An Introduction to Kähler Manifold

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April 26, 2025

1 Complex manifold

Def 1.1. A complex manifold M is a differentiable manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all α, β.

Def 1.2. A function on an open set $U \subset M$ is holomorphic if for all α, $f \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n$.

$z = (z_1, \cdots, z_n)$ of functions on $U \subset M$ is said to be a holomorphic coordinate system if $\varphi_\alpha \circ z^{-1}$ and $z \circ \varphi_\alpha^{-1}$ are holomorphic on $z(U \cap U_\alpha), \varphi_\alpha(U \cap U_\alpha)$ resp.

A map $f : M \to N$ of complex manifolds is holomorphic if it is given in terms of local holomorphic coordinates on N by holomorphic functions.

Exam 1.1. *One-dimensional complex manifold is called a Riemann surface*

$\mathbb{P}^n = \{$*lines through the origin in* $\mathbb{C}^{n+1}\}$ *is the complex projective space, with homogeneous coordinates* $z = [z_0 : \cdots : z_n]$.

$\mathbb{C}^n/\Lambda$ *is the complex torus, where* $\Lambda = \mathbb{Z}^k \subset \mathbb{C}^k$ *is a discrete lattice.*

Def 1.3. $T_{\mathbb{R},p}(M)$ is the real tangent space to M at p, it can be realized as the space of $\mathbb{R}$-linear derivations on the ring of real-valued C^∞ functions in a neighborhood of p. If we write $z_j = x_j + iy_j$, then

$$T_{\mathbb{R},p}(M) = \mathbb{R}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\}$$

$T_{\mathbb{C},p}(M) = T_{\mathbb{R},p}(M) \otimes_{\mathbb{R}} \mathbb{C}$ is called the complexified tangent space to M at p, it can be realized as the space of $\mathbb{C}$-linear derivations on the ring of complex-valued C^∞ functions in a neighborhood of p. We can write

$$T_{\mathbb{C},p}(M) = \mathbb{C}\left\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right\} = \mathbb{C}\left\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\right\},$$

where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

$T'_p(M) = \mathbb{C}\left\{\frac{\partial 1}{\partial z_j}\right\}$ is called the holomorphic tangent space, it can be realized as the subspace of $T_{\mathbb{C},p}(M)$ consisting of derivations that vanish on antiholomorphic functions. And it is independent of the holomorphic coordinate system chosen.

The subspace $T''_p(M) = \mathbb{C}\left\{\frac{\partial}{\partial \bar{z}_j}\right\}$ is called the antiholomorphic tangent space, and we have

$$T_{\mathbb{C},p}(M) = T'_p(M) \oplus T''_p(M).$$

Prop 1.1. *A map $f: M \to N$ is holomorphic iff*

$$f_*(T'_p(M)) \subset T'_{f(p)}(N)$$

for all $p \in M$.

Prop 1.2. $T''_p(M) = \overline{T'_p(M)}$ *with operation of conjugation* $\frac{\partial}{\partial z_j} \mapsto \frac{\partial}{\partial \bar{z}_j}$.

$T_{\mathbb{R},p}(M) \to T_{\mathbb{C},p}(M) \to T'_p(M)$ *is an isomorphism*

Exam 1.2. *For a smooth curve $z(t) = x(t) + iy(t)$ in $\mathbb{C}$, the tangent to the are can be taken either as*

$$x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y} \in T_{\mathbb{R}}(\mathbb{C})$$

or

$$z'(t)\frac{\partial}{\partial z} \in T'(\mathbb{C})$$

Remark 1.1. Let $(z_1, \cdots, z_n), (w_1, \cdots, w_m)$ be the holomorphic coordinates centered at $p \in M, q \in N$ resp. and $f: M \to N$ is a holomorphic map with $f(p) = q$.

If we write $z_j = x_j + iy_j, w_\alpha = u_\alpha + iv_\alpha$, then $f_*: T_{\mathbb{R},p}(M) \to T_{\mathbb{R},q}(N)$ is given by

$$J_{\mathbb{R}}(f) = \left[\begin{array}{c|c} \dfrac{\partial u_\alpha}{\partial x_j} & \dfrac{\partial u_\alpha}{\partial y_j} \\ \hline \dfrac{\partial v_\alpha}{\partial x_j} & \dfrac{\partial v_\alpha}{\partial y_j} \end{array}\right]$$

And $f_*: T_{\mathbb{C},p}(M) \to T_{\mathbb{C},q}(N)$ is given by

$$J_{\mathbb{C}}(f) = \begin{bmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{bmatrix}$$

where

$$J(f) = \left(\frac{\partial w_\alpha}{\partial z_j}\right)$$

If $m = n$, then $\det J_{\mathbb{R}}(f) = |\det J(f)|^2 \geqslant 0$, *i.e.* holomorphic maps are orientation preserving.

