \circ \mathrm{d}^i.$$

Exam 1.1.2. $\sin_0(X) = \{$*points in* $X\}, \sin_1(X) = \{$*paths in* $X\}$.
The map $\mathrm{d}_0, \mathrm{d}_1 : \sin_1(X) \to \sin_0(X)$ maps γ to its terminal point and start point resp.



Def 1.1.4. $S_n(X) = \mathbb{Z}\sin_n(X)$ is the free abelian group generated by $\sin_n(X)$:

$$S_n(X) = \left\{ \text{formal linear combinations } \sum_{i=1}^{R} a_i\sigma_i \middle| a_i \in \mathbb{Z}, \sigma_i \in \sin_n(X) \right\}$$

An element in $S_n(X)$ is called a singular n-chain in X.

Def 1.1.5. The boundary operator $\mathrm{d} : S_n(X) \to S_{n-1}(X)$ is a group homomorphism with

$$d(\sigma) := \sum_{i=0}^{n} (-1)^i \mathrm{d}_i\sigma.$$

Exam 1.1.3. *When $n = 1$:*

$$d(\sigma) = \sigma\big|_{e_1} - \sigma\big|_{e_0}.$$

When $n = 2$:

$$d(\sigma) = \sigma\big|_{e_1e_2} - \sigma\big|_{e_0e_2} + \sigma\big|_{e_0e_1}$$

Lemma 1.1.1. *The composition $S_n(X) \xrightarrow{\mathrm{d}} S_{n-1}(X) \xrightarrow{\mathrm{d}} S_{n-2}(X)$ is zero.*

Proof. By linearity, it suffices to check $d \circ d(\sigma) = 0$ for any $\sigma \in \sin_n(x)$.

$$\mathrm{d} \circ \mathrm{d}(\sigma) = \sum_{i,j} (-1)^{i+j} \mathrm{d}_i \circ \mathrm{d}_j(\sigma) = \sum_{i,j} (-1)^{i+j} (\sigma \circ \mathrm{d}^i \circ \mathrm{d}^j) = 0.$$

The last equality holds since

$$\mathrm{d}^i \circ \mathrm{d}^j = \begin{cases} \mathrm{d}^{j+1} \circ \mathrm{d}^i & i < j \\ \mathrm{d}^j \circ \mathrm{d}^{i+1} & i \geqslant j \end{cases}.$$

□

Def 1.1.6. For $n \geqslant 0$, we define the n-th singular homology of X:

$$H_n(X) = \frac{\ker(\mathrm{d} : S_n(X) \to S_{n-1}(X))}{\mathrm{im}(\mathrm{d} : S_{n+1}(X) \to S_n(X))}.$$

It is not hard to see that $H_n(X)$ is abelian.

Def 1.1.7. A chain complex is a graded abelian group

$$C = \bigoplus_{n \in \mathbb{Z}} C_n,$$

together with a homomorphism $\mathrm{d} : C \to C$ of degree -1 such that $\mathrm{d}^2 = 0$.
d is called the boundary operator and elements in C_n are called n-chains, define

$$Z_n = \ker\left(C_n \xrightarrow{d} C_{n-1}\right) = \{n\text{-cycles}\} = \{\text{closed } n\text{-chains}\},$$

$$B_n = \mathrm{Im}\left(C_{n+1} \xrightarrow{d} C_n\right) = \{n\text{-boundaries}\} = \{\text{exact } n\text{-chains}\}.$$

Since $\mathrm{d}^2 = 0$, so $B_n \subset Z_n$.
Homology of the chain complex is

$$H_n(C, d) = H_n(C) := \frac{Z_n}{B_n}.$$

Remark 1.1.1.

$$\left(S_*(X) = \bigoplus_{n \geqslant 0} S_n(X), d = \sum_{i=0}^{n} (-1)^i \mathrm{d}_i \right)$$

is a chain complex called the singular chain complex of X and

$$H_n(X) = H_n(S_*(X), d).$$

The singular chain complex is very big, so it is hard for us to compute for us to compute the singular homology now.

Exam 1.1.4. *X is a point, then*

$$\sin_n(X) = \{constant\ \sigma_n : \Delta^n \to X\}, S_n(X) = \mathbb{Z} \cdot \sigma_n.$$

$$\mathrm{d}(\sigma_n) = \sum_{i=0}^{n} (-1)^i \mathrm{d}_i(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} \sigma_{n-1} & 2|n \\ 0 & 2 \nmid n \end{cases}$$

So the chain complex $S_*(X)$ is

$$S_{-1}(X) = 0 \longleftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\mathrm{Id}} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow \cdots$$

Hence the singular homology of a point is

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

More generally, we have

Prop 1.1.1. *For any space X,*

$$H_0(X) \cong \mathbb{Z}\Pi_0(X),$$

where $\Pi_0(X) = \{path\ components\ of\ X\}$.

Proof.

$$Z_0(X) = S_0(X) = \left\{ \sum_{i=1}^{k} a_i x_i \middle| a_i \in \mathbb{Z}, x_i \in X \right\}.$$

$B_0(X)$ is generated by

$$\big\{\sigma(e_1) - \sigma(e_0)\big|\text{path } \sigma : \Delta^1 \to X\big\} = \{x_1 - x_0 | [x_1] = [x_0] \in \Pi_0(X)\}.$$

So 0-th singular homology of X is

$$H_0(X) = \frac{Z_0(X)}{B_0(X)} \cong \mathbb{Z}\Pi_0(X).$$

□

Thm 1.1.1. $H_n(\mathbb{S}^k) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & k = n = 0 \\ \mathbb{Z} & k > 0, n = 0, k \\ 0 & otherwise \end{cases}$

Proof. We need to develop a lot of theories to prove this theorem. □

Thm 1.1.2. *Let X be a path-connected space, then there is a surjective homomorphism $\pi_1(X) \to H_1(X)$ that sends each loop to its homology class.*

Proof. Let $\gamma_1, \gamma_2 : \Delta^1 \to X$ be two loops starting at p that are homotopic relative to boundary and $H : [0,1] \times [0,1] \to X$ is their homotopy, then

$$H(0,s) = \gamma_1(0), H(1,s) = \gamma_1(1), H(t,0) = \gamma_1(t), H(t,1) = \gamma_2(t).$$

So consider the singular 2-simplex

$$\sigma : \Delta^2 \to X, (t_0e_0 + t_1e_1 + t_2e_2) \mapsto H\left(1 - t_0, \frac{t_2}{1 - t_0}\right).$$

Then $\sigma(e_0) = \gamma_1(0)$ is well-defined and

$$\sigma(t_0e_0 + (1 - t_0)e_1) = H(1 - t_0, 0) = \gamma_1(1 - t_0),$$
$$\sigma(t_0e_0 + (1 - t_0)e_2) = H(1 - t_0, 1) = \gamma_2(1 - t_0),$$
$$\sigma(t_1e_1 + (1 - t_1)e_2) = H(1, 1 - t_1) = \gamma_1(1).$$

Therefore the boundary of σ is

$$\mathrm{d}(\sigma) = \mathrm{d}_0\sigma - \mathrm{d}_1\sigma + \mathrm{d}_2\sigma = \sigma_1(1) - \sigma_2 + \sigma_1.$$

And $\sigma_1(1)$ is the boundary of the singular 2-simplex $\sigma_c : \Delta_2 \to X, x \to g(1)$.
So $\sigma_1 - \sigma_2 = \mathrm{d}(\sigma - \sigma_c) \in B_1(X)$
And since $d(\gamma_1) = d(\gamma_2) = p - p = 0$.
Therefore $[\gamma_1] = [\gamma_2] \in Z_1(X)/B_1(X) = H_1(X)$.
Let $f : \pi_1(X) \to H_1(X)$ be the map that sends each loop to its homology class.
We now assume γ_1, γ_2 are two arbitrary loops start at p, consider the map

$$\sigma_1 : \Delta^2 \to X, (t_0e_0 + t_1e_1 + t_2e_2) \mapsto \begin{cases} \gamma_1(1 + t_2 - t_0) & t_0 \geqslant t_2 \\ \gamma_2(t_2 - t_0) & t_2 > t_0 \end{cases}$$

Then σ_1 is continuous since $\gamma_1(1) = \gamma_2(0) = p$ and its boundary is

$$\mathrm{d}(\sigma_1) = \gamma_1 + \gamma_2 - (\gamma_1 \circ \gamma_2).$$

So $f(\gamma_1 \circ \gamma_2) = f(\gamma_1) + f(\gamma_2)$.
And $f(\gamma_1) + f\left(\overline{\gamma_1}\right) = f\left(\gamma_1 \circ \overline{\gamma_1}\right) = 0, i.e. f\left(\overline{\gamma_1}\right) = -f(\gamma_1)$.
Therefore f is a homomorphism.
We now prove that f is surjective.
For any element

$$[c] = \left[\sum_{i=1}^{n} a_i\gamma_i\right] \in H_1(X),$$

WLOG, assume $a_i = 1$ for any i.
If some γ_i is not a loop, then there must be another γ_j whose start point is the terminal point of γ_i since c is closed.
Then we can replace γ_i, γ_j by $\gamma_i \cdot \gamma_j$.
WLOG, we can assume every γ_i is a loop.
Since X is path-connected.
For a fixed point p, we can find a path f_i from p to the start point of γ_i, and then

$$\left[f_i \cdot \gamma_i \cdot \overline{f_i}\right] = [\gamma_i] \in H_1(X).$$

WLOG, we can assume every γ_i start at p.
Hence we obtain

$$\left[\sum_{i=1}^{n} \gamma_i\right] = f(\gamma_1 \cdot \gamma_2 \cdot \ \cdots \ \cdot \gamma_n),$$

i.e. f is surjective. $\square$

More generally, we have

Thm 1.1.3 (Poincarè)**.** $H_1(X) \cong \bigoplus_{[b]\in\Pi_0(X)} \mathrm{Ab}(\pi_1(X,b))$.

