Introduction to Quasiconformal Maps and Teichmuller Spaces Lecture Notes

Note taker: Jacky567

Qiuzhen College, Tsinghua University

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Contents

Chapter 1

Quasiconformal maps

1.1 Differentiable quasiconformal maps

Def 1.1.1. Let $(M_1, d_1), (M_2, d_2)$ be two Riemannian manifold and $f : M_1 \to M_2$ is a homeomorphism, define the dilatation

$$D_f : M_1 \to \mathbb{R}, x \mapsto \overline{\lim_{r\to 0}} \frac{\beta(f(B_r(x)))}{\alpha(f(B_r(x)))}$$

where $\alpha(f(B_r(x)))$ is the diameter of the largest ball that can be inscreted in $f(B_r(x))$ and $\beta(f(B_r))$ is the diameter of smallest ball circumscribe $f(B_r(x))$.

Def 1.1.2. We say that f is k-quasiconformal if $D_f(x) \leqslant k$ for any $x \in M_1$.

Exam 1.1.1. *(1) $M_1 = M_2 = \mathbb{R}$, then $D_f(x) = 1$ for every $x \in \mathbb{R}$.*

(2) $M_1 = M_2 = \mathbb{R}^n$ and f is bi-Lipschitz map, then $\beta(f(B_r)) \leqslant Lr, \alpha(f(B_r)) \geqslant \frac{r}{L}$, i.e. *$f$ is L^2-quasiconformal.*

(3) $M_1 = M_2 = \mathbb{C}$ and $f(re^{i\theta}) = r^2 e^{i\theta}$, then f is 2-quasiconformal but not bi-Lipschitz on every neighborhood of 0.

Prop 1.1.1. *Suppose $U \subset \mathbb{R}^k$ is a domain and $f : U \to f(U)$ is C^1-diffeomorphism, then the dilatation can given by*

$$D_f(p) = \frac{\sup_{\|v\|=1} \|\mathrm{d}f_p(v)\|}{\inf_{\|v\|=1} \|\mathrm{d}f_p(v)\|}$$

Proof. Since around p,

$$f(q) = f(p) + \mathrm{d}f_p(q - p) + o(\|q - p\|).$$

So for sufficiently small r, $f(B_r(p))$ is closed to the ellisoid

$$E = f(p) + \mathrm{d}f_p(B_r(0)) \subset T_{f(p)}f(U).$$

And notice that $\alpha(E) = 2r \sup_{\|v\|=1} \|\mathrm{d}f_p(v)\|, \beta(E) = 2r \sup_{\|v\|=1} \|\mathrm{d}f_p(v)\|$.

Hence

$$D_f(p) = \overline{\lim_{r\to 0}} \frac{\beta(f(B_r(x)))}{\alpha(f(B_r(x)))} = \frac{\sup_{\|v\|=1} \|\mathrm{d}f_p(v)\|}{\inf_{\|v\|=1} \|\mathrm{d}f_p(v)\|}$$

□

Coro 1.1.1. *Suppose $U \subset \mathbb{R}^k$ is a domain and $f : U \to f(U)$ is C^1-diffeomorphism, then on every compact subset of U, D_f is continuous and f is quasiconformal.*

Prop 1.1.2. *Let* $f : \mathbb{C} \to \mathbb{C}$*, then*

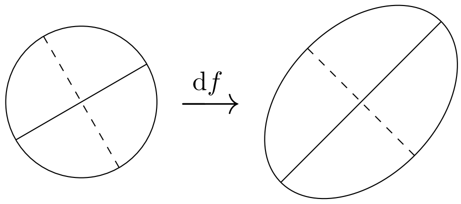
$$D_f(z_0) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}(z_0).$$

Proof. Since $\mathrm{d}f = f_z \mathrm{d}z + f_{\bar{z}} \mathrm{d}\bar{z}$.

So we have

$$(|f_z| - |f_{\bar{z}}|)|\mathrm{d}z| \leqslant |\mathrm{d}w| \leqslant (|f_z| + |f_{\bar{z}}|)|\mathrm{d}z|.$$

Geometrically, $\mathrm{d}f$ maps the unit circle to a ellipse:



and two axes of the ellipse are $|f_z| + |f_{\bar{z}}|, |f_z| - |f_{\bar{z}}|$.

Hence we have

$$D_f(z) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}$$

□

1.2 Extremal length

Def 1.2.1. A function $\rho : \mathbb{C} \to \mathbb{R}$ is called allowable if

(1) $\rho \geqslant 0$ and measurable,

(2) $A(\rho) = \int_{\mathbb{C}} \rho^2(z)\mathrm{d}x\mathrm{d}y \neq 0, \infty$.

Let Γ be a family of curves, each $\gamma \in \Gamma$ is a countable union of open arcs which are rectifiable, define

$$L_\gamma(\rho) = \int_\gamma \rho(z)|\mathrm{d}z|, L(\rho) = \inf_{\gamma \in \Gamma} L_\gamma(\rho).$$

The extermal length of Γ is defined as

$$\lambda(\Gamma) = \sup_\rho \frac{L^2(\rho)}{A(\rho)}$$

Def 1.2.2. We say $\Gamma_1 < \Gamma_2$ if every γ_2 contains γ_1.

Exam 1.2.1. *If* $\Gamma_1 \subset \Gamma_2$*, then* $\Gamma_2 < \Gamma_1$*.*

Prop 1.2.1. *(1) If* $\Gamma_1 < \Gamma_2$*, then* $\lambda(\Gamma_1) \leqslant \lambda(\Gamma_2)$*.*

(2) Let $\Gamma_1 + \Gamma_2 = \{\gamma_1 + \gamma_2 | \gamma_i \in \Gamma_i\}$*, then* $\lambda(\Gamma_1 + \Gamma_2) \geqslant \lambda(\Gamma_1) + \lambda(\Gamma_2)$*.*

(3) If $\Gamma_1 \cap \Gamma_2 = \varnothing$*, then* $\lambda(\Gamma_1 \cup \Gamma_2)^{-1} \geqslant \lambda(\Gamma_1)^{-1} + \lambda(\Gamma_2)^{-1}$*.*

Proof. (1) If $\gamma_1 \subset \gamma_2$, then $L_{\gamma_1}(\rho) \leqslant L_{\gamma_2}(\rho)$.

So $\lambda(\Gamma_1) \leqslant \lambda(\Gamma_2)$.

(2) WLOG, assume $L_i(\rho_i) = A(\rho_i)$.
Let $\rho = \max(\rho_1, \rho_2)$, then

$$L(\rho) \geqslant L_1(\rho_1) + L_2(\rho_2) = A(\rho_1) + A(\rho_2)$$
$$A(\rho) \leqslant A(\rho_1) + A(\rho_2)$$

So

$$\lambda(\Gamma_1 + \Gamma_2) = \sup_\rho \frac{L^1(\rho)}{A(\rho)} \geqslant A(\rho_1) + A(\rho_2) = \lambda(\Gamma_1) + \lambda(\Gamma_2).$$

(3) Let E_1, E_2 be two complementary measurable sets with $\Gamma_i \subset E_i$.
For allowable ρ, take $\rho_i = \rho \cdot \chi_{E_i}$.
Then $L_1(\rho_1) \geqslant L(\rho), L_2(\rho_2) \geqslant L(\rho)$ and $A(\rho) = A(\rho_1) + A(\rho_2)$.
So

$$\frac{A(\rho)}{L^2(\rho)} \geqslant \frac{A(\rho_1)}{L_1^2(\rho_1)} + \frac{A(\rho_2)}{L_2^2(\rho_2)}$$

Hence $\lambda(\Gamma_1 \cup \Gamma_2)^{-1} \geqslant \lambda(\Gamma_1)^{-1} + \lambda(\Gamma_2)^{-1}$

□

Exam 1.2.2. *Let Γ be the set of all arcs in an annulus $r_1 \leqslant |z| \leqslant r_2$ which join the boundary circles, then*

$$\int_0^{2\pi} \int_{r_1}^{r_2} \rho\left(re^{i\theta}\right) \mathrm{d}r\mathrm{d}\theta \geqslant \int_0^{2\pi} L(\rho)\mathrm{d}\theta = 2\pi L(\rho).$$

And by Cauchy inequality,

$$4\pi^2 L^2(\rho) \leqslant \int_0^{2\pi} \int_{r_1}^{r_2} \frac{1}{r}\mathrm{d}r\mathrm{d}\theta \cdot \int_0^{2\pi} \int_{r_1}^{r_2} \rho^2 r\mathrm{d}r\mathrm{d}\theta \leqslant 2\pi \log \frac{r_2}{r_1} A(\rho)$$

Hence $\lambda(\Gamma) = \frac{1}{2\pi} \log \frac{r_2}{r_1}$.