Prop 1.3. $\mathbb{C}^n$ *has natural orientation given by $2n$-form*

$$\left(\frac{i}{2}\right)^n (\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1) \wedge \cdots \wedge (\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n) = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$$

Moreover, any complex manifold has natural orientation.

2 Submanifolds and subvarieties

Thm 2.1 (Implicit Function Theorem). *Given holomorphic functions $f_1, \cdots, f_k: U \to \mathbb{C}$ with a neighborhood U of 0 such that*

$$\det\left(\frac{\partial f_i}{\partial z_j}\right) = 0$$

Then there exists functions $w_1, \cdots, w_k$ defined in a neighborhood of $0 \in \mathbb{C}^{n-k}$ such that in a neighborhood of 0 in $\mathbb{C}^n$,

$$f_1(z) = \cdots = f_k(z) = 0 \Leftrightarrow z_i = w_i(z_{k+1}, \cdots, z_n).$$

Def 2.1. A complex submanifold S of a complex manifold M is a subset $S \subset M$ given locally either as the zeros of a collection $f_1, \cdots, f_k$ of holomorphic functions with rank $J(f) = k$, or as the image of an open set U in $\mathbb{C}^{n-k}$ under a map $f : U \to M$ with rank $J(f) = n - k$.

Remark 2.1. By the implicit function theorem, two conditions are equivalent, and S has naturally the structure of a complex manifold of dimension $n - k$.

Def 2.2. An analytic subvariety V of a complex manifold is a subset given locally as the zeros of a finite collection of holomorphic functions.

A point $p \in V$ is called a smooth point of V if V is a submanifold of M near p, and the locus of smooth points of V is denoted by V^*, a point $p \in V - V^*$ is called a singular point of V, $V_s = V - V^*$ is called the singular locus.

V is called smooth or nonsingular if $V = V^*$.

Prop 2.1. *smooth analytic subvariety V is a submanifold of M.*

Def 2.3. An analytic variety V is called irreducible, if there does not exists proper subvariety V_1, V_2 of V such that $V_1 \cup V_2 = V$.

Prop 2.2. *An analytic variety V is irreducible iff V^* is connected.*

Proof. $\Leftarrow$: Suppose $V = V_1 \cup V_2$.

Then $V_1 \cap V_2 \subset V_s, i.e. V^*$ is disconnected, contradiction! □

3 Calculus on Complex Manifolds

Def 3.1. For $p + q = n$, define $\Omega^{p,q} = \{\varphi \in \Omega^n | \varphi(z) \in \bigwedge^p (T'_z)^*(M) \otimes \bigwedge^q (T''_z)^*(M)\}$, the form in $\Omega^{p,q}$ is said to be of type (p, q).

Prop 3.1. *For $\varphi \in \Omega^{p,q}$, $\mathrm{d}\varphi \in \Omega^{p+1,q} \oplus \Omega^{p,q+1}$.*

Def 3.2. $\partial = \pi^{p+1,q} \circ \mathrm{d} : \Omega^{p,q} \to \Omega^{p+1,q}, \bar{\partial} = \pi^{p,q+1} \circ \mathrm{d} : \Omega^{p,q} \to \Omega^{p,q+1}$ and then $\mathrm{d} = \partial + \bar{\partial}$.

Prop 3.2. *For $\varphi = \varphi_{IJ}(z)\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J$ with $|I| = p, |J| = q$, the operators ∂ and $\bar{\partial}$ is given by*

$$\partial\varphi = \frac{\partial}{\partial z^i}\varphi_{IJ}(z)\mathrm{d}z^i \wedge \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J$$

$$\bar{\partial}\varphi = \frac{\partial}{\partial \bar{z}^j}\varphi_{IJ}(z)\mathrm{d}\bar{z}^j \wedge \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J$$

Def 3.3. A hermitian metric on M is given by a positive definite hermitian inner product

$$(\bullet, \bullet)_z : T'_z(M) \otimes \overline{T'_z(M)} \to \mathbb{C}$$

on the holomorphic tangent space at z for each $z \in M$, depending smoothly on z, that is, such that for local coordinates z on M, the functions $h_{ij}(z) = \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)_z$ are smooth.