Let us first look at an example:

Exam 1.1.5. *When $X = \mathbb{S}^1$, we want to show that $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.*
Let $f : \mathbb{R} \to \mathbb{S}^1$ be the universal cover, for a path $\sigma : \Delta^1 \to \mathbb{S}^1$, pick any lift $\tilde{\sigma} : \Delta^1 \to \mathbb{R}$.
Define winding number $w(\sigma) := \tilde{\sigma}(e_1) - \tilde{\sigma}(e_0) \in \mathbb{R}$.
For an 1-chain $c = \sum a_i\sigma_i$, define

$$w(c) = \sum_{i=1}^{k} a_i\omega(\sigma_i),$$

this gives a morphism $w : S_1(X) \to \mathbb{R}$.
We then have the following facts:

Prop 1.1.2. *(1) $c \in Z_1(X) \Rightarrow w(c) \in \mathbb{Z}$*

(2) $c \in B_1(X) \Rightarrow w(c) = 0$

(3) $w : H_1(X) \to \mathbb{Z}$ is an isomorphism.

Proof. (1) Define

$$g : \mathbb{S}^1 \to [0,1), e^{2\pi i\theta} \mapsto \theta,$$
$$h : S_0(\mathbb{S}^1) \to \mathbb{R}, \sum_{i=1}^{n} a_ic_i \mapsto \sum_{i=1}^{n} a_if(c_i).$$

Then we have

$$h(\mathrm{d}(\sigma)) = g(\sigma(e_1)) - g(\sigma(e_0)) = \tilde{\sigma}(e_1) - \tilde{\sigma}(e_0) + n = w(\sigma) + n$$

for some $n \in \mathbb{Z}$

So $w(c) + n = h(\mathrm{d}(c)) = h(0) = 0$ for some $n \in \mathbb{Z}$, $i.e. w(c) = -n \in \mathbb{Z}$.

(2) For a map $\sigma : \Delta^2 \to \mathbb{S}^1$, let $\gamma_1 = \mathrm{d}_0\sigma, \gamma_2 = \mathrm{d}_1\sigma, \gamma_3 = \mathrm{d}_2\sigma$.
Let $\tilde{\sigma}$ be a lift of σ and $\tilde{\gamma}_1 = \mathrm{d}_0\tilde{\sigma}, \tilde{\gamma}_2 = \mathrm{d}_1\sigma, \tilde{\gamma}_3 = \mathrm{d}_2\sigma$.
Then $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(0), \tilde{\gamma}_2(1) = \tilde{\gamma}_3(0), \tilde{\gamma}_3(1) = \tilde{\gamma}_1(0)$, moreover,

$$\begin{aligned} w(d(\sigma)) =& w(\gamma_1) - w(\gamma_2) + w(\gamma_3) \\ =& \tilde{\gamma}_1(1) - \tilde{\gamma}_1(0) + \tilde{\gamma}_2(1) - \tilde{\gamma}_2(0) + \tilde{\gamma}_3(1) - \tilde{\gamma}_3(0) = 0 \end{aligned}$$

Hence

$$w(c) = 0 \text{ for } c = d\left(\sum_{i=1}^{n} a_i\sigma_i\right) \in B_1(\mathbb{S}^1).$$

(3) Consider the map $g : H_1(\mathbb{S}^1) \to \mathbb{Z}, c \mapsto w(c)$.
By (1) and (2), g is well-defined homomorphism.
And for loop $\gamma : I \to \mathbb{S}^1, \theta \mapsto e^{2n\pi i\theta}$, we have $g(\gamma) = n$.
So g is surjective.
By theorem 1.1.2, there is a surjective homomorphism from $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ to $H_1(\mathbb{S}^1)$.
Hence $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

∎

Proof of theorem 1.1.3. We first assume X is path-connected.

By theorem 1.1.2, let $f : \pi_1(X) \to H_1(X)$ sends each loop to its homology class.

Since $H_1(X)$ is abelian.

So $[\pi_1(X), \pi_1(X)] \subset \ker f$.

It suffices to proof that $\tilde{f} : \mathrm{Ab}(\pi_1(X)) \to H_1(X)$ is injective.

Let $[\gamma] \in \mathrm{Ab}(\pi_1(X))$ is in the kernel of $\tilde{f}$.

Then $\tilde{f}([\gamma])$ is the boundary of a 2-chain $c = \sum a_i\sigma_i$.

WLOG, we assume $a_i = \pm 1$ and let $\mathrm{d}_j\sigma_i = \tau_{ij}$, then

$$\tilde{f}([\gamma]) = \partial c = \sum a_i(\tau_{i0} - \tau_{i1} + \tau_{i2}).$$

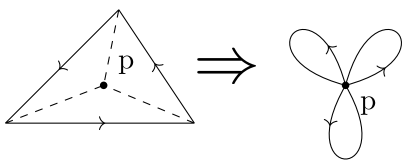
And since $\tilde{f}([\gamma])$ is a singular 1-cycle.

So after identify the canceling pairs and glue together the 2-simplices, we can get a Δ-complex K and a map $\sigma : K \to X$.

Let A be the vertices of K with the segment corresponding to γ.

For a point p in X, we can slide the image of each vertex along a path from its original image to p by a homotopy.

For example, when K is a triangle, the map is:



Then by the homotopy extension property, we can extend the homotopy to all of K.

Therefore we can assume that $\tilde{f}([\gamma])$ is the boundary of c such that τ_{ij} are loops at p.

Moreover, we obtain

$$[\gamma] = \sum a_i([\tau_{i0}] - [\tau_{i1}] + [\tau_{i2}]) = \sum a_i[\tau_{i0} \cdot \bar{\tau}_{i1} \cdot \tau_{i2}] = 0,$$

since the map $\sigma : \Delta^2 \to X$ natural gives a homotopy $\tau_{i0} \cdot \bar{\tau}_{i1} \cdot \tau_{i2} \simeq 0$.

Hence $H_1(X) \cong \mathrm{Ab}(\pi_1(X))$.

For general X, we have

$$S_n(X) \cong \bigoplus_{Y \in \Pi_0(X)} S_n(Y).$$

So

$$H_1(X) \cong \bigoplus_{Y \in \Pi_0(X)} H_1(Y) \cong \bigoplus_{[b] \in \Pi_0(X)} \mathrm{Ab}(\pi_1(X, b)).$$

$\square$

Def 1.1.8. Let $(C_*, d), (D_*, d)$ be two chain complexes, a chain map is a group homomorphism $f : C_* \to D_*$ of degree-0 that commutes with d:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\mathrm{d}"] \arrow[d, "f"] & C_n \arrow[r, "\mathrm{d}"] \arrow[d, "f"] & C_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\mathrm{d}"] & D_n \arrow[r, "\mathrm{d}"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Prop 1.1.3. *A chain map f sends cycles to cycles and boundary to boundary.*

So a chain map f induces a homomorphism $f_* : H_n(C) \to H_n(D)$.

Proof. For $c \in Z_n(C)$, $\mathrm{d}f(c) = f(\mathrm{d}c) = 0$.
For $c = \mathrm{d}c_0 \in B_n(C)$, $f(c) = f(\mathrm{d}c_0) = \mathrm{d}f(c_0)$.
So $f(Z_n(C)) = Z_n(D), f(B_n(C)) = B_n(D)$.
And the homomorphism $f_* : H_n(C) \to H_n(D)$ is well-defined. □

Prop 1.1.4. *Given a continuous $f : X \to Y$, it induces a map*

$$f_* : \sin_n(X) \to \sin_n(Y)$$
$$(\sigma : \Delta^n \to X) \mapsto (f \circ \sigma : \Delta^n \to Y)$$

And $f_* : S_*(X) \to S_*(Y)$ is a chain map.

Proof.

$$\mathrm{d}_i(f \circ \sigma) = f \circ \sigma \circ \mathrm{d}_i = f \circ \mathrm{d}_i(\sigma).$$

So $\mathrm{d}f_* = f_*\mathrm{d}$, *i.e.* f_* is a chain map. □

Exam 1.1.6. $\mathrm{Id}_* = \mathrm{Id}, (f \circ g)_* = f_* \circ g_*$.

1.2 Basis category theory

Def 1.2.1. A category $\mathcal{C}$ consist of:

(1) a class of objects $\mathrm{ob}(\mathcal{C})$

(2) For any $X, Y \in \mathrm{ob}(\mathcal{C})$, a set of morphisms, denoted by $\mathcal{C}(X, Y)$

(3) For any $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, a map $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z), (f, g) \mapsto g \circ f$.