Thm 1.2.1. *Suppose $\Gamma \subset U \subset \mathbb{R}^2, f : U \to U'$ is diffeomorphism and $\Gamma' = f(\Gamma)$, if f is k-quasiconformal, then $\frac{\lambda(\Gamma)}{k} \leqslant \lambda(\Gamma') \leqslant k\lambda(\Gamma)$.*

Proof. Given density ρ on U_1, we define

$$\rho'(\zeta) = \left(\frac{\rho}{|f_z| - |f_{\bar{z}}|}\right)(z)$$

with $f(z) = \zeta$, then

$$|\mathrm{d}\zeta| \geqslant (|f_z| - |f_{\bar{z}}|)|\mathrm{d}z|.$$

So

$$\int_{\gamma'} \rho'|\mathrm{d}\zeta| \geqslant \int_\gamma \rho|\mathrm{d}z|.$$

Thus $L_{\gamma'}(\rho') \geqslant L_\gamma(\rho)$, *i.e.* $L(\rho') \geqslant L(\rho)$.
On the other hand,

$$A(\rho') = \int_{U'} \rho'\mathrm{d}\zeta_1\mathrm{d}\zeta_2 = \int_U (\rho' \circ f)^2 J_f \mathrm{d}x\mathrm{d}y = \int_U \rho^2 \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}\mathrm{d}x\mathrm{d}y \leqslant kA(\rho)$$

Therefore $\lambda(\Gamma') \geqslant \frac{\lambda(\Gamma)}{k}$.
And consider

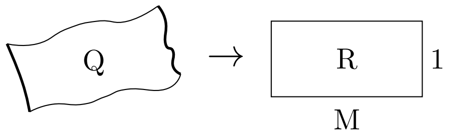
$$\rho''(\zeta) = \left(\frac{\rho}{|f_z| + |f_{\bar{z}}|}\right)(z),$$

we have $\lambda(\Gamma') \leqslant k\lambda(\Gamma)$. □

Coro 1.2.1. *$\lambda(\Gamma)$ is a conformal invariant.*

1.3 Quadrilateral

Def 1.3.1. Q-quadrilateral is a Jordan domain with a pair of disjoint closed arcs on the boundary(the b-sides).



Lemma 1.3.1. *There exists $M > 0$ and a conformal map $f : Q \to R$, where R is a rectangle, such that the b-sides are mapped to the vertical sides.*

Proof. follows from Riemann mapping theorem and Schwarz-Christoffel formula. □

Remark 1.3.1. In this note, we assume the width of R is 1 and the length is M.

Prop 1.3.1. *If $f : R \to R'$ is conformal and mapping corner to corner, then the length $M = M'$ and f is identity.*

Proof. By reflecting, we can extend f to whole $\mathbb{C}$.
And since
$$\lim_{|z|\to\infty} \frac{|f(z)|}{|z|} < +\infty$$
So f must be a degree-1 polynomial.
Notice that $f(0) = 0, f(M) = M', f(i) = i$.
Hence f must be identity and $M = M'$. □

Def 1.3.2. The modular of a rectangle R is M, denoted by $m(R) = M$.
More generally, given Q, we define $m(Q) = m(R)$ when we have conformal map $f : Q \to R$.

Prop 1.3.2. *$\lambda(\Gamma) = m(Q)$, where Γ is the family of arcs connecting the b-sides of Q.*

Proof. By corollary 1.2.1, we only need to prove this for $Q = R$ is a rectangle, so
$$\int_0^1 \int_0^M \rho(x + iy)\mathrm{d}x\mathrm{d}y \geqslant \int_0^1 L(\rho)\mathrm{d}y = L(\rho).$$
And by Cauchy inequality,
$$L^2(\rho) \leqslant \int_0^1 \int_0^M \mathrm{d}x\mathrm{d}y \cdot \int_0^1 \int_0^M \rho^2\mathrm{d}x\mathrm{d}y \leqslant MA(\rho).$$
Hence $\lambda(\Gamma) = M = m(Q)$. □

Def 1.3.3. Let Γ be the family of arcs connecting the b-side, define
$$s_b = \inf_{\gamma\in\Gamma} L_\gamma(\rho)$$
where ρ is the Euclidean density, and similarly, we can define s_a.

Thm 1.3.1 (Rengel inequality)**.**
$$\frac{s_b^2(Q)}{A(Q)} \leqslant m(Q) \leqslant \frac{A(Q)}{s_a^2(Q)}.$$
The equality holds iff Q is a rectangle.

Proof. Let Γ_b be the family of arcs connecting the b-sides of Q, then

$$\frac{s_b^2(Q)}{A(Q)} \leqslant \lambda(\Gamma_b) = m(Q).$$

On the other hand, we consider quadrilateral Q' whose b-sides are the a-sides of Q, then

$$\frac{s_a^2(Q)}{A(Q)} = \frac{s_b^2(Q')}{A(Q')} = m(R') = \frac{1}{m(R)}.$$

Hence we have

$$\frac{s_b^2(Q)}{A(Q)} \leqslant m(Q) \leqslant \frac{A(Q)}{s_a^2(Q)}.$$

□

1.4 Quasiconformal maps

Def 1.4.1. Let $\Omega \subset \mathbb{C}$ be a domain, and suppose $f : \Omega \to f(\Omega)$ is a homeomorphism, we say that f is k-quasiconformal if

$$\frac{m(Q)}{k} \leqslant m(f(Q)) \leqslant km(Q)$$

for every quadrilateral $Q \subset \Omega$.

Prop 1.4.1. *(1) If f is k-quasiconformal, f^{-1} is k-quasiconformal*

(2) If f_1 is k_1-quasiconformal and f_2 is k_2-quasiconformal, then $f_1 \circ f_2$ is k_1k_2-quasiconformal.

(3) Conformal map is 1-quasiconformal.

Proof. (1)

$$\frac{m\left(f^{-1}(Q)\right)}{k} \leqslant m(Q) \leqslant km\left(f^{-1}(Q)\right).$$

(2)

$$\frac{m(Q)}{k_1k_2} \leqslant \frac{m(f_2(Q))}{k_1} \leqslant m(f_1 \circ f_2(Q)) \leqslant k_1 m(f_2(Q)) \leqslant k_1k_2m(Q).$$

(3) Let R be a rectangle and WLOG, assume $f(R) = R'$ is also an rectangle otherwise we can composite a conformal map.

Then by proposition 1.3.1, f is identity, *i.e.* f is 1-quasiconformal.

□

Thm 1.4.1. *Every 1-quasiconformal map is conformal.*

Proof. Let R be a rectangle and WLOG, assume $f(R) = R'$ is also an rectangle otherwise we can composite a conformal map.

Consider a vertical line l that divides R into Q_1 and Q_2 and let $Q_i' = f(Q_i)$.
Then $m(Q_i') = m(Q_i)$ and $f(l)$ is an arc connecting the a-sides.
By Rengel inequality,

$$m(Q_i') \leqslant \frac{A(Q_i')}{s_a^2(Q_i')} = A(Q_i').$$

Suppose $f(l)$ is not a vertical line, *i.e.* Q_1, Q_2 are not rectangle.
Then $m(Q_1') + m(Q_2') < A(Q_1') + A(Q_2') = A(R') = A(R)$, contradiction!
So f is identity.
For general cases, f must be conformal. □

Prop 1.4.2. *If $f : \Omega \to f(\Omega)$ is C^1, then f is k-quasiconformal in the C^1 sense iff f is k-quasiconformal in geometric sense.*

Proof. $\Rightarrow$: Let Q be a quadrilateral in Ω.

WLOG, we assume $R = Q$ and $R' = f(Q)$ are rectangles, then

$$|f_x|^2 \leqslant (|f_z| + |f_{\bar{z}}|)^2 \leqslant kJ_f$$

So

$$\begin{aligned} m(R') = M' &= \int_R J_f(z)\mathrm{d}x\mathrm{d}y \\ &\geqslant \frac{1}{k}\int_R |f_x|^2(z)\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{Mk}\int_0^1 \left(\int_0^M \mathrm{d}x \cdot \int_0^M |f_x|^2(z)\mathrm{d}x\right)\mathrm{d}y \\ &\geqslant \frac{1}{Mk}\int_0^1 \left(\int_0^M |f_x|(x)\mathrm{d}x\right)^2 \mathrm{d}y \geqslant \frac{1}{Mk}(M')^2 \end{aligned}$$

Hence $M' \leqslant kM$, *i.e.* f is k-quasiconformal in geometric sense.