Remark 3.1. Writing $(\bullet, \bullet)_z$ in terms of the basis $\{\mathrm{d}z_i \otimes \mathrm{d}\bar{z}_j\}$ for $\left(T'_z(M) \otimes \overline{T'_z(M)}\right)^*$, the hermitian matrix is given by

$$\mathrm{d}s^2 = \sum_{i,j} h_{ij}(z)\mathrm{d}z_i \otimes \mathrm{d}\bar{z}_j$$

Def 3.4. A coframe for the hermitian metric is an n-tuple of forms $(\varphi_1, \cdots, \varphi_n)$ of type $(1, 0)$ such that

$$\mathrm{d}s^2 = \sum_i \varphi_i \otimes \bar{\varphi}_i,$$

i.e. $(\varphi_1(z), \cdots, \varphi_n(z))$ is an orthonormal basis for $(T'_z)^*(M)$.

Def 3.5. By the $\mathbb{R}$ linear isomorphism $T_{\mathbb{R},z}(M) \to T'_z(M)$,

$$\mathbb{R}\mathrm{d}s^2 : T_{\mathbb{R},z}(M) \otimes T_{\mathbb{R},z}(M) \to \mathbb{R}$$

is a Riemannian metric and the quadratic form

$$\operatorname{Im} \mathrm{d}s^2 : T_{\mathbb{R},z}(M) \otimes T_{\mathbb{R},z}(M) \to \mathbb{R}$$

is alternating, *i.e.* it represents a real differential form of degree 2.
$\omega = -\frac{1}{2} \operatorname{Im} \mathrm{d}s^2$ is called the associated $(1,1)$-form of the metric.

Prop 3.3. $\omega = \frac{i}{2} \sum \varphi_j \wedge \bar{\varphi}_j$

Proof. Let $\varphi_j = \alpha_j + i\beta_j$.
Then

$$\mathrm{d}s^2 = \sum (\alpha_j \otimes \alpha_j + \beta_j \otimes \beta_j + i \sum (-\alpha_j \otimes \beta_j + \beta_j \otimes \alpha_j).$$

So

$$\omega = -\frac{1}{2} \operatorname{Im} \mathrm{d}s^2 = \sum \alpha_j \wedge \beta_j = \frac{i}{2} \sum \varphi_j \wedge \bar{\varphi}_j.$$

□

Def 3.6. A real differential form ω of type $(1,1)$ is positive if $i\langle \omega(z), v \wedge \bar{v} \rangle > 0$ for any $v \in T'_z(M)$.

Prop 3.4. *Let $z = (z_1, \cdots, z_n)$ be a local holomorphic coordinate on M, a form ω is positive if*

$$\omega(z) = \frac{i}{2} \sum h_{jk}(z) \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$$

with $H(z) = (h_{ij}(z))$ a positive definite hermitian matrix for each z.

Exam 3.1. *Let $z_0, \cdots, z_n$ be the coordinates on $\mathbb{C}^{n+1}$, and let $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ be the standard projection map.*
Let $U \subset \mathbb{P}^n$ be an open set and $Z : U \to \mathbb{C}^{n+1} \backslash \{0\}$ be a lifting of U, i.e. $\pi \circ Z = \mathrm{Id}$.
Consider the differential form $\omega = \frac{i}{2\pi} \partial\bar{\partial} \log \|Z\|^2$.
We claim that ω is well-defined and positive.
For another lifting $Z' : U \to \mathbb{C}^{n+1} \backslash \{0\}$, where f is holomorphic, let $Z' = fZ$, then

$$\frac{i}{2\pi} \partial\bar{\partial} \log \|Z'\|^2 = \frac{i}{2\pi} \partial\bar{\partial} \left(\log \|Z\|^2 + \log f + \log \bar{f} \right) = \omega + \frac{i}{2\pi} \left(\partial\bar{\partial} \log f - \bar{\partial}\partial \log \bar{f} \right) = \omega$$