(4) For any $X \in \mathrm{ob}(\mathcal{C})$, an element $1_X \in \mathcal{C}(X, X)$.

And they satisfy:

(1) For any $f \in \mathcal{C}(Y, X)$, $1_X \circ f = f, f \circ 1_Y = f$.

(2) $(h \circ g) \circ f = h \circ (g \circ f)$.

Remark 1.2.1. (1) If $\mathrm{ob}(\mathcal{C})$ is a set, we call $\mathcal{C}$ a small category.

(2) We often write $X \in \mathcal{C}$ instead of $X \in \mathrm{ob}(\mathcal{C})$ and write $f \in \mathcal{C}(X, Y)$ as $f : X \to Y$

(3) Given $f : X \to Y$, we say f is an isomorphism if $\exists g : Y \to X$, *s.t.* $f \circ g = 1_Y, g \circ f = 1_X$.
In this case, we write $f : X \xrightarrow{\cong} Y$, write g as f^{-1}.

Exam 1.2.1. *(1) Given a group G, $\mathcal{C}_G$:* $\mathrm{ob}(\mathcal{C}_G) = \{*\}, \mathcal{C}(*, *) = G, \circ : G \times G \to G, (g, f) \mapsto gf$.

(2) $\mathcal{C} = \mathrm{Set}, \mathrm{ob}(\mathcal{C}) = \{\textit{all sets}\}, \mathcal{C}(X, Y) = \{\textit{all maps } X \to Y\}$.

(3) $\mathcal{C} = \mathrm{Top}, \mathrm{ob}(\mathcal{C}) = \{\textit{all topological spaces}\}, \mathcal{C}(X, Y) = \{\textit{all continuous maps } X \to Y\}$

(4) $\mathcal{C} = \mathrm{Ab}, \mathrm{ob}(\mathcal{C}) = \{\textit{all abelian groups}\}, \mathcal{C}(X, Y) = \{\textit{group homomorphisms } X \to Y\}$

(5) $\mathcal{C} = \mathrm{Gp}, \mathrm{ob}(\mathcal{C}) = \{\textit{all groups}\}, \mathcal{C}(X, Y) = \{\textit{group homomorphisms } X \to Y\}$

(6) $\mathcal{C} = \Delta$ *is the simplicial category,* $\mathrm{ob}(\Delta) = \{\{0\}, \{0, 1\}, \cdots\} = \{[0], [1], \cdots\}$ *and*
$\Delta([m], [n]) = \{\textit{order preserving maps } [m] \xrightarrow{f} [n]\}$

(7) *alternative definition of* Δ*:* $\mathrm{ob}(\Delta') = \{\Delta^n\}_{n\in N}$ *and*
$\Delta'(\Delta^m, \Delta^n) = \{f : \Delta^m \to \Delta^n | f \text{ is linear, } f(e_i) = e_{i'}, i \leqslant j \Rightarrow i' \leqslant j'\}$.

(8) *Given a space* X*, we define fundamental groupoid* $\Pi_1(X)$*:* $\mathrm{ob}(\Pi_1(X)) = \{\text{points in } X\}$ *and* $\mathcal{C}(p, q) = \{\text{paths } I \xrightarrow{\gamma} X \text{ from } p \text{ to } q\}/\{\text{homotopy related to } \partial I\}, [\gamma] \circ [\eta] = [\gamma \cdot \eta]$.

Remark 1.2.2. (1) $\Pi_1(X)(p, p) = \pi_1(X, p)$.

(2) all morphisms in $\mathcal{C}$ are isomorphisms, we call such category a groupoid(a groupoid with single object is group).

Def 1.2.2. Let $\mathcal{C}, \mathcal{D}$ be categories, a (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ consists of:

(1) an assignment $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$

(2) for all $X, Y \in \mathrm{ob}(\mathcal{C})$, a map $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ satisfying that:

  (a) For any $X \in \mathrm{ob}(\mathcal{C})$, $F(1_X) = 1_{F(X)}$

  (b) For any $f \in \mathcal{C}(X, Y), g \in \mathcal{C}(Y, Z)$, $F(g \circ f) = F(g) \circ F(f)$.

Exam 1.2.2. *(1) For* $n \in \mathbb{N}$*, we have a functor* $H_n(-) : \mathrm{Top} \to \mathrm{Ab}, X \mapsto H_n(X)$ *and morphism* $(f : X \to Y) \mapsto (f_* : H_n(X) \to H_n(Y))$.

(2) $\pi_1(-) : \mathrm{Top}_* \to \mathrm{Gp}, (X, b) \mapsto \pi_1(X, b)$.

(3) *Given space* X*, we have a functor* $F : \Pi_1(X) \to \mathrm{Gp}, F(b) = \pi_1(X, b)$*, for* $[\gamma] \in \Pi_1(X)(b, b')$*, set* $F([\gamma]) : \pi_1(X, b) \to \pi_1(X, b'), [\alpha] \mapsto [\gamma \circ \alpha \circ \bar{\gamma}]$.

Def 1.2.3. Given category $\mathcal{C}$, define its opposite category $\mathcal{C}^{op}$ as $\mathrm{ob}(\mathcal{C}^{op}) = \mathrm{ob}(\mathcal{C})$, the morphism $\mathcal{C}^{op}(X, Y) := \mathcal{C}(Y, X), (f^{op} : Y \to X) \leftrightarrow (f : Y \to X), (f^{op} \circ g^{op}) = (g \circ f)^{op}$.

Def 1.2.4. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is defined as a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$. That is, $F : \mathcal{C} \to \mathcal{D}$ consists of:

(1) an assignment $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$

(2) a map $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(Y), F(X))$, $F(1_X) = 1_{F(X)}, F(f \circ g) = F(g) \circ F(f)$.

Exam 1.2.3. *(1) Fix any space* Y*, we can define a contravariant functor* $F : \mathrm{Top} \to \mathrm{Set}$ *with* $F(X) = \mathrm{Map}(X, Y)$ *and*

$$\begin{aligned} F : \mathrm{Top}(X_1, X_2) &\to \mathrm{Set}(F(X_2), F(X_1)) \\ \mathrm{Map}(X_1, X_2) &\to \{\text{all maps } \mathrm{Map}(X_2, Y) \to \mathrm{Map}(X, Y)\} \\ f &\mapsto (g \mapsto g \circ f) \end{aligned}$$

(2) *Given any* $\mathcal{C}$ *and any* $Y \in \mathrm{ob}(\mathcal{C})$*, we can define a contravariant functor* $F : \mathcal{C} \to \mathrm{Set}$ *with* $F(X) = \mathcal{C}(X, Y)$ *and*

$$\begin{aligned} F : \mathcal{C}(X_1, X_2) &\to \mathrm{Set}(\mathcal{C}(X_2, Y), \mathcal{C}(X_1, Y)) \\ f &\mapsto (g \mapsto (g \circ f)) \end{aligned}$$

(3) *We also have a covariant functor* $F' : \mathcal{C} \to \mathrm{Set}$ *with* $F'(X) = \mathcal{C}(Y, X)$.

We call F *and* F' *the functor represented by the object* Y.

Def 1.2.5. A simplicial set is a contravariant functor $K : \Delta \to \mathrm{Set}$.
More concretely, a simplical set consists of:

(1) a set $K([n])$ for all $n \in \mathbb{N}$

(2) a map $K(f) : K([n]) \to K([m])$ for all order preserving $f : [m] \to [n]$, such that $K(f \circ g) = K(g) \circ K(f)$ for any $[m] \xrightarrow{g} [n] \xrightarrow{f} [l]$.

Exam 1.2.4. *Let X be a space, then we have a simplicial set $K = \sin_*(X)$ with*

$$K : [m] \mapsto \sin_n(X) = \mathrm{Map}(\Delta^n, X).$$

And for any $f : [m] \to [n]$, we have $i_f : \Delta^m \to \Delta^n, e_k \mapsto e_{f(k)}$, define

$$K(f) : \sin_n(X) \to \sin_m(X), \sigma \mapsto \sigma \circ f'.$$

Remark 1.2.3. $\mathrm{d}^k : \sin_n(X) \to \sin_{n-1}(X)$ is exactly $K(f)$ for $f : [n-1] \hookrightarrow [n]$.

So $H_*(X)$ can be recovered from K.

Actually, the simplicial set $K = \sin_*(X)$ contains strictly more information that is enough to recover the (weak) homotopy type of X.

But $H_*(\mathbb{CP}^2) \cong H_*(\mathbb{S}^2 \wedge \mathbb{S}^4)$ although $\mathbb{CP}^2 \not\simeq \mathbb{S}^2 \wedge \mathbb{S}^4$.

Def 1.2.6. Given covariant functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$, a natural transformation T from F_1 to F_2 is an assignment $T(X) \in \mathcal{D}(F_1(X), F_2(X))$ such that the diagram

$$\begin{tikzcd} F_1(X) \arrow[r, "T(X)"] \arrow[d, "F_1(f)"'] & F_2(X) \arrow[d, "F_2(f)"] \\ F_1(Y) \arrow[r, "T(Y)"] & F_2(Y) \end{tikzcd}$$

commutes for any $f : X \to Y$.

If $T(X)$ is an isomorphism $\forall X \in \mathrm{ob}(\mathcal{C})$, we call T a natural isomorphism.