$\Leftarrow$: We first assume $f(z) = |f_z(0)|z + |f_{\bar{z}}(0)|\bar{z} + o(z)$.

Consider square R_δ with length δ and $R'_\delta = f(R_\delta)$.

Then

$$\begin{aligned} s_b(R'_\delta) &= \delta(|f_z(0)| + |f_{\bar{z}}(0)|) + o(\delta) \\ A(R'_\delta) &= \delta^2(|f_z(0)|^2 - |f_{\bar{z}}(0)|^2) + o(\delta^2). \end{aligned}$$

So by Rengel inequality,

$$m(R'_\delta) \geqslant \frac{s_b^2(R'_\delta)}{A(R'_\delta)} = \frac{\delta^2\left(|f_z(0)| + |f_{\bar{z}}(0)|\right)^2 + o(\delta^2)}{\delta^2\left(|f_z(0)| - |f_{\bar{z}}(0)|^2\right) + o(\delta^2)} = D_f(0) + O(1).$$

Hence $D_f(0) \leqslant k$, *i.e.* f is k-quasiconformal in C^1-sense.

For the general case, Let $\arg f_z(0) = \theta, \arg f_{\bar{z}}(0) = \varphi$ and

$$g(z) = e^{-i\frac{\theta+\varphi}{2}} f\left(e^{i\frac{\varphi-\theta}{2}} z\right).$$

Then

$$\begin{aligned} g_z(0) &= e^{-i\frac{\theta+\varphi}{2}} f_z(0) \cdot e^{i\frac{\varphi-\theta}{2}} = |f_z(0)| \\ g_{\bar{z}}(0) &= e^{-i\frac{\theta+\varphi}{2}} f_{\bar{z}}(0) \cdot \overline{e^{i\frac{\varphi-\theta}{2}}} = |f_{\bar{z}}(0)| \end{aligned}$$

And since the rotations are conformal, *i.e.* 1-quasiconformal.

Hence f is k-quasiconformal in C^1-sense as g is. □

1.5 Holder continuity

Def 1.5.1. Let Ω be a topological annulus(doubly connected region) and C_1, C_2 be the bounded, unbounded region of the component resp.

We say the closed curve γ in G separates C_1 and C_2 if γ has non-zero winding number about the points of C_1.

We denote $\Gamma = \{\gamma | \gamma \text{ separates } C_1 \text{ and } C_2\}$ and define the modular $m(\Omega) = \lambda(\Gamma)^{-1}$.

Prop 1.5.1. *Let Ω be a topological annulus and f maps Ω conformally to annulus $\{r_1 < |z| < r_2\}$, then*

$$m(\Omega) = \inf_{\rho} \frac{A(\rho)}{L^2(\rho)} = \frac{1}{2\pi} \log \frac{r_2}{r_1}$$

and reach the maximum at

$$\rho_0 = \left| \frac{f'}{f} \right|.$$

Proof. For any allowable ρ, take $\rho_1(f(z)) = \rho(z) \cdot \rho_0(z)^{-1}$.

For $\gamma_r \subset \Omega$ such that $f(\gamma_r) = \{|z| = r\}$, we have

$$L_{\gamma_r}(\rho) = \int_{\gamma_r} \rho |\mathrm{d}z| = \int_{|z|=r} \rho_1 \frac{1}{r} |\mathrm{d}z| = \int_0^{2\pi} \rho_1 \left(re^{i\theta} \right) \mathrm{d}\theta.$$

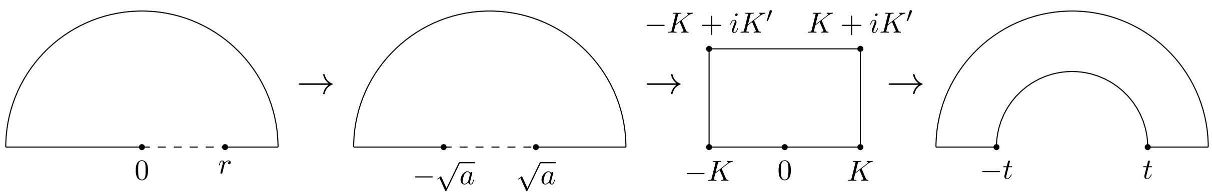
$$\begin{aligned} \iint_{\Omega} \rho^2 \mathrm{d}x\mathrm{d}y &= \int_{r_1}^{r_2} \int_0^{2\pi} \rho_1^2 \left(re^{i\theta} \right) \frac{1}{r} \mathrm{d}\theta \mathrm{d}r \\ &\geqslant \frac{1}{2\pi} \int_{r_1}^{r_2} \frac{1}{r} \left(\int_0^{2\pi} \rho_1 \left(re^{i\theta} \right) \mathrm{d}\theta \right)^2 \mathrm{d}r \\ &\geqslant \frac{1}{2\pi} \int_{r_1}^{r_2} \frac{L^2(\rho)}{r} \mathrm{d}r = \frac{L^2(\rho)}{2\pi} \log \frac{r_2}{r_1} \end{aligned}$$

Hence we obtain the conclusion. □

Coro 1.5.1. *Consider $\Omega_r = \{|z| \leqslant 1\} \backslash [0, r]$, prove that*

$$m(\Omega_r) = \frac{\mu(r)}{2\pi} = \frac{1}{4} \frac{I\left(\sqrt{1-r^2}\right)}{I(r)}, I(r) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-r^2x^2)}}.$$

Proof.



Consider mobius transformation

$$f(z) = \frac{z - \sqrt{a}}{-\sqrt{a}z + 1}, \frac{2\sqrt{a}}{a+1} = r$$

Then $f(\Omega_r) = \{|z| \leqslant 1\} \backslash [-\sqrt{a}, \sqrt{a}]$.

Let $\Omega^+ = f(\Omega_r) \cap \{\operatorname{Im} z \geqslant 0\}$ and consider

$$g(z) = \int_0^{\frac{z}{\sqrt{a}}} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-a^2t^2)}}.$$

We can verify that $g(\Omega_r^+) = \{-K \leqslant \operatorname{Re} z \leqslant K, 0 < \operatorname{Im} z < K'\}$ is a rectangle with

$$K = I(a), K' = \frac{1}{2} I(\sqrt{1-a^2}).$$

And $g\left((\sqrt{a}, 1)\right) = (K, K + iK'), g\left(-1, -\sqrt{a}\right) = (-K, -K + iK')$.

Consider

$$h(z) = te^{\frac{i\pi}{2K}(K-z)}.$$

Then $h(g(\Omega_r^+)) = \{t < |z| < 1, \operatorname{Im} z \geqslant 0\}$ with

$$t = e^{-\frac{K'\pi}{2K}}.$$

Notice that $h \circ g$ maps $(-1, -\sqrt{a}) \cup (\sqrt{a}, 1)$ to $(-1, -t) \cup (t, 1)$.

So we obtain a conformal map g_0 by reflect $h \circ g$ along real axis, which maps $f(\Omega_r)$ to annulus $\{t < |z| < 1\}$.

By proposition 1.5.1,

$$m(\Omega_r) = -\frac{1}{2\pi}\log t = \frac{1}{8}\frac{I\left(\sqrt{1-a^2}\right)}{I(a)}.$$

And since $I(r) = (1+a)I(a), I\left(\sqrt{1-r^2}\right) = I\left(\frac{1-a}{1+a}\right) = \frac{1}{2}(1+a)I\left(\sqrt{1-a^2}\right)$.

Hence

$$m(\Omega_r) = \frac{1}{8}\frac{I\left(\sqrt{1-a^2}\right)}{I(a)} = \frac{1}{4}\frac{I\left(\sqrt{1-r^2}\right)}{I(r)}.$$

□

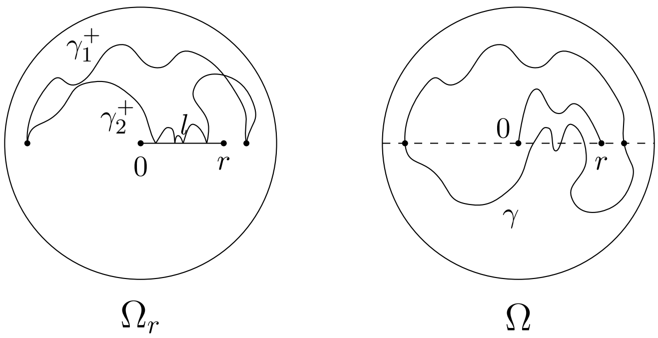
Remark 1.5.1. We call Ω_r the Grötzsch extremal region.