In open set $U_0 = \{z_0 \neq 0\}$, consider coordinate $w_i = \frac{z_i}{z_0}$ and lifting $Z = (1, w_1, \cdots, w_n)$

$$\omega = \frac{i}{2\pi} \partial\bar{\partial} \log \left(1 + \sum \omega_i \bar{\omega}_i \right) = \frac{i}{2\pi} \partial \left(\frac{\sum w_i \mathrm{d}\bar{w}_i}{1 + \sum w_i \bar{w}_i} \right)$$

$$= \frac{i}{2\pi} \left(\frac{\sum \mathrm{d}w_i \wedge \mathrm{d}\bar{w}_i}{1 + w_i \bar{w}_i} - \sum \frac{(\sum \bar{w}_i \mathrm{d}w_i) \wedge (\sum w_i \mathrm{d}\bar{w}_i)}{(1 + \sum w_i \bar{w}_i)^2} \right)$$

At point $[1 : 0 : \cdots : 0]$, $\omega = \frac{i}{2\pi} \sum \mathrm{d}w_i \wedge \mathrm{d}\bar{w}_i > 0$.
So ω defines a Hermitain metric on $\mathbb{P}^n$, called the Fubini-Study metric.

4 Holomorphic Vector Bundles

Def 4.1. A holomorphic vector bundle $E \xrightarrow{\pi} M$ is a complex vector bundle together with the structure of a complex manifold on E, *i.e.* the trivialization $\varphi_U : E_U \to U \times \mathbb{C}^k$ is biholomorphic. (or equivalently, it has holomorphic transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{C})$)

Prop 4.1. *dual, direct sum, tensor, alternating product of holomorphic vector bundles are holomorphic.*

Def 4.2. Take a local holomorphic frame $\{e^1, \cdots, e^k\}$ of E over U, define

$$\bar{\partial} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E), \omega_i \otimes e^i \mapsto \bar{\partial}\omega_i \otimes e^i.$$

Prop 4.2. *$\bar{\partial}$ is well-defined.*

Proof. For another holomorphic frame $\left\{(e')^1, \cdots, (e')^k\right\}$ of E over U, let $e^i = g^i_j (e')^j$.

Then $\sigma = g^i_j \omega_i \otimes (e')^j$.

So

$$\bar{\partial}\sigma = \bar{\partial}(g_{ij}\omega^i) \otimes (e')^j = g^i_j \cdot \partial\omega_i \otimes (e')^j = \bar{\partial}\omega_i \otimes e^i$$

since g_{ij} is holomorphic. □

Exam 4.1. *$T'(M)$ is called holomorphic tangent bundle.*

5 Metrics, connections and curvature

Def 5.1. A hermitian metric on E is a hermitian inner product on each fiber E_x of E varying smoothly with $x \in M$, *i.e.* for a frame $\zeta = \{\zeta_1, \cdots, \zeta_k\}$ of E, $h_{ij}(x) = (\zeta_i(x), \zeta_j(x))$ is smooth.

The frame ζ is called unitary if $\zeta_1(x), \cdots, \zeta_k(x)$ is orthonormal basis for E_x.

A holomorphic vector bundle with a hermitian metric is called a hermitian vector bundle.

Def 5.2. A connection D on $E \to M$ is a map $D : \Omega^0(E) \to \Omega^1(E)$ satisfying the Leibnitz rule:

$$D(f\zeta) = \mathrm{d}f \otimes \zeta + f \cdot D(\zeta)$$

for all sections $\zeta \in \Omega^0(E)(U), f \in C^\infty(U)$.

Let $e = \{e^1, \cdots, e^n\}$ be a frame for E over U, write $De^i = \theta^i_j e^j$, where $\theta = (\theta_{ij})$ is a matrix of 1-forms, called the connection matrix of D w.r.t. e.

Remark 5.1. For general $\sigma = \sigma_i e^i$, we have

$$D\sigma = \mathrm{d}\sigma_i \cdot e^i + \sigma_i \cdot De^i = \left(\mathrm{d}\sigma_j + \sigma_i \theta^i_j\right) e^j$$

For another frame $e' = \left\{(e')^1, \cdots, (e')^n\right\}$ with $(e')^i = \sum g^i_j e^j$, then

$$D(e')^i = \sum \mathrm{d}g^i_j \cdot e_j + \sum g^i_k \theta^k_j e^j$$

So

$$\theta_{e'} = \mathrm{d}g \cdot g^{-1} + g \cdot \theta_e \cdot g^{-1}.$$

Def 5.3. If E is hermitian, we say connection D on E is compatible with the complex structure if $D'' = \bar{\partial}$, where $D = D' + D''$ with $D' : \Omega^0(E) \to \Omega^{1,0}(E), D'' : \Omega^0(E) \to \Omega^{0,1}(E)$.