Remark 1.2.4. Similar definition works for contravariant functors.

Exam 1.2.5. *Given continuous map $f : X \to Y$, we have a natural transformation f_* from $\sin_*(X) : \Delta \to \mathrm{Set}$ to $\sin_*(Y) : \Delta \to \mathrm{Set}$ and $f_*([n]) : \sin_n(X) \to \sin_n(Y), \sigma \mapsto \rho \circ \sigma$.*

1.3 Homotopy invariance of homology

Def 1.3.1. Let C_*, D_* be chain complexes, $f_{0*}, f_{1*} : C_* \to D_*$ be two chain maps.

A chain homotopy h from f_{0*} to f_{1*} is a degree-1 map $h : C_* \to D_{*+1}$ such that

$$dh + hd = f_{1*} - f_{0*}.$$

In this case, we say f_0, f_1 are chain homotopic, denoted by $f_{0*} \overset{h}{\simeq} f_{1*}$.

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\mathrm{d}"] \arrow[d, shift right, "f_{0*}"'] \arrow[d, shift left, "f_{1*}"] & C_n \arrow[r, "\mathrm{d}"] \arrow[d, shift right, "f_{0*}"'] \arrow[d, shift left, "f_{1*}"] \arrow[dl, "h"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{0*}"'] \arrow[d, shift left, "f_{1*}"] \arrow[dl, "h"'] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "\mathrm{d}"] & D_n \arrow[r, "\mathrm{d}"] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Lemma 1.3.1. *If $f_{0*} \overset{h}{\simeq} f_{1*}$, then $f_{0*} = f_{1*} : H_n(C_*) \to H_n(D_*)$ for any n.*

Proof. Take $[c] \in H_n(C_*)$, then $\mathrm{d}c = 0$.

So $\mathrm{d}h(c) = \mathrm{d}h(c) + h\,\mathrm{d}(c) = f_{1*}(c) - f_{0*}(c) \in B_n(D_*) = \mathrm{im}\left(D_{n+1} \xrightarrow{d} D_n\right)$

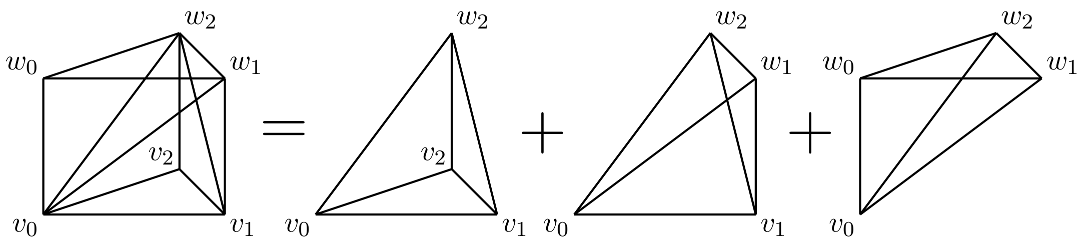
Hence $[f_{1*}(c)] = [f_{0*}(c)] \in H_n(D_*)$. $\square$

Prop 1.3.1. *$f_0, f_1 : X \to Y, f_0 \overset{H}{\simeq} f_1$, then $f_{0*}, f_{1*} : S_*(X) \to S_*(Y)$ are chain homotopic.*

The ideal is to subdivide $\Delta^n \times I$(domain of the restriction of H in a simplex) into simplices, denote the map by P. Once we have this, we know the domain $f_{1*} - f_{0*}$ are the top and bottom of this triangular prism, which is the difference of the boundary of prism and the side of prism. In particular, side of prism is given by $\partial\Delta^n \times I$. So we have

$$\partial P + P\partial = f_{1*} - f_{0*}.$$

This means that P is a chain homotopy. And to subdivide $\Delta^n \times I$, we let the bottom surface of prism to be the "bottom" of the first simplex. Every time, we vertically drag a vertex in the bottom to the top, and let the new surface be the "top" of previous simplex, as well as the "bottom" of next simplex. For example, for $n = 2$:



Proof. Given $\sigma : \Delta^n \to X$, we have $\Delta^n \times I \xrightarrow{\sigma\times\mathrm{Id}} X \times I \xrightarrow{H} Y$.

Let $\Delta^n = \{e_0, \cdots, e_n\}$ and denote $v_i = e_i \times \{0\}, w_i = e_i \times \{1\}$.

Consider the singular simplex $\eta_i : \Delta^{n+1} \to \Delta^n \times I$ such that

$$e_0 \mapsto v_0, \cdots, e_i \mapsto v_i, e_{i+1} \mapsto w_i, \cdots, e_{n+1} \mapsto w_n.$$

Then we have

$$\Delta^n \times I = \bigcup_{i=0}^{n} \eta_i(\Delta^n).$$

Denote the simplex $\Delta^{n+1} \xrightarrow{\eta_i} \Delta^n \times I \xrightarrow{\sigma\times\mathrm{Id}} X \times I$ by

$$(\sigma \times \mathrm{Id})\Big|_{[v_0,\cdots,v_i,w_i,\cdots,w_n]} \in \sin_n(X \times I).$$

Define prism operator $P : S_n(X) \to S_{n+1}(Y)$ by

$$P(\sigma) := \sum_{i=0}^{n}(-1)^i \left(H \circ (\sigma \times \mathrm{Id})\Big|_{[v_0,\cdots,v_i,w_i,\cdots,w_n]}\right).$$

We claim that P is a chain homotopy: $\mathrm{d}P(\sigma) + P\,\mathrm{d}(\sigma) = f_{1*}(\sigma) - f_{0*}(\sigma)$.

$$\begin{aligned}\mathrm{d}P(\sigma) =& \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{i+j}\left(H \circ (\sigma \times \mathrm{Id})\Big|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]}\right)\\ &+\sum_{i=0}^{n}\sum_{j=i}^{n}(-1)^{i+j+1}\left(H \circ (\sigma \times \mathrm{Id})\Big|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]}\right)\end{aligned}$$

$$\begin{aligned}P\,\mathrm{d}(\sigma) =& \sum_{j=0}^{n}\sum_{i=0}^{j-1}(-1)^{j+i}\left(H \circ (\sigma \times \mathrm{Id})\Big|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]}\right)\\ &+\sum_{i=0}^{n}\sum_{i=j+1}^{n}(-1)^{i+j-1}\left(H \circ (\sigma \times \mathrm{Id})\Big|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]}\right)\end{aligned}$$

So we obtain that

$$\begin{aligned}
\mathrm{d}P(\sigma) + P\,\mathrm{d}(\sigma) =& \sum_{i=0}^{n} \left(H_0 \; (\sigma \times \mathrm{Id})\Big|_{[v_0,\cdots,v_{i-1},w_i,\cdots,w_n]} \right) \\
&- \sum_{i=0}^{n} \left(H \circ (\sigma \times \mathrm{Id})\Big|_{v_0,v\cdots,v_i,w_{i+1},\cdots w_n} \right) \\
=& H \circ (\sigma \times \mathrm{Id})\Big|_{[w_0,\cdots,w_n]} - H \circ (\sigma \times \mathrm{Id})\Big|_{[v_0,\cdots,v_n]} \\
=& f_{1*}(\sigma) - f_{0*}(\sigma)
\end{aligned}$$

□

Coro 1.3.1. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \xrightarrow{\cong} H_n(Y)$*

Proof. Let g be the homotopy inverse of f.
Then $g \circ f \simeq \mathrm{Id}_X, f \circ g \cong \mathrm{Id}_Y$.
So $g_* \circ f_* = \mathrm{Id}_{H_*(X)}, f_* \circ g_* = \mathrm{Id}_{H_*(Y)}$.
Hence $g_* = f_*^{-1}$ are isomorphism. □

Exam 1.3.1. *If X is contractible, then* $H_n(X) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$

If Y is a deformation retract of X, then $H_n(Y) \cong H_n(X)$ for any n.

Def 1.3.2. $H_0(\mathrm{Top})$ is the homotopy category of Top with
$\mathrm{obj}(H_0(\mathrm{Top})) = \mathrm{obj}(\mathrm{Top}) = \{\text{spaces}\}$,
$H_0(\mathrm{Top})(X, Y) = [X, Y] := \mathrm{Map}(X, Y)/\text{homotopy} = \{\text{homotopy classes of maps } X \to Y\}$.

Prop 1.3.2. *The functor $H_n(-) : \mathrm{Top} \to \mathrm{Ab}$ can be factorized as $\mathrm{Top} \to H_0(\mathrm{Top}) \to \mathrm{Ab}$.*

Proof. Follows by the homotopy invariance of $H_n(-)$. □

Def 1.3.3. For any X, define its reduced homology $\tilde{H}_n(X) := \ker(f_* : H_n(X) \to H_n(*))$ where $f : X \to *$ is the constant map.

We can also define $\tilde{H}_n(X)$ as the homology group of chain complex

$$\cdots \longrightarrow C_2(X) \xrightarrow{\mathrm{d}} C_1(X) \xrightarrow{\mathrm{d}} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\varepsilon \left(\sum n_i \sigma_i\right) = \sum n_i$.