By the prove of the above corollary, we have

$$\frac{\mu(r)}{2} = \mu\left(\frac{2\sqrt{r}}{1+r}\right).$$

More details about the properties of elliptic functions can be found in *Conformal Invariants, Inequalities, and Quasiconformal Maps.*

Prop 1.5.2. *Let $\Omega = \{|z| \leqslant 1\}\backslash c$ where c is an arc connecting 0 and r, then $m(\Omega) \leqslant m(\Omega_r)$.*



Proof. By corollary 1.5.1, we can consider the conformal maps $f : \overline{\Omega_r^+} \to \{t \leqslant |z| \leqslant 1, \operatorname{Im} z \geqslant 0\}$ and the allowable function ρ_0.

Let γ be an arbitrary curve in Γ and divides it into two arcs γ_1 and γ_2 which both have one endpoint on each of the segments $(-1, 0)$ and $(r, 1)$.

Let γ_i^+ be obtained by reflecting the part of γ_i below the real axis.

Since it is well-defined to extend f on upper half disc.

So $f(\gamma_i^+)$ is an arc which has one endpoint on each of the segments $(-1, -t)$ and $(t, 1)$.

Notice that ρ is symmetric w.r.t. real axis.

So $L_{\gamma_i}(\rho_0) = L_{\gamma_i^+}(\rho_0) \geqslant \pi$.

Hence

$$\lambda(\Gamma) \geqslant \frac{\inf_{\gamma\in\Gamma} L_\gamma^2(\rho_0)}{-2\pi\log t} \geqslant \frac{(2\pi)^2}{-2\pi\log t} = \lambda(\Gamma_r)$$

□

Prop 1.5.3. *$d(0, w) = \log\left(\frac{1+|w|}{1-|w|}\right)$ for $w \in \mathbb{D}$.*

Prop 1.5.4. *Suppose $f : \mathbb{D} \to \mathbb{D}$ and f is 2-quasiconformal, then*

$$\frac{1}{2}d(z, w) - \log 2 \leqslant d(f(z), f(w)) \leqslant 2d(z, w) + 2\log 2.$$

Proof. WLOG, assume $w = f(w) = 0$.

Since $m(\Omega) \geqslant \frac{1}{2}m(\Omega_2)$, we have

$$\frac{\mu(|z|)}{2} \leqslant \mu(|f(z)|).$$

And so

$$|f(z)| \leqslant \mu^{-1}\left(\frac{\mu(|z|)}{2}\right) = \frac{2\sqrt{|z|}}{1+|z|}.$$

Hence

$$d(0, |f(z)|) \leqslant d\left(0, \frac{2\sqrt{|z|}}{1+|z|}\right) = 2d\left(0, \sqrt{z}\right) \leqslant 2d(0, z) + 2\log 2.$$

And the other side of inequality follows from 2-quasiconformal map f^{-1}. □

Prop 1.5.5. *Suppose $f : \mathbb{D} \to \mathbb{D}$ and f is 2^n-quasiconformal, then*

$$\frac{1}{2^n}d(z, w) - \left(2 - 2^{1-n}\right)\log 2 \leqslant d(f(z), f(w)) \leqslant 2^n d(z, w) + \left(2^{n+1} - 2\right)\log 2.$$

Proof. Let

$$\phi(z) = \mu^{-1}\left(\frac{\mu(|z|)}{2}\right) = \frac{2\sqrt{|z|}}{1+|z|}$$

Similar to proposition 1.5.4, we have

$$|f(z)| \leqslant \mu^{-1}\left(\frac{\mu(|z|)}{2^n}\right) = \phi^{(n)}(z),$$

which is the n-th iterate of the function ϕ.

Then

$$\begin{aligned} d(0, |f(z)|) &\leqslant d\left(0, \phi^{(n)}(z)\right) \\ &\leqslant 2d\left(0, \phi^{(n-1)}(z)\right) + 2\log 2 \\ &\leqslant \cdots \\ &\leqslant 2^n d(0, z) + \left(2^{n+1} - 2\right)\log 2 \end{aligned}$$

And the other side of inequality follows from 2^n-quasiconformal map f^{-1}. □

Lemma 1.5.1. *Given k and a point p on a hyperbolic geodesic γ, there is an L-quasiconformal diffeomorphism $f : \mathbb{H}^2 \to \mathbb{H}^2$ which fixes p and sends γ to itself, stretching hyperbolic distances along γ by a factor k and stretching all other distances by a factor between 1 and k.*

Proof. WLOG, assume $p = i$ and γ is the image axis, then take

$$f\left(re^{i\theta}\right) = r^k e^{i\theta}.$$

Then

$$|f_z| = \frac{k+1}{2}|z|^{k-1}, |f_{\bar{z}}| = \frac{k-1}{2}|z|^{k-1}$$

$$D_f(z) = \frac{\frac{k+1}{2}|z|^{k-1} + \frac{k-1}{2}|z|^{k-1}}{\frac{k+1}{2}|z|^{k-1} - \frac{k-1}{2}|z|^{k-1}} = k$$

□

Thm 1.5.1. *Suppose $f : \mathbb{D} \to \mathbb{D}$ and f is k-quasiconformal, then*

$$\frac{1}{k}d(z,w) - A \leqslant d(f(z), f(w)) \leqslant kd(z,w) + A,$$

where $A = A(k)$ only depends on k.

Proof. WLOG, assume $w = f(w) = 0$ and let $2^{n-1} < k \leqslant 2^n$.

Then by lemma 1.5.1, there exists a $\frac{2^n}{k}$-quasiconformal g that fix 0 and maps the geodesic connecting 0 and $f(z)$ to itself.

So by proposition 1.4.1, $g \circ f$ is 2^n-quasiconformal, *i.e.* by proposition 1.5.5,

$$\frac{2^n}{k}d(f(z), 0) = d(g \circ f(z), 0) \leqslant 2^n d(z, 0) + \left(2^{n+1} - 2\right)\log 2.$$

Therefore

$$d(0, f(z)) \leqslant kd(0, z) + k\left(2 - 2^{1-n}\right)\log 2.$$

And the other side of inequality follows from k-quasiconformal map f^{-1}. □

Lemma 1.5.2. *Consider topological annulus*

$$\Omega_p' = \mathbb{C}\backslash([-1, 0] \cup [p, +\infty)),$$

let Ω be a topological annulus and C_1, C_2 be the two components of its complement, if

$$\{-1, 0\} \subset C_1, \{p, \infty\} \subset C_2,$$

then

$$m(\Omega) \leqslant m(\Omega_p') = \frac{1}{2\pi}\mu\left(2p + 1 - 2\sqrt{p^2 + p}\right).$$

Proof. Let $f : \mathbb{D}^2 \to C_1 \cup \Omega$ be a conformal map with $f(0) = 0$ and $f(a) = -1$.

Then by Koebe quarter theorem, $|f'(0)| = 4p$.

And by Koebe distortion theorem,

$$1 = |f(a)| \leqslant \frac{|a||f'(0)|}{(1-|a|)^2} \leqslant \frac{4p|a|}{(1-|a|)^2}.$$

So $|a| \geqslant |a_0|$ where a_0 belongs to the case $\Omega = \Omega_p'$.

by proposition 1.5.2, $m(\Omega) = m(f^{-1}(\Omega)) \leqslant m(\Omega_a) \leqslant m(\Omega_{a_0}) = m(\Omega_p')$.

Moreover, we have

$$a_0 = 2p + 1 - 2\sqrt{p^2 + p}, m(\Omega_p') = \frac{1}{2\pi}\mu\left(2p + 1 - 2\sqrt{p^2 + p}\right).$$

□

Remark 1.5.2. Ω_p is called the Teichmüller extremal region.

Let Ω_r^- be the reflection of Ω_r along $\{|z|=1\}$ and $\Omega = \Omega_r \cup \Omega_r^- \cup \{|z|=1\}$.

Then $f(\Omega) = \Omega'_{r^{-2}-1}$ with $f(z) = \frac{z}{r} - 1$ and so

$$2\mu(r) = \mu\left(2r^{-2} - 1 - 2\sqrt{r^{-4} - r^{-2}}\right).$$

$$2\mu\left(\frac{2\sqrt{a}}{a+1}\right) = \mu(a).$$

We can actually use this conclusion in corollary 1.5.1 without creating circular argument.