D is said to be compatible with the metric if $\mathrm{d}(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta)$.

Thm 5.1. *If E is hermitian, there is a unique connection D on E compatible with both the metric and the complex structure.*

Proof. Let $e = \{e_1, \cdots, e_n\}$ be a holomorphic frame and $h_{ij} = (e_i, e_j)$.
If such D exists, then $D'' e_i = \bar{\partial} e_i = 0$.
So connection matrix θ must have type $(1,0)$, and

$$\mathrm{d}h_{ij} = \mathrm{d}\,(e_i, e_j) = \theta_i^k h_{kj} + \bar{\theta}_j^k h_{ik}.$$

Therefore $\partial h_{ij} = \theta_i^k h_{kj}$, $i.e.\partial h = \theta h$ and $\bar{\partial} h_{ij} = \bar{\theta}_j^k h_{ik}$, $i.e.\bar{\partial} h = h\bar{\theta}^T$.
Hence $\theta = \partial h \cdot h^{-1}$ is the unique solution of these two equations. □

Def 5.4. This unique connection is called the associated(or metric) connection.

Remark 5.2. If $e_1, \cdots, e_n$ is a unitary frame, then $\theta_i^j + \bar{\theta}_j^i = \mathrm{d}\,(e_i, e_j) = 0$, *i.e.* the connection matrix w.r.t. a unitary frame is skew-hermitian.

6 Kähler manifold

Def 6.1. A compact complex manifold M with a hermitian metric $\mathrm{d}s^2$ is called Kähler if its associated $(1,1)$-form ω is d-closed(Sympletic structure)

Def 6.2. We say a metric $\mathrm{d}s^2$ on M osculates to order k to the Euclidean on $\mathbb{C}^n$ if for every point $p \in M$, we can find a holomorphic coordinate system (z) in a neighborhood of p for which

$$\mathrm{d}s^2 = \sum (\delta_{ij} + g_{ij}) \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j$$

where g_{ij} vanishes up to order k at p, written as

$$\mathrm{d}s^2 = \sum (\delta_{ij} + [k]) \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j.$$

Prop 6.1. $\mathrm{d}s^2$ *is Kähler iff it osculates to order* 2 *to the Euclidean metric everywhere.*

Proof. $\Leftarrow$ is trivial.
$\Rightarrow$: Let $\omega = \frac{i}{2} \sum (\delta_{jk} + a_{jkl} z^l + a_{jk\bar{l}} \bar{z}_l + [2]) \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$
Then $a_{jk\bar{l}} = \overline{a_{jkl}}$ and $a_{jkl} = a_{lkj}$ since $\mathrm{d}\omega = 0$.
Let $z^l = w^l + \frac{1}{2} \sum b_{mn}^l w^m w^n$ with $b_{mn}^l = -a_{lmn}$.
We claim that $\omega = \frac{i}{2} \sum (\delta_{ij} + [2]) \mathrm{d}w^i \wedge \mathrm{d}\bar{w}^j$. □

Exam 6.1. *Any metric on a compact Riemann surface is Kähler since* $\mathrm{d}\omega$ *is* 3*-form.*
The complex torus $T = \mathbb{C}^N/\Lambda$ *is Kähler with Euclidean metric* $\mathrm{d}s^2 = \sum \mathrm{d}z_i \otimes \mathrm{d}\bar{z}_i$.
If M, N *are Kähler, then* $M \times N$ *is Kähler with product metric.*
If $S \subset M$ *is a submanifold and* M *is Kähler, then* S *is Kähler with pull-back* $(1,1)$*-form* $i^*\omega$
Consider Fubini-Study metric on $\mathbb{P}^n$*, its associated* $(1,1)$*-form is*

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z\|^2 = \frac{\sqrt{-1}}{4\pi} (\partial + \bar{\partial})(\bar{\partial} - \partial) \log \|Z\|^2 = \frac{\sqrt{-1}}{4\pi} \mathrm{d}\left((\bar{\partial} - \partial) \log \|Z\|^2 \right)$$

So ω *is closed,* i.e. *Fubini-Study metric is Kähler.*

Prop 6.2. *Any compact manifold that can be embedded in projective space* $\mathbb{P}^n$ *is Kähler.*