Chapter 2

Properties of Homology

2.1 Relative homology

Def 2.1.1. Let $C_* = \bigoplus_n C_n$ be a chain complex, a subcomplex is a graded subgroup $B_* = \bigoplus_n B_n$ such that $B_n \subset C_n, \mathrm{d}(B_n) \subset B_{n-1}$.

In this case, we define the quotient chain complex

$$\frac{C_*}{B_*} := \bigoplus_n \frac{C_n}{B_n}, \mathrm{d}[a] = [\mathrm{d}a].$$

It is easy to check that $\left(\frac{C_*}{B_*}, \mathrm{d}\right)$ is still a chain complex.

Exam 2.1.1. *Let (X, A) be a space pair,* i.e. *A is a subspace of X.*
Then $\sin_n(A) = \{\sigma \in \sin_n(X) | \sigma(\Delta^n) \subset A\} \subset \sin_n(X)$ and $S_*(A)$ is a subcomplex of $S_*(X)$.

Def 2.1.2. Define $S_*(X, A) = \frac{S_*(X)}{S_*(A)}$, called relative singular chain complex.
And we define the relative homology $H_n(X, A) := H_n(S_*(X, A))$.

Remark 2.1.1. $H_n(-,-)$ is a functor $\mathrm{Top}_2 \to \mathrm{Ab}$, where $\mathrm{obj}(\mathrm{Top}_2) = \{(X, A) | A \subset X\}$ with $\mathrm{Top}_2((X, A), (Y, B)) = \{f : X \to Y | f(A) \subset B\}$.

For $f : X \to Y$, then $f_* : S_*(X) \to S_*(Y)$ maps $S_*(A)$ into $S_*(B)$.

So we can induce the maps $f_* : S_*(X, A) \to S_*(Y, B), f_* : H_*(X, A) \to H_*(Y, B)$.

Given $f_0, f_1 : (X, A) \to (Y, B)$ with $f_0 \overset{H}{\simeq} f_1$, then $f_{0*} = f_{1*} : H_n(X, A) \to H_n(Y, B)$.

And we will prove that

Prop 2.1.1. *If A is a subcomplex of a CW complex X, then $H_n(X, A) \cong \tilde{H}_n(X/A)$.*

Def 2.1.3. A sequence of abelian group consists of maps:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \longrightarrow \cdots$$

such that $f_n \circ f_{n+1} = 0$, we say the sequence is exact at C_n if $\ker f_n = \mathrm{im} f_{n+1}$.

Remark 2.1.2. A sequence of abelian group is a chain complex, and exact at C_n iff $H_n(C_*) = 0$.

Def 2.1.4. We say the sequence is exact if it is exact everywhere.
A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0$$

Prop 2.1.2. *i is injective(*i.e. *A is subgroup of B), q is surjective and $B/A \cong C$.*

Proof. $\ker i = \mathrm{im}(0 \to A) = \{0\}, \mathrm{im}q = \ker(C \to 0) = C$.
So $C = \mathrm{im}q \cong B/\ker q = B/\mathrm{im}i \cong B/(A/\ker i) = B/A$ □

Def 2.1.5. A short exact sequence of chain complexes consists of

$$0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0$$

such that f, g are chain maps and $0 \longrightarrow A_n \xrightarrow{f} B_n \xrightarrow{g} C_n \longrightarrow 0$ is exact for every n.

Exam 2.1.2. *$0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$ is a short exact sequence of chain complexes.*

Prop 2.1.3. *Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ be a short exact sequence, show that the following three sets are in bijection with one another:*

(1) The set of homomorphisms $\sigma : C \to B$ such that $p\sigma = 1_C$.

(2) The set of homomorphisms $\pi : B \to A$ such that $\pi i = 1_A$.

(3) The set of homomorphisms $\alpha : A \oplus C \to B$ such that $\alpha(a, 0) = ia, p\alpha(a, c) = c$ for all $a \in A, c \in C$.

Moreover, show that any homomorphism as in (3) is an isomorphism.

Proof. $(i) \Rightarrow (ii)$: Since $p(\mathrm{Id}_B - \sigma p) = p - p = 0$ and $A = \ker p$.
So there is a morphism $\pi : B \to A$, such that $i\pi = \mathrm{Id}_B - \sigma p$.
Therefore $i\pi i = i - \sigma pi = i$.
And since i is injective.
Hence $\pi i = \mathrm{Id}_A$.
Similarly, we have $(ii) \Rightarrow (i)$.
$(i) \Rightarrow (iii)$: Let $\alpha : A \oplus C \to B, (a, c) \mapsto i(a) + \sigma(c)$.
Then $\alpha(a, 0) = ia, p\alpha(a, c) = pi(a) + p\sigma(c) = c$.
$(iii) \Rightarrow (i)$: Let $\sigma : C \to B, c \mapsto \alpha(0, c)$.
Then $p\sigma(c) = p\alpha(0, c) = c$.
Moreover, let $\beta : B \mapsto A \oplus C, b \mapsto (\pi(b), p(b))$.
Then $\beta\alpha(a, c) = \beta(ia + \sigma c) = (a, c)$.
And $\alpha\beta(b) = \alpha(\pi(b), p(b)) = i\pi(b) + \sigma p(b) = b - \sigma p(b) + \sigma p(b) = b$.
Hence α is isomorphism. □

Def 2.1.6. Any one of those structures in the previous proposition is a splitting of the short exact sequence, and the sequence is then said to be split.

Prop 2.1.4. *$\tilde{H}_n(X) = H_n(X)$ for $n > 0$ and $H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$.*

Proof. Since $H_n(*) = 0$ for $n > 0$.
So it sufficient to prove $0 \longrightarrow \tilde{H}_0(X) \xrightarrow{f} H_0(X) \xrightarrow{g} \mathbb{Z} \longrightarrow 0$ is split.
Let $a \in H_0(X)$ such that $g(a) = 1$.
Then $h : \mathbb{Z} \to H_0(X), 1 \mapsto a$ satisfies that $hg = 1_C$.
By proposition 2.1.3, $H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$. □

Lemma 2.1.1 (Snake)**.** *Let $0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{q} C_* \longrightarrow 0$ be a short exact sequence of chain complexes.*

Then there is a well-define map $\partial : H_n(C_*) \to H_{n-1}(A_*)$ that fits into the long exact sequence:

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(C_*) \arrow[dll, "\partial"'] \\
H_n(A_*) \arrow[r, "i_*"] & H_n(B_*) \arrow[r, "q_*"] & H_n(C_*) \arrow[dll, "\partial"'] \\
H_{n-1}(A_*) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. Take $[C] \in H_n(C_*)$, pick $b \in q^{-1}(C)$.

Then $q(\mathrm{d}b) = \mathrm{d}c = 0, i.e. \mathrm{d}b \in \ker q = \operatorname{Im} i$.

So there exists $a \in A_{n-1}$ such that $i(a) = db$.

And since $i(\mathrm{d}a) = \mathrm{d}(ia) = 0$ and i is injective.

Therefore $\mathrm{d}a = 0, i.e. [a] \in H_{n-1}(A_*)$.

This define the map $\partial : H_n(C_*) \to H_{n-1}(A_*), [c] \mapsto [a]$.

$$\begin{tikzcd}
 & b \arrow[r, "q"] \arrow[d, "\mathrm{d}"] & c \arrow[d, "\mathrm{d}"] \\
a \arrow[r, "i"] \arrow[d, "\mathrm{d}"] & \mathrm{d}b \arrow[r, "q"] \arrow[d, "\mathrm{d}"] & 0 \\
0 \arrow[r, "i"] & 0 &
\end{tikzcd}$$

If we choose another $b' \in q^{-1}(c)$, then $b - b' \in \ker q = \operatorname{Im} i$.

So there exists $\tilde{a} \in A_n$ such that $b - b' = i(\tilde{a})$.

Therefore $i(a - a') = \mathrm{d}b - \mathrm{d}b' = i(\mathrm{d}\tilde{a}), i.e. [a] = [a']$.

$$\begin{tikzcd}
\tilde{a} \arrow[r, "i"] \arrow[d, "\mathrm{d}"] & b - b' \arrow[r, "q"] \arrow[d, "\mathrm{d}"] & 0 \\
\mathrm{d}\tilde{a} \arrow[r, "i"] & \mathrm{d}b - \mathrm{d}b' &
\end{tikzcd}$$

If we choose c' with $[c'] = [c]$, then $c' - c = \mathrm{d}\tilde{c}$.

Pick any $\tilde{b} \in q^{-1}(\tilde{c})$ and let $b' = b + \mathrm{d}\tilde{b}$.

So $q(b') = q(b) + q(\mathrm{d}\tilde{b}) = c + \mathrm{d}\tilde{c} = c'$.

Therefore $\mathrm{d}b' = \mathrm{d}b + \mathrm{d}^2\tilde{b} = \mathrm{d}b, i.e. a' = a$.

$$\begin{tikzcd}
\tilde{b} \arrow[r, "q"] \arrow[d, "\mathrm{d}"] & \tilde{c} \arrow[d, "\mathrm{d}"] \\
b' - b \arrow[r, "q"] \arrow[d, "\mathrm{d}"] & c' - c \\
0 &
\end{tikzcd}$$

Since $q \circ i = 0$.

So $q_* \circ i_* = 0$.