Lemma 1.5.3. *Consider topological annulus*

$$\Omega''_\lambda = \mathbb{C}\backslash\left(\left\{|z|=1, |\arg z - \pi| \leqslant \arcsin\frac{\lambda}{2}\right\} \cup [0,+\infty)\right),$$

let Ω be a topological annulus and C_1, C_2 be the two components of its complement, if

$$\operatorname{diam}(C_1 \cap \{|z| \leqslant 1\}) \geqslant \lambda, \{0,\infty\} \subset C_2,$$

then

$$m(\Omega) \leqslant m(\Omega''_\lambda) = \frac{1}{2\pi}\mu\left(\frac{\sqrt{4+2\lambda} - \sqrt{4-2\lambda}}{4}\right).$$

Proof. Consider the map $f(z) = \sqrt{z}$.

On the image plane, we get a figure which is symmetric w.r.t. the origin with two component images of C_1 and two of C_2, denote them by $C_1^+, C_1^-, C_2^+, C_2^-$.

Take $\Omega' = \mathbb{C}\backslash(C_1^+ \cup C_1^-)$.

Then $m(\Omega) \leqslant \frac{1}{2}m(\Omega')$.

And since there exists z_1, z_2 such that $|z_1|, |z_2| \leqslant 1, |z_1 - z_2| \geqslant \lambda$.

Let $w_1, w_2 \in C_1^+$ are the image and

$$g(z) = \frac{z+w_1}{z-w_1}\frac{w_1+w_2}{w_1-w_2}.$$

Then

$$g(-w_1) = 0, g(-w_2) = 1, g(w_1) = \infty, g(w_2) = -u^2 = -\left(\frac{w_1+w_2}{w_1-w_2}\right)^2.$$

Notice that

$$|z_1+z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right) - |z_1-z_2|^2 \leqslant 4 - \lambda^2.$$

So we can obtain

$$|u| - \frac{1}{u} = \frac{2(z_1+z_2)}{z_2-z_1} \leqslant \frac{2}{\lambda}\sqrt{4-\lambda^2}, |u| \leqslant \frac{2+\sqrt{4-\lambda^2}}{\lambda},$$

Hence by lemma 1.5.2,

$$m(\Omega) \leqslant \frac{1}{2}m\left(\Omega'\right) \leqslant \frac{1}{2}m\left(\Omega'_{|u|^2}\right) = \frac{1}{2\pi}\mu\left(\sqrt{\frac{1}{|u|^2+1}}\right) \leqslant \frac{1}{2\pi}\mu\left(\frac{\sqrt{4+2\lambda}-\sqrt{4-2\lambda}}{4}\right).$$

And the equality holds when $\Omega = \Omega''_\lambda$, $i.e. m(\Omega) \leqslant m\left(\Omega''_\lambda\right)$ □

Lemma 1.5.4.

$$2\log\frac{1+\sqrt{1-r^2}}{r} \leqslant \mu(r) \leqslant \log\frac{4}{r}.$$

Proof. By mobius transformation

$$f(z) = \frac{z-a}{a-a^2z}, \frac{2a}{a^2+1} = r,$$

we map Ω_r to $\Omega = \{|z| \leqslant a\} \backslash [-1, 1]$.
And take conformal map

$$g(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Then it maps annulus $\{1 < |z| < \rho\}$ to ellipse E_ρ with axes $\rho \pm \rho^{-1}$ and slit along $[-1, 1]$.
When $\rho + \rho^{-1} \leqslant 2a$, $E_\rho \subset \Omega$ and so

$$\frac{\log \rho}{2\pi} = m(E_\rho) \leqslant m(\Omega) = \frac{\mu(r)}{2\pi}.$$

When $\rho - \rho^{-1} \geqslant 2a$, $E_\rho \subset \Omega$ and so

$$\frac{\log \rho}{2\pi} = m(E_\rho) \geqslant m(\Omega) = \frac{\mu(r)}{2\pi}.$$

Notice that $\rho = 2a - r, 4r^{-1}$ satisfies the condition of first and second case resp.
So

$$2 \log \frac{1 + \sqrt{1 - r^2}}{r} \leqslant \mu(r) \leqslant \log \frac{4}{r}.$$

□

Thm 1.5.2 (Mori)**.** *Suppose $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$ and f is k-quasiconformal, then*

$$|f(z_1) - f(z_2)| \leqslant 16|z_1 - z_2|^{\frac{1}{k}}$$

for $|z_1|, |z_2| \leqslant 1$.

Proof. We first consider the case that f can be continuously extended to the closed disk.
And then we can extend f to a k-quasiconformal map on $\mathbb{C}$ by

$$f\left(\frac{1}{\bar{z}}\right) = \frac{1}{\bar{f}(z)}.$$

If $z_1 - z_2 \geqslant 1$, then it is trivial.
So we assume $z_1 - z_2 < 1$.
Construct an annulus A whose inner circle has the segment z_1, z_2 for diameter and whose outer circle is a concentric circle of radius $\frac{1}{2}$.
Let $w_1 = f(z_1), w_2 = f(z_2)$ and C_1, C_2 be the two components of complement of A.
If $A \subset \mathbb{D}^2$, then consider

$$g(z) = \frac{z - w_1}{1 - \bar{w}_1 z}.$$

So $g \circ f(A)$ is a topological annulus and $0, g(f(z_2)) \in \partial(g \circ f(A))$ and we can obtain

$$\begin{aligned}\frac{1}{2\pi} \log \frac{1}{|z_2 - z_1|} &= m(A) \leqslant km(g \circ f(A)) \\ &\leqslant \frac{k}{2\pi} \mu\left(\left|\frac{w_2 - w_1}{1 - \bar{w}_1 w_2}\right|\right) \\ &\leqslant \frac{k}{2\pi} \log \frac{8}{|w_2 - w_1|}\end{aligned}$$

If $A\backslash\mathbb{D}^2 \neq \varnothing$, then $0 \notin A$.
So $f(A)$ is a topological annulus and $\{w_1, w_2\} \subset f(C_1) \cap \{|z| \leqslant 1\}$, $\{0, \infty\} \subset f(C_2)$.
Therefore by lemma 1.5.3,

$$\begin{aligned}\frac{1}{2\pi}\log\frac{1}{z_2 - z_1} &= m(A) \leqslant km(f(A)) \\ &\leqslant \frac{k}{2\pi}\mu\left(\frac{\sqrt{4+2|w_1-w_2|}-\sqrt{4-2|w_1-w_2|}}{4}\right) \\ &\leqslant \frac{k}{2\pi}\log\frac{4\sqrt{8+2\sqrt{16-4|w_1-w_2|}}}{|w_1-w_2|} \\ &< \frac{k}{2\pi}\log\frac{16}{|w_1-w_2|}.\end{aligned}$$

Hence we obtain the desire formula.
For the general case, we consider $\mathbb{D}_r = \{|z| < r\}$ and $g_r(z) = r^{-1}z$.
Then $g_r \circ f(rz)$ is K-quasiconformal and continuous on $|z| = 1$.
By the above case, we have

$$|g_r \circ f(rz_2) - g_r \circ f(rz_1)| < 16|z_2 - z_1|^{\frac{1}{k}}.$$

As $r \to 1$, $g_r \to \mathrm{Id}$, *i.e.* we obtain

$$|f(z_2) - f(z_1)| \leqslant 16|z_2 - z_1|^{\frac{1}{k}}$$

So f is Hölder continuous, *i.e.* it can be extended to boundary continuously. □

Coro 1.5.2. *Suppose $f : \mathbb{D} \to \mathbb{D}$ and f is k-quasiconformal, then f can be extended to a homeomorphism of the closed disk.*

Proof. By Mori theorem, f is Hölder continuous.
So it can be continuously extended to the closed disk.
And since continuous bijection from compact set to Hausdorff space is homeomorphism.
Hence $f : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ is homeomorphism. □

Coro 1.5.3. *Suppose $f_n : \mathbb{D} \to \mathbb{D}, f_n(0) = 0$, if f_n are k-quasiconformal, then $\{f_n\}$ has a subsequence such that $f_{i_n} \rightrightarrows f$ and f is k-quasiconformal.*

Proof. By Mori theorem, $\{f_n\}$ are equicontinuous.
So by Ascoli-Arzela theorem, $\{f_n\}$ has a subsequence $f_{i_n} \rightrightarrows f$.
Similarly, there exists a subsequence $f_{j_n}^{-1} \rightrightarrows g$.
WLOG, assume $i_n = j_n$ otherwise, we can take a subsequence in $\{i_n\} \cap \{j_n\}$.
So $f^{-1} = g$, *i.e.* f is homeomorphic.
For an arbitrary quadrilateral $Q \subset \mathbb{D}$, take a sequence of $Q_n \subset Q$ such that $f_{i_n}(Q_n) \subset f(Q)$.
Then $m\left(f_{i_n}(Q_n)\right) \leqslant km(Q_n)$.
And since $f_{i_n} \rightrightarrows f$, so we have

$$m(f(Q)) = \lim_{n\to\infty} m\left(f_{i_n}(Q_n)\right) \leqslant \lim_{n\to\infty} km(Q_n) = km(Q)$$

Hence f is k-quasiconformal. □

Prop 1.5.6. *Suppose $f : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ is k-quasiconformal and fixes $p, q, r \in \mathbb{S}^1$, then $d(0, f(0)) \leqslant C$, where $C = C(k)$ only depends on k.*

Proof. Suppose $f_n : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ is k-quasiconformal, f_n fixes p, q, r and $d(0, f_n(0)) \to \infty$.