Take $[b] \in \ker q_*, [q(b)] = 0 \in H_n(C_*)$.

Then there exists $c \in C_{n+1}$ such that $\mathrm{d}c = q(b)$.

Pick $b' \in q^{-1}(c)$.

Then $q(b - \mathrm{d}b') = \mathrm{d}c - \mathrm{d}(q(b')) = \mathrm{d}c - \mathrm{d}c = 0$.

So there exists $a \in A_n$ such that $i(a) = b - \mathrm{d}b', i.e. i(\mathrm{d}a) = \mathrm{d}b - \mathrm{d}^2b' = 0$.

Therefore $\mathrm{d}a = 0, i.e. i_*[a] = [b - \mathrm{d}b'] = [b] \in \operatorname{Im} i_*$.

Consider $[b] \in H_n(B_*)$ with $b \in q^{-1}(c)$.

So $q_*[b] = [q(b)]$ and $\mathrm{d}b = 0$, *i.e.* $\partial q_*[b] = 0$.
If $\partial[c] = 0$, then there exists $\tilde{a} \in A_n$ such that $a = \mathrm{d}\tilde{a}$.
Therefore $\mathrm{d}(b - i(\tilde{a})) = i(a) - i(a) = 0$.
Hence $[b - i(\tilde{a})] \in H_n(B_*)$, $i.e. q_*[b - i(\tilde{a})] = [q(b)] = [c] \in \operatorname{Im} q_*$. □

Remark 2.1.3. Happy year of the Snake!

Prop 2.1.5. *Triple (X, A, B) with $B \subset A \subset X$, then $\sin_n(B) \subset \sin_n(A) \subset \sin_n(X)$.*
We have exact sequence

$$0 \to \frac{S_*(A)}{S_*(B)} \xrightarrow{i_*} \frac{S_*(X)}{S_*(B)} \xrightarrow{j_*} \frac{S_*(X)}{S_*(A)} \to 0.$$

And long exact sequence

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A, B) \xrightarrow{i_*} H_n(X, B) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \to \cdots$$

Proof. Follows by Snake lemma. □

Coro 2.1.1. *If we set $B = *$, we get*

$$\cdots \to H_{n+1}(X, A) \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X, A) \to \cdots$$

Remark 2.1.4. $\partial : H_{n+1}(X, A) \to H_n(A)$ is "natural", that is, for any $f : (X, A) \to (Y, B)$, the following diagram is commutes:

$$\begin{array}{ccc} H_{n+1}(X, A) & \xrightarrow{f_*} & H_{n+1}(Y, B) \\ \partial\big\downarrow & & \big\downarrow\partial \\ H_n(A) & \xrightarrow{f_*} & H_n(B) \end{array}$$

Lemma 2.1.2 (Five)**.** *Given commutative diagram*

$$\begin{array}{ccccccccc} A_4 & \xrightarrow{\mathrm{d}} & A_3 & \xrightarrow{\mathrm{d}} & A_2 & \xrightarrow{\mathrm{d}} & A_1 & \xrightarrow{\mathrm{d}} & A_0 \\ \big\downarrow f_4 & & \big\downarrow f_3 & & \big\downarrow f_2 & & \big\downarrow f_1 & & \big\downarrow f_0 \\ B_4 & \xrightarrow{\mathrm{d}} & B_3 & \xrightarrow{\mathrm{d}} & B_2 & \xrightarrow{\mathrm{d}} & B_1 & \xrightarrow{\mathrm{d}} & B_0 \end{array}$$

assume first row si exact at A_3, A_2, A_1, second row exact at B_3, B_2, B_1, then

(1) f_0 injective, f_1, f_3 surjective then f_2 surjective.

(2) f_4 surjective, f_1, f_3 injective then f_2 injective.

(3) f_0, f_1, f_3, f_4 isomorphism then f_2 is isomorphism.

Proof. (2) If $f_2(a_2) = 0$, let $a_1 = \mathrm{d}a_2$.
Then $f_1(a_1) = \mathrm{d}f_2(a_2) = 0$.
So $a_1 = 0, i.e. a_2 = \mathrm{d}a_3$ for $a_3 \in A_3$.
Let $b_3 = f_3 a_3$.
Then $\mathrm{d}b_3 = f_2(a_2) = 0$, *i.e.* there exists $b_4 \in B_4$, such that $\mathrm{d}b_4 = b_3$.
Take $a_4 \in A_4, s.t. f_4(a_4) = b_4$.
Let $a_3' = \mathrm{d}a_4$.
Then $f_3(a_3') = \mathrm{d}b_4 = b_3 = f_3(a_3)$.
So $a_3' = a_3 = \mathrm{d}a_4, i.e. a_2 = \mathrm{d}a_3 = 0$.
(1) is similar and (3) follows by (1)+(2). □

Def 2.1.7. A map $f:(X,A)\to(Y,B)$ is called a homology isomorphism if $f_*:H_n(X,A)\to H_n(Y,B)$ is an isomorphism for any n.

Coro 2.1.2. *Given $f:(X,A)\to(Y,B)$, if 2 out of the 3 maps*

$$f:(X,A)\to(Y,B), f':X\to Y, f\big|_A:A\to B$$

are homology isomorphisms, then so is the third one.

Proof. WLOG, assume $f'_*, \left(f\big|_A\right)_*$ are isomorphism, then

$$\begin{tikzcd}
{} \arrow[r] & H_n(A) \arrow[r] \arrow[d, "\cong"] & H_n(X) \arrow[r] \arrow[d, "\cong"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(X) \arrow[r] \arrow[d, "\cong"] & {} \\
{} \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & H_{n-1}(Y) \arrow[r] & {}
\end{tikzcd}$$

The proof is complete by five lemma. □

Exam 2.1.3. *For triple (X,A,B) such that B is a deformation retract of A, then we have the inclusion $\mathrm{Id}_X:(X,B)\to(X,A)$ that $B\hookrightarrow A$.*
So $H_*(X)\cong H_*(X), H_*(B)\cong H_*(A), \text{i.e.} H_*(X,B)\to H_*(X,A)$ is an isomorphism.
This can also be proved by long exact sequence of triple, since $H_n(A,B)=0$.

Def 2.1.8. A triple (X,A,U) is called excisive if $\bar{U}\subset \mathrm{int}(A)$.
In this case, the inclusion $(X-U,A-U)\hookrightarrow(X,A)$ is called an excision.

Thm 2.1.1. *Any excision is a homology isomorphism.*

This theorem is powerful but hard as well, so we will prove it in the next section. We now first give some of its corollaries.

Coro 2.1.3. *Given (X,A), suppose $B\subset X, \bar{A}\subset \mathrm{int}(B)$ and A is a deformation retract of B, then the quotient map $q:(X,A)\to(X/A,*)$ is a homology isomorphism.*

Proof. We have the following commutative diagram, and we claim that they are all isomorphism.

$$\begin{tikzcd}
(X,A) \arrow[r, "(1)"] \arrow[d, "(5)"] & (X,B) \arrow[r, "(2)"] \arrow[d] & (X-A,B-A) \arrow[d, "(6)"] \\
(X/A,*) \arrow[r, "(3)"] & (X/A,B/A) \arrow[r, "(4)"] & (X/A-*,B/A-*)
\end{tikzcd}$$

(6) is a homeomorphism so is a homology isomorphism.
(2),(4) are excision.
(1) By example 2.1.3
(3) $*\hookrightarrow B/A$ is deformation retract so by example.
So (5) is isomorphism. □

Prop 2.1.6. $\tilde{H}_m(\mathbb{S}^n)=\begin{cases}\mathbb{Z}\langle[\iota_n]\rangle & m=n\\ 0 & m\neq n\end{cases}$
where $\iota_n\in S_n(\mathbb{D}^n,\partial\mathbb{D}^n)$, $[\iota_n]\in H_n(\mathbb{D}^n,\partial\mathbb{D}^n)\cong\tilde{H}_n(\mathbb{S}^n)$.

Proof. When $n=0$, $\tilde{H}_m(\mathbb{S}^0) \cong \tilde{H}_m(\mathbb{D}^0/\varnothing) \cong H_m(\mathbb{D}^0, \varnothing) = H_m(*) = \begin{cases} \mathbb{Z}\langle [l_0] \rangle & m=0 \\ 0 & m \neq 0 \end{cases}$.

Consider triple $(D^n, S^{n-1}, *)$, by corollary 2.1.3, $H_m(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_m(\mathbb{S}^n, *) = \tilde{H}_m(\mathbb{S}^n)$. Then we have

$$\begin{array}{ccccccc} H_m(\mathbb{D}^n,*) & \longrightarrow & H_m(\mathbb{D}^n,\mathbb{S}^{n-1}) & \longrightarrow & H_{m-1}(\mathbb{S}^{n-1},*) & \longrightarrow & H_{m-1}(\mathbb{D}^n,*) \\ \downarrow= & & \downarrow\cong & & \downarrow\cong & & \downarrow= \\ 0 & \longrightarrow & \tilde{H}_m(\mathbb{S}^n) & \longrightarrow & \tilde{H}_m(\mathbb{S}^{n-1}) & \longrightarrow & 0 \end{array}$$

So $\tilde{H}_m(\mathbb{S}^n) \cong \tilde{H}_{m-1}(\mathbb{S}^{n-1})$ and this concludes the desired formula. □

Coro 2.1.4. *(1) For $n \neq m$, $\mathbb{S}^n \not\simeq \mathbb{S}^m, \mathbb{R}^n \not\simeq \mathbb{R}^m$.*

(2) $\mathbb{S}^{n-1}$ is not a retraction of $\mathbb{D}^n$.

Proof. (1) $\tilde{H}_n(\mathbb{S}^n) \cong \mathbb{Z} \ncong 0 \cong \tilde{H}_n(\mathbb{S}^m)$.

So $\mathbb{S}^n \not\simeq \mathbb{S}^m$ and $\mathbb{R}^n \backslash \{0\} \cong \mathbb{S}^{n-1} \not\simeq \mathbb{S}^{m-1} \cong \mathbb{R}^m \backslash \{0\}$.

Hence $\mathbb{R}^n \not\simeq \mathbb{R}^m$.

(2) $\mathbb{Z} \cong \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \to \tilde{H}_{n-1}(\mathbb{D}^n) \cong 0$ has no left inverse.

So $\mathbb{S}^{n-1} \to \mathbb{D}^n$ has no left inverse.

□

Thm 2.1.2 (Brouwer fixed point). *$f: \mathbb{D}^n \to \mathbb{D}^n$ is continuous, then* $\mathrm{Fix}(f) \neq \varnothing$.

Proof. Suppose $f(x) \neq x$ for any $x \in \mathbb{D}^n$.

Define $\tilde{f}: \mathbb{D}^n \to \mathbb{S}^{n-1}$ that map x to the intersection of $\mathbb{S}^{n-1}$ and the ray l_x from x to $f(x)$. Then $\tilde{f}\Big|_{\mathbb{S}^{n-1}} = \mathrm{Id}$, *i.e.* f is a retraction, contradiction! □

2.2 Locality principle of homology

Def 2.2.1. X is a space and $\mathcal{A} = \{\text{a collection of subsets of } X\}$, we say $\mathcal{A}$ is a cover of X if

$$X = \bigcup_{A \in \mathcal{A}} \mathrm{int}(A).$$

And we can define a chain complex:

$$\sin_n^{\mathcal{A}}(X) = \{\sigma : \Delta^n \to X | \sigma(\Delta^n) \in A \text{ for some } A \in \mathcal{A}\} = \{\mathcal{A}\text{-small simplices}\} \subset \sin_n(X).$$

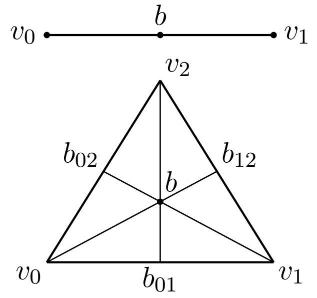
$$S_n^{\mathcal{A}}(X) = \mathbb{Z}\sin_n^{\mathcal{A}}(X) = \{\mathcal{A}\text{-small chains}\} \subset S_n(X).$$

Then $S_*^{\mathcal{A}}$ is a subcomplex of $S_*(X)$.

Thm 2.2.1. *The inclusion $S_*^{\mathcal{A}}(X) \xrightarrow{i} S_*(X)$ is a quasi-isomorphism, that is, it induces an isomorphism on homology group:*

$$i_* : H_n(S_*^{\mathcal{A}}(X)) \xrightarrow{\cong} H_n(S_*(X)).$$

The idea to prove this theorem is to prove that we can divide a simplex into some small piece that are as small as we want. This operator is done by the barycentric subdivision. For a simplex $[v_0, \cdots, v_n]$, we decomposite it into some n-simplex $[b, w_0, \cdots, w_{n-1}]$ where b is the barycenter and $[w_0, \cdots, w_{n-1}]$ is the barycentric subdivision of a face $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$. In particular, the figures below show the cases when $n = 1, 2$:



Using this idea, we now try to prove this theorem rigorously.

Proof. We define $\$: S_*(X) \to S_*(X)$ called barycentric subdivision, which is natural chain map, naturally homotopic to Id.

By linearity, suffices to define $\$(\sigma)$ for $\sigma \in \sin_n(X)$.

Let $\sigma_* : \sin_n(\Delta^n) \to \sin_n(X)$, by naturality, $\$\sigma = \sigma_*(\$\iota_n)$, where $\iota_n : \Delta^n \overset{\mathrm{Id}}{=\!=} \Delta^n \in \sin_n(\Delta^n)$ is the universal singular simplex.

So we just need to define $\$\iota_n$.

Given star shaped (X, b) and $\sigma : \Delta^n \to X$, define

$$b * \sigma : \Delta^{n+1} \to X, (x_0, \cdots, x_{n+1}) \mapsto x_0 \cdot b + (1 - x_0)\sigma\left(\frac{x_1}{1-x_0}, \cdots, \frac{x_{n+1}}{1-x_0}\right).$$

This extends to a linear map $b * - : S_n(X) \to S_{n+1}(X)$.

We inductively define $\$\iota_n$ as $\$c_0 = c_0, \$\iota_n = b_n * (\$\mathrm{d}\iota_n)$, for example,

$$\$\iota_1 = [b, v_1] - [b, v_0],$$

$$\$\iota_2 = [b, b_{01}, v_1] - [b, b_{01}, v_0] + [b, b_{12}, v_2] - [b, b_{12}, v_1] + [b, b_{02}, v_0] - [b, b_{02}, v_2].$$

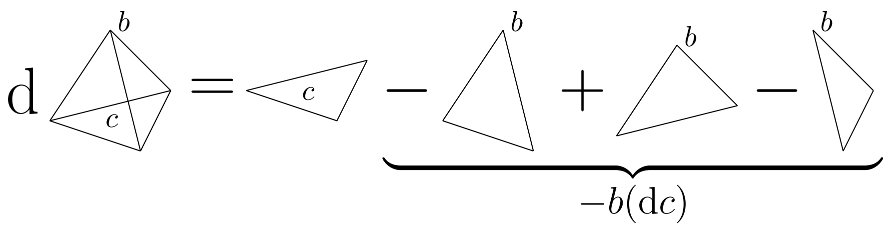
For barycentric subdivision, we have the following propositions: □

Prop 2.2.1. $\$: S_*(X) \to S_*(X)$ *is a chain map.*

Proof. By naturality of $\mathrm{d}, \$$, it suffices to check $\mathrm{d}\$(\iota_n) = \$\,\mathrm{d}(\iota_n)$.

When $n = 1$, $\$\mathrm{d}\iota_1 = \$(e_1 - e_0) = e_1 - e_0$ and $\mathrm{d}(\$\iota_1) = \mathrm{d}(b_1 e_1 - b_1 e_0) = e_1 - e_0$.

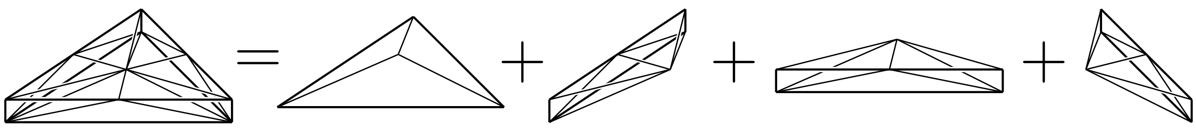
For $n \geqslant 2$, we have $\mathrm{d}(b * c) = c - b * \mathrm{d}c$ for $c \in S_{\geqslant 1}(X)$.



So $\mathrm{d}\$\iota_n = \mathrm{d}(b_n * \$\mathrm{d}\iota_n) = \$\mathrm{d}\iota_n - b_n * \mathrm{d}\$\mathrm{d}\iota_n = \$\mathrm{d}\iota_n - b_n * \$\mathrm{d}^2\iota_n = \$\mathrm{d}\iota_n$. □

Prop 2.2.2. *There exists a natural chain homotopy* $T : S_*(X) \to S_{*+1}(X)$ *from* $\$$ *to* Id.

Proof. Set $T\iota_n = b_n * (\iota_n - T\mathrm{d}\iota_n)$.



Geometrically, this formula inductively defined a subdivision of $\Delta^n \times I$ by join all simplices of $\Delta^n \times \{0\} \cup \partial\Delta^n \times I$ to the barycenter of $\Delta^n \times \{1\}$, as the figure above shown.

We now prove that $\mathrm{d}T + T\mathrm{d} = \mathrm{Id} - \$$ by induction.
It suffices to prove $\mathrm{d}T\iota_n + T\mathrm{d}\iota_n = \iota_n - \$\iota_n$:

$$\begin{aligned}\mathrm{d}T\iota_n =& \mathrm{d}b_n * (\iota_n - T\mathrm{d}\iota_n)\\ =&\iota_n - T\mathrm{d}\iota_n - b_n * (\mathrm{d}\iota_n - \mathrm{d}T_n\mathrm{d}\iota_n)\\ =&\iota_n - T\mathrm{d}\iota_n - b_n * (\$\mathrm{d}\iota_n + T\mathrm{d}^2\iota_n)\\ =&\iota - T\mathrm{d}\iota_n - \$\iota_n\end{aligned}$$

□

Lemma 2.2.1. *Given any cover $\mathcal{A}$ of Δ^n, there exists m such that $\$^m\iota_n \in S_n^{\mathcal{A}}(\Delta^n)$.*

Proof. For $\sigma \in \sin_n(\Delta^n)$, we define $\mathrm{diam}(\sigma) := \sup\limits_{x,y\in\Delta^n} |\sigma(x) - \sigma(y)|$.