WLOG, we assume $\{f_n(0)\}$ converge to a point on $\partial\mathbb{D}$ and it is not p, q.

Let $A_n \in \mathrm{Isom}(\mathbb{D})$ such that $A_n(f_n(0)) = 0$ and $g_n = A_n \circ f_n$.

Then we obtain

$$|g_n(p) - g_n(q)| = \frac{\left(1 - |f_n(0)|^2\right)|p-q|}{\left|1 - \overline{f_n(0)}p\right|\left|1 - \overline{f_n(0)}q\right|} = \frac{\left(1 - |f_n(0)|^2\right)|p-q|}{|p - f_n(0)||q - f_n(0)|} \to 0.$$

By Mori theorem, $|p - q| \leqslant 16|g_n(p) - g_n(q)|^{\frac{1}{k}}$, contradiction! □

Thm 1.5.3. *Given a compact set $E \subset \mathbb{C}$ and two point $p \neq q \in E$, there exists $C = C(E)$, such that if $f : \mathbb{C} \to \mathbb{C}$, $f(p) = p, f(q) = q$ and f is k-quasiconformal, then for $z_1, z_2 \in E$,*

$$|f(z_1) - f(z_2)| \leqslant C|z_1 - z_2|^{\frac{1}{k}}$$

Proof. Take a simply connected region $\Omega \supset E$.

Let $g : \mathbb{D} \to \Omega, h : f(\Omega) \to \mathbb{D}$ be conformal maps such that $g(0) = p, g(\alpha) = q, h(p) = 0, h(q) = \beta$ with $\alpha, \beta \in \mathbb{R}$ and denote $\phi = h \circ f \circ g$.

Then by Mori theorem,

$$|\phi(z_1) - \phi(z_2)| \leqslant 16|z_1 - z_2|^{\frac{1}{k}}.$$

And since E is compact.

Let $\left|g^{-1}(z)\right| \leqslant r_0 < 1$ for $z \in E$.

Then by Koebe distortion theorem,

$$\begin{aligned}
|g(z_1) - g(z_2)| &\geqslant \frac{|g'(z_1)|}{4}\left|\frac{z_1 - z_2}{1 - z_1\bar{z}_2}\right| \\
&\geqslant \frac{(1 - |z_1|)|g'(0)|}{64}|z_1 - z_2| \\
&\geqslant \frac{(1 - r_0)(1 - |\alpha|)^2}{64|\alpha|}|p - q||z_1 - z_2| \\
&\geqslant \frac{(1 - r_0)^3}{64|\alpha|}|p - q||z_1 - z_2|
\end{aligned}$$

On the other hand, for $|z| \leqslant r_0$, by Mori theorem, we obtain

$$1 - r_0 = 1 - |z| \leqslant 16(1 - |\phi(z)|)^{\frac{1}{k}}, |\alpha| \leqslant 16|\beta|^{\frac{1}{k}}.$$

So $|\phi(z)| \leqslant \rho_0 < 1$, where ρ_0 only depends on r_0.

So similarly, we have

$$|z_1 - z_2| \leqslant \frac{4}{(1 - \rho_0)^6|\beta|}|p - q||h(z_1) - h(z_2)|.$$

Hence

$$\begin{aligned}
|f(z_1) - f(z_2)| &\leqslant \frac{4}{(1 - \rho_0)^6|\beta|}|p - q||h \circ f(z_1) - h \circ f(z_2)| \\
&\leqslant \frac{2^{6+4k}}{(1 - \rho_0)^6|\alpha|^k}|p - q|\left(\frac{64|\alpha|}{(1 - r_0)^3|p - q|}|z_1 - z_2|\right)^{\frac{1}{k}} \\
&\leqslant C|z_1 - z_2|
\end{aligned}$$

where C only depends on E. □

1.6 quasiconformal maps on Riemann surface

Def 1.6.1. S_1, S_2 are Riemann surface, then $f : S_1 \to S_2$ is k-quasiconformal if $\tilde{f} : \mathbb{D} \to \mathbb{D}$ is k-quasiconformal.

Lemma 1.6.1. *Suppose $f : S_1 \to S_2$ is a k-quasiconformal map and $\gamma_1 \subset S_1$ a closed geodesic on S_1, let $\gamma_2 \subset S_2$ be the closed geodesic which is homotopic to $f(\gamma_1)$, then $l(\gamma_2) \leqslant kl(\gamma_1)$.*

Proof. Consider $f_* : \pi_1(S_1, *_1) \to \pi_1(S_2, *_2)$.

After identifying π_1 with $\Gamma_i < \mathrm{Aut}(\mathbb{D})$, we obtain a lift $\tilde{f} : \mathbb{D} \to \mathbb{D}$ such that

$$\tilde{f} \circ A_1 = A_2 \circ \tilde{f}, f_*(A_1) = A_2.$$

Let γ_i be the axis of A_i.

We first assume $\tilde{f}(p) \in \gamma_2$ where $p \in \gamma_1$, then

$$\begin{aligned} nl(\gamma_2) &\leqslant d\left(\tilde{f}(p), A_2^n\left(\tilde{f}(p)\right)\right) = d\left(\tilde{f}(p), \tilde{f}(A_1^n(p))\right) \\ &\leqslant kd(p, A_1^n(p)) + C = knl(\gamma_1) + C. \end{aligned}$$

So $l(\gamma_2) \leqslant kl(\gamma_1) + \frac{C}{n}$, *i.e.* $l(\gamma_2) \leqslant kl(\gamma_1)$ as $n \to \infty$.

For the general case, take a fixed point $q \in \gamma_2$, then

$$\left| d\left(\tilde{f}(p), A_2^n\left(\tilde{f}(p)\right)\right) - d\left(q, A_2^n(q)\right) \right| \leqslant 2d(\tilde{f}(p), q) = C_1.$$

□

Prop 1.6.1. *There exists no quasiconformal map $f : \mathbb{D} \to \mathbb{C}$.*

Proof. Let $\Omega = \{|z| \leqslant r_0 < 1\}$ and $f(\Omega) \subset \{|z| \leqslant R\}$.

Then by definition,

$$m(\mathbb{C} \backslash f(\Omega)) \leqslant km(\{r_0 < |z| < 1\}) = \frac{1}{2\pi} \log \frac{1}{r_0}.$$

On the other hand, for arbitrary M,

$$m(\mathbb{C} \backslash B_R) \geqslant m(\{R \leqslant |z| \leqslant M\}) = \frac{1}{2\pi} \log \frac{M}{R}.$$

Take $M \geqslant \frac{R}{r_0}$, contradiction! □

1.7 Topological definition of quasiconformal maps

Def 1.7.1. $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a homeomorphism, define

$$N_f = \{A \circ f \circ B | A, B \in \mathrm{Aut}(\mathbb{S}^2), A \circ f \circ B \text{ fixes } p, q, r \in \mathbb{S}^2\},$$

$$\mathrm{Homeo}(\mathbb{S}^2, p, q, r) = \{g : \mathbb{S}^2 \to \mathbb{S}^2 | g \text{ is homeomorphism}, g \text{ fixes } p, q, r\}.$$

Lemma 1.7.1. *If f is quasiconformal, then N_f is sequentially precompact in* $\mathrm{Homeo}(\mathbb{S}^2, p, q, r)$.

Proof. Consider the stereographic projection $g : \mathbb{S}^2 \to \bar{\mathbb{C}}$ with $g(p) = \infty$.

Then for a sequence $\{A_n \circ f \circ B_n\} \subset N_f$, $g \circ A \circ f \circ B \circ g^{-1}$ is a k-quasiconformal map on $\mathbb{C}$.

So $\{g \circ A_n \circ f \circ B_n \circ g^{-1}\}$ converges to a k-quasiconformal map $f_0 : \mathbb{C} \to \mathbb{C}$.

Take $h = g^{-1} \circ f_0 \circ g, h(p) = p$.

Hence h is a homeomorphism in $\mathrm{Homeo}(\mathbb{S}^2, p, q, r)$. □

Exam 1.7.1. *Take*

$$f(z) = \left(e^{|z|} - 1\right) e^{i\arg(z)}, B_n(z) = nz, A_n(z) = \frac{z}{e^n - 1}.$$

Then $A_n \circ f \circ B_n$ *fixes* $0, 1, \infty$ *but maps* 2 *to*

$$\frac{e^{2n} - 1}{e^n - 1} \to \infty$$

Lemma 1.7.2. *If* f *is not quasiconformal, then for any* $n > 0$*, there exists quadrilateral* Q_n *with* $m(Q_n) = 1$ *such that* $nm(f(Q_n)) < 1$

Proof. Since f is not quasiconformal.

Take quadrilateral Q such that $2nm(f(Q)) < m(Q)$.

WLOG, we assume Q is a rectangle and $m(Q) \geqslant 1$, otherwise we swap the a-sides with b-sides.

If $m(Q) = 1$, then there is nothing need to be proved, so we only consider $m(Q) > 1$.

Let $k < m(Q) \leqslant k+1$ where $k \in \mathbb{Z}^+$ and divide Q into k squares $R_1, \ldots, R_k$ and a rectangle R_{k+1}, which may also be a square, then

$$m(Q) > 2nm(f(Q)) \geqslant 2n \sum_{i=1}^{k+1} m(f(R_i)) > 2n \sum_{i=1}^{k} m(f(R_i)).$$

So there exists some $i \leqslant k$ such that

$$m(f(R_i)) < \frac{m(Q)}{2nk} \leqslant \frac{k+1}{2nk} \leqslant \frac{1}{n}.$$

Hence take $Q_n = R_i$, we have $nm(f(Q_n)) < 1$ and $m(Q) = m(R_i) = 1$. □

Thm 1.7.1 (Topological definition). *A homeomorphism* $f : \mathbb{S}^2 \to \mathbb{S}^2$ *is quasiconformal iff* N_f *is sequentially precompact subset of* $\text{Homeo}(\mathbb{S}^2, p, q, r)$.

Proof. The "only if" part is proved in lemma 1.7.1, we now prove the "if" part.

WLOG, we assume p, q, r are $0, 1, \infty$ resp.

For $z \in \mathbb{C}$, take $x, y \in \partial B_r(z)$ and $\hat{f} = A \circ f \circ B \in N_f$ such that

$$B(0) = z, B(1) = x, A(f(z)) = 0, A(f(x)) = 1, B(\infty) = A(\infty) = \infty.$$

Since A, B are affine map, we have

$$d(0, B^{-1}(y)) = 1, \frac{d(f(y), f(z))}{d(f(x), f(z))} = d(0, A(f(y))).$$

Suppose f is not quasiconformal, then there exists z_n, x_n, y_n such that

$$\left|B_n^{-1}(y_n)\right| = 1, \left|\hat{f}_n\left(B_n^{-1}(y_n)\right)\right| > n.$$

WLOG, assume $\hat{f}_n$ converges to $g \in \text{Homeo}(\mathbb{S}^2, 0, 1, \infty)$ and $B_n^{-1}(y_n)$ converges to y.

Then $|g(y)| \geqslant \lim_{n\to\infty} n = \infty$.

But $g(\infty) = \infty$ and g is injective, contradiction! □

1.8 Analytical properties of quasiconformal maps

Def 1.8.1. Given a quasiconformal map $f : U \to f(U)$, we say f is absolutely continuous on lines if in every rectangle $R \subset U$, f is absolutely continuous on almost every horizontal and almost every vertical line.

Lemma 1.8.1. *Let $f : U \to f(U)$ be quasiconformal, then f is absolutely continuous on lines.*

Proof. Take disjoint intervals $(a_1, b_1), \dots, (a_n, b_n)$ in $[\alpha, \beta]$.

Let $A(y)$ be the image area under f of the rectangle $[\alpha, \beta] \times [0, y]$.

Then A is increasing, *i.e.* derivative $A'(y)$ exists a.e.

WLOG, assume $A'(y_0)$ exists.

Consider $Q = [\alpha, \beta] \times [y_0, y_0 + \delta], Q_i = [a_i, b_i] \times [y_0, y_0 + \delta]$ and $Q' = f(Q), Q_i' = f(Q_i)$.

Let γ_0 be the line connecting a_i and b_i, $l_i = l(\gamma_0) = |b_i - a_i|, l_i' = l(f(\gamma_0))$.

We first show that for sufficiently small δ, the length of any arc γ connecting the b-side of Q_i' is near by l_i'.

Take a partition $a_i = t_0 < t_1 < \cdots < t_m = b_i$ and $\zeta_i = f(t_k, y_0)$ such that

$$\sum_{k=1}^{m} |\zeta_k - \zeta_{k-1}| \geqslant l_i' - \frac{\varepsilon}{2}.$$

Take sufficiently small δ such that the variation of f on vertical segments $\{t_k\} \times [y_0, y_0 + \delta]$ is less than $\frac{\varepsilon}{4n}$, then we have

$$l(\gamma) \geqslant \sum_{k=1}^{m} |\zeta_k - \zeta_{k-1}| - \frac{\varepsilon}{2} \geqslant l_i' - \varepsilon$$

So take $\varepsilon < \frac{1}{2}\min\{l_i'\}$, we obtain

$$\frac{(l_i')^2}{4A(Q_i)} \leqslant m_i(Q_i') \leqslant k\frac{l_i}{\delta},$$

$$\left(\sum_{i=1}^{n} l_i'\right)^2 \leqslant \sum_{i=1}^{n} \frac{(l_i')^2}{l_i} \cdot \sum_{i=1}^{n} l_i \leqslant 4K\frac{A(y_0 + \delta) - A(y_0)}{\delta} \sum_{i=1}^{n} l_i.$$

And since

$$\lim_{\delta \to 0} \frac{A(y_0 + \delta) - A(y_0)}{\delta} = A'(y_0) < +\infty,$$

Hence as $\sum l_i \to 0$, $\sum l_i' \to 0$, *i.e.* f is absolutely continuous on horizontal line.

Similarly, f is absolutely continuous on vertical line. □

Thm 1.8.1. *Let $f : U \to f(U)$ be quasiconformal, then f is differentiable a.e. on U and it is differentiable in then sense of distribution.*

Proof. By lemma 1.8.1, f is absolutely continuous on lines.

So f has partial derivatives f_x, f_y a.e.

By Egoroff theorem, the limits

$$f_x(z) = \lim_{\delta \to 0} \frac{f(z + \delta) - f(z)}{\delta}, f_y(z) = \lim_{\delta \to 0} \frac{f(z + \delta i) - f(z)}{\delta}.$$

are taken uniformly except on $U \backslash E$ of arbitrary small measure.

Then it is sufficient to prove that f is differentiable a.e. on E.

Notice that almost every point $x_0 + y_0 i \in E$ is point of density for $E \cap (\{x = x_0\} \cup \{y = y_0\})$.

So it is sufficient to prove that f is differentiable at such a point $z_0 = x_0 + y_0 i$.
And WLOG, we assume $z_0 = 0$.
Then there exists $\delta > 0$ such that for $|x|, |y|, |\delta_1|, |\delta_2| < \delta$,

$$|f_x(z) - f_x(0)| < \varepsilon, |f_y(z) - f_y(0)| < \varepsilon,$$

$$\left|\frac{f(z+\delta_1) - f(z)}{\delta_1} - f_x(z)\right| < \varepsilon, \left|\frac{f(z+\delta_2 i) - f(z)}{\delta_2} - f_y(z)\right| < \varepsilon.$$

So

$$f(z) - f(0) - xf_x(0) - yf_y(0) = (f(z) - f(x) - yf_y(x)) + (f(x) - f(0) - xf_x(0)) + y\,(f_y(x) - f_y(0))$$

If $x \in E$ or $y \in E$, we have

$$|f(z) - f(0) - xf_x(0) - yf_y(0)| \leqslant 3\varepsilon|z|.$$

Now we want to prove the case when $x, y \notin E$.
Since $\frac{m(E\cap(-x,x))}{2|x|} \to 1$ as $x \to 0$.
Take δ sufficiently small such that for $x < \delta$

$$m(E \cap (-x, x)) > \frac{2+\varepsilon}{1+\varepsilon}|x|.$$

Then $E \cap \left(\frac{x}{1+\varepsilon}, x\right) \neq \varnothing$.
Similarly, we apply this process to y.
So if $|z| < \frac{\delta}{1+\varepsilon}$, there exists $x_1, x_2, y_1, y_2 \in E$ such that

$$\frac{x}{1+\varepsilon} < x_1 < x < x_2 < (1+\varepsilon)x, \frac{y}{1+\varepsilon} < y_1 < y < y_2 < (1+\varepsilon)y.$$