And for a chain$c = \sum a_i\sigma_i \in S_n(\Delta^n)$, $\mathrm{diam}(c) = \max \mathrm{diam}(\sigma_i)$.
Then by induction, $\mathrm{diam}(\$\sigma) \leqslant \max\{\frac{n-1}{n}\mathrm{diam}(\sigma), |\sigma(b_n) - \sigma(v_0)|\} \leqslant \frac{n}{n+1}\mathrm{diam}(\sigma)$.
So by Lebesgue lemma, there exists m such that $\$^m(\iota_n) \in S_*^{\mathcal{A}}(\Delta^n)$. □

Coro 2.2.1. *For any cover $\mathcal{A}$ of X and any $c \in S_*(X)$, there exists m such that $\$^m c \in S_*^{\mathcal{A}}(X)$.*

Proof. By finiteness, it suffices to assume $c = \sigma : \Delta^n \to X$.
Let $\mathcal{A}' = \{\sigma^{-1}(A) | A \in \mathcal{A}\}$.
Then $\mathcal{A}'$ is a cover of Δ^n.
By lemma 2.2.1, $\exists m >> 0, s.t. \$^m\iota_n \in S_n^{\mathcal{A}'}(\Delta^n)$.
So $\$^m\sigma = \$^m\sigma_*(\iota_n) = \sigma_*(\$^m\iota_n) \in S_*^{\mathcal{A}}(X)$. □

proof of theorem 2.2.1. Consider $i_* : H_n(S_*^{\mathcal{A}}(X)) \to H_n(S_*(X))$.
Surjectivity: take $[C] \in H_n(S_*(X))$ with $\mathrm{d}c = 0$, then there exists m such that $\$^m c \in S_*^{\mathcal{A}}(X)$ and $\mathrm{d}(\$^m c) = \$^m \mathrm{d}c = 0$.
So $[\$^m c] \in H_n(S_*^{\mathcal{A}}(X))$ and $i_*[\$^m c] = [\$^m c] = [c]$ since $\$ \overset{T}{\simeq} \mathrm{Id}$.
Injectivity: Take $[c] \in \ker i_*$, then $c \in S_n^{\mathcal{A}}(X)$ and $\mathrm{d}\alpha = c$ for some $\alpha \in S_{n+1}(X)$.
Then there exists m, such that $\$^m\alpha \in S_{n+1}^{\mathcal{A}}(X)$.
So $\mathrm{d}(\$^m\alpha) = \$^m\mathrm{d}\alpha = \$^m c$, *i.e.* $[c] = [\$^m c] = 0 \in H_*(S_*^{\mathcal{A}}(X))$.
Hence i_* is isomorphism. □

Coro 2.2.2. *If (X, A, U) is excisive,* i.e. $\bar{U} \subset \mathrm{int}(A)$, *then the excision $(X\backslash U, A\backslash U) \hookrightarrow (X, A)$ induces isomorphism on homology.*

Proof. Let $B = X\backslash U$, $\mathcal{A} = \{A, B\}$ is a cover of X, then $(X\backslash U, A\backslash U) = (B, A \cap B)$.
So $\sin_n^{\mathcal{A}}(X) = \sin_n(A) \cup \sin_n(B)$, $S_*^{\mathcal{A}}(X) = S_*(A) + S_*(B)$ and $S_*(A) \cap S_*(B) = S_*(A \cap B)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_*(A) & \longrightarrow & S_*^{\mathcal{A}}(X) & \longrightarrow & S_*(B)/S_*(A\cap B) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{i_*} & & \downarrow{j_*} & & \\ 0 & \longrightarrow & S_*(A) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X)/S_*(A) & \longrightarrow & 0 \end{array}$$

By 5-lemma, j_* is an isomorphism. □

Thm 2.2.2 (Mayer-Vietoris sequence). *$\mathcal{A} = \{A_1, A_2\}$ is a cover of X, then we have a long exact sequence*

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X) \arrow[dll, "\partial"'] \\
H_n(A_1 \cap A_2) \arrow[r, "(j_{1*}, i_{2*})"'] & H_1(A_1) \oplus H_1(A_2) \arrow[r, "i_{1*} - i_{2*}"] & H_n(X) \arrow[dll, "\partial"] \\
H_{n-1}(A_1 \cap A_2) \arrow[r] & \cdots &
\end{tikzcd}$$

where

$$\begin{tikzcd}
A_1 \cap A_2 \arrow[r, "j_1"] \arrow[d, "j_2"'] & A_2 \arrow[d, "i_2"] \\
A_1 \arrow[r, "i_1"'] & X
\end{tikzcd}$$

Proof. $S_*^{\mathcal{A}}(X) = S_*(A_1) + S_*(A_2), S_*(A_1) \cap S_*(A_2) = S_*(A_1 \cap A_2)$.

So we have a short exact sequence

$$0 \longrightarrow S_*(A_1 \cap A_2) \xrightarrow{(i_{1*}, i_{2*})} S_*(A_1) \oplus S_*(A_2) \longrightarrow S_*^{\mathcal{A}}(X) \longrightarrow 0$$

And the proof is complete by snake lemma. $\square$

Prop 2.2.3. *If X is a CW complex, $X = A_1 \cup A_2$ are subcomplex, then we also have the MV sequence*

2.3 The Eilen-Steenrod axioms

Def 2.3.1. A homology theory is

(1) a sequence of functors $\{h_n : \mathrm{Top}_2 \to \mathrm{Ab}\}_{n \in \mathbb{Z}}$

(2) a natural transformation $\partial : h_n(X, A) \to h_{n-1}(A, \varnothing)$ that satisfy:

   (a) (homotopy invariance axiom)$f \simeq g \Rightarrow f_* = g_* : h_n(X, Y) \to h_n(X', Y')$ for any n and $f, g : (X, Y) \to (X', Y')$.

   (b) (excision axiom)Any excision $(X - U, A - U) \to (X, A)$ induces isomorphisms on $h_n(-)$ for any n.

   (c) (long exact sequence) $\cdots \to h_{q+1}(X, A) \xrightarrow{\partial} h_q(A) \to h_q(X) \to h_q(X, A) \xrightarrow{\partial} \cdots$ is exact.

   (d) (dimension axiom)$h_q(*) \cong \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \neq 0 \end{cases}$

   (e) (Milnor axiom)Given a collection of spaces $\{X_k\}_{k \in I}$, the inclusion maps

$$i_k : X_k \to \bigsqcup_{k \in I} X_k, i_{k,*} : h_n(X_k) \to h_n\left(\bigsqcup_{k \in I} X_k\right)$$

   induces an isomorphism

$$\alpha := \bigoplus_{k \in I} i_{k,*} : \bigoplus_{k \in I} h_n(X_k) \to h_n\left(\bigsqcup_{k \in I} X_k\right).$$

Thm 2.3.1 (Milnor). *Let $h_n(-)$ be a homology theory, then for any CW pairs (X, A), we have natural isomorphism $h_n(X, A) \cong H_n(X, A)$,* i.e. *these axioms determines homology theory.*

Def 2.3.2. A generalized homology theory consists of a sequence of functors $\{h_n : \mathrm{Top}_2 \to \mathrm{Ab}\}_{n\in\mathbb{Z}}$ and natural transformations $\partial : h_{n+1}(X, A) \to h_n X$ that satisfy all axioms other than the dimension axiom.

Exam 2.3.1. *For an abelian group G and space X, we define the chain complex*

$$S_n(X; G) = S_n(X) \otimes G, \mathrm{d} : S_n(X; G) \to S_{n-1}(X; G).$$

And the homology with G-coefficient:

$$H_n(X; G) := H_n(S_*(X; G)), H_n(X, A; G) := H_n\left(\frac{S_*(X; G)}{S_*(A; G)}\right).$$

The dimension axiom is replaced by

$$H_n(*; G) = \begin{cases} G & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Exam 2.3.2. *For a space X and $n \geqslant 0$, define*

$$\Omega_n(X) := \{(M, f) | M \textit{ is smooth } n\textit{-dimensional closed manifold, } f : M \to X \textit{continuous}\}/\sim.$$

And $(M, f) \sim (N, g)$ if there is some $(n+1)$-dim compact manifold W and $h : W \to X$ such that $\partial W \cong M \sqcup N$, $h\big|_{\partial W} = f \sqcup g$.

The group structure is given by $[(M, f)] + [(N, g)] = [M \sqcup N.f \sqcup g]$.

$\tilde{\Omega}_n(X) := \ker(\Omega_n(X) \to \Omega_n(*)), \Omega_n(X, A) = \tilde{\Omega}_n(X/A)$ *for CW pairs (X, A).*

$\Omega_n(-)$ is a generalized cobordism theory.

And $\Omega_n(*) \cong \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \mathbb{Z}/2 \oplus \mathbb{Z}/2, \cdots$ with generators $*, 0, [\mathbb{RP}^2], 0, ([\mathbb{RP}^2 \times \mathbb{RP}^2], [\mathbb{RP}^4])$.