Consider rectangle $(x_1, x_2) \times (y_1, y_2)$.
By the maximal principle, there exists a point z^* on the perimeter such that

$$\begin{aligned}|f(z) - f(0) - xf_x(0) - yf_y(0)| &\leqslant |f(x^* + iy^*) - f(0) - xf_x(0) - yf_y(0)| \\ &\leqslant 3\varepsilon|z^*| + |x - x^*||f_x(0)| + |y - y^*||f_y(0)| \\ &\leqslant 3\varepsilon(1+\varepsilon)|z| + \varepsilon|f_x(0)||z| + \varepsilon|f_y(0)||z|\end{aligned}$$

Hence f is differentiable a.e.
Moreover, for compact $K \subset U$, we have

$$\int_K J_f \mathrm{d}x\mathrm{d}y \leqslant A(f(K)).$$

And since f is k-quasiconformal.
Similar to proposition 1.4.2, we can prove that

$$|f_{\bar{z}}| \leqslant K|f_z|, |f_z|^2 \leqslant \frac{J}{1-K^2} \text{ with } K = \frac{k-1}{k+1}.$$

So f_x, f_y are locally square integrable.
Take a test function ϕ, which is C^1 with compact support.
By the integration over horizontal lines and Fubini theorem,

$$\iint f_x\phi \mathrm{d}x\mathrm{d}y = -\iint f\phi_x \mathrm{d}x\mathrm{d}y, \iint f_y\phi \mathrm{d}x\mathrm{d}y = -\iint f\phi_y \mathrm{d}x\mathrm{d}y.$$

□

Chapter 2

Boundary Correspondence

2.1 Quasi-isometry maps

Def 2.1.1. E is a convex subset of $\mathbb{H}^n$, then one can define the nearest point retraction map $\pi : \mathbb{H}^n \to E$: for $p \in \mathbb{H}^n$, there exists a unique point $q \in E$ such that $d(p, E) = d(p, q)$ and we set $\pi(p) = q$.

Lemma 2.1.1. $\|\mathrm{d}\pi(z)\| \leqslant \cosh(r)^{-1}$, *where* $r = d(z, E)$.

Proof. Let $r = d(z, E)$ and consider a smooth curve γ in the level set

$$\Sigma_r = \{w \in \mathbb{H}^n | d(w, E) = r\}.$$

with $\gamma(0) = z, \gamma'(0) = v \in T_z\Sigma_r$.
Then $\pi \circ \gamma$ is a curve in E with $(\pi \circ \gamma)'(0) = \mathrm{d}\pi_z(v)$.
Let $\alpha(t, -)$ be the unit-speed geodesic from $\pi(\gamma(t))$ to $\gamma(t)$ and Consider the Jacobi field

$$J(s) = \alpha\left(\frac{\partial}{\partial t}\right)\bigg|_{t=0}.$$

Then $\alpha(t, -)$ is perpendicular to ∂E and $J(0) = \mathrm{d}\pi_z(v), J(r) = v$.
And since $\mathbb{H}^n$ has constant curvature -1, so we obtain

$$J(s) = \mathrm{d}\pi(z)(v)\cosh(s) + J'(0)\sinh(s).$$

On the other hand,

$$J'(0) = \hat{\nabla}_{\frac{\partial}{\partial s}}\alpha_*\left(\frac{\partial}{\partial t}\right)\bigg|_{s=t=0} = \hat{\nabla}_{\frac{\partial}{\partial t}}\alpha_*\left(\frac{\partial}{\partial s}\right)\bigg|_{s=t=0} = \hat{\nabla}_{\frac{\partial}{\partial t}}N\bigg|_{t=0} = \nabla_{J(0)}N$$

where N is the unit normal vector to E pointing outward.
Notice that E is convex.
So the second fundamental form

$$B(J(0), J(0)) = \left\langle \nabla_{J(0)}N, J(0)\right\rangle = \langle J'(0), J(0)\rangle \geqslant 0.$$

Therefore we have

$$\begin{aligned} &|v|^2 = |J(r)|^2 \\ &= \cosh^2(r)|\mathrm{d}\pi_z(v)|^2 + \sinh^2(r)\left|J'(0)\right|^2 + 2\cosh(r)\sinh(r)\langle \mathrm{d}\pi_z(v), J'(0)\rangle \\ &\geqslant \cosh^2(r)|\mathrm{d}\pi_z(v)|^2 \end{aligned}$$

And since $\mathrm{d}\pi_z$ vanishes in the direction perpendicular to Σ_r.
Hence $\|\mathrm{d}\pi(z)\| \leqslant \cosh(r)^{-1}$. □

Def 2.1.2. A map $\gamma : \mathbb{R} \to \mathbb{H}^n$ is called (L, A)-quasigeodesic if

$$\frac{|x-y|}{L} - A \leqslant d(\gamma(x), \gamma(y)) \leqslant L|x-y| + A.$$

Lemma 2.1.2. *For $L, A > 0$, there exists A_1, L_1, D_1 depending on (L, A) such that for every (L, A)-quasigeodesic γ, there exists a (L, A_1)-quasigeodesic γ_1 such that*

(1) $l_{\mathbb{H}^n}(\gamma_1([a, b])) \leqslant L_1|a-b|$.

(2) $d_{\mathbb{H}^n}(\gamma(x), \gamma_1(x)) \leqslant D_1$ for every x.

Proof. Let $\gamma_1\big|_{[n,n+1]}$ be the geodesic from $\gamma(n)$ to $\gamma(n+1)$ with $|\gamma'(t)| \equiv C$.
Then for $n \leqslant a < n+1, m \leqslant b < m+1$,

$$\begin{aligned}
&l_{\mathbb{H}^n}(\gamma_1([a, b])) \\
&= l_{\mathbb{H}^n}(\gamma_1([a, n+1])) + \sum_{i=n+1}^{m-1} l_{\mathbb{H}^n}(\gamma_1([i, i+1])) + l_{\mathbb{H}^n}(\gamma_1[m, b]) \\
&= (n+1-a)d(\gamma(n), \gamma(n+1)) + \sum_{i=n+1}^{m-1} d(\gamma(i), \gamma(i+1)) + (b-m)d(\gamma(m), \gamma(m+1)) \\
&\leqslant (L+A)|b-a|
\end{aligned}$$

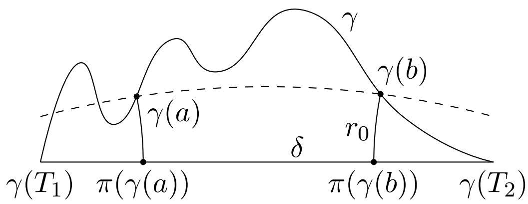
And for $n \leqslant x < n+1$,

$$d(\gamma(x), \gamma_1(x)) \leqslant d(\gamma(x), \gamma(n)) + d(\gamma_1(n), \gamma_1(x)) \leqslant 2(L+A).$$

□

Thm 2.1.1 (Morse lemma)**.** *Suppose $\gamma : I \to \mathbb{H}^n$ is an (L, A)-quasigeodesic where I is an interval either finite or infinite, then $\gamma(I)$ is within a bounded distance R from a geodesic in $\mathbb{H}^n$, where R only depends on L and A.*

Proof.



by lemma 2.1.2 we can assume γ is (L, A)-quasigeodesic and $l(\gamma([a, b])) \leqslant L|a-b|$.
We first consider the case that $I = [T_1, T_2]$ is finite.
Let δ be the geodesic arc between $\gamma(T_1)$ and $\gamma(T_2)$ and $(a, b) \subset [T_1, T_2]$ be a maximal interval such that $\gamma((a, b))$ lies outside the cylinder $B(\delta, r_0)$ where

$$\frac{2r_0L^2 + LA}{\cosh(r_0) - L^2} \leqslant 1.$$

Then we have

$$\begin{aligned}
\cosh(r_0)d(\pi(\gamma(a)), \pi(\gamma(b))) &\leqslant l(\gamma(a, b)) \leqslant L|a-b| \\
&\leqslant L(Ld(\gamma(a), \gamma(b)) + A) \\
&\leqslant L^2(2r_0 + d(\pi(\gamma(a)), \pi(\gamma(b)))) + LA
\end{aligned}$$

So

$$d(\pi(\gamma(a)), \pi(\gamma(b))) \leqslant \frac{2r_0L^2 + LA}{\cosh(r_0) - L^2} \leqslant 1.$$

Therefore $d(\gamma(a), \gamma(b)) \leqslant \cosh(r_0)$, *i.e.* $|a-b| \leqslant L\cosh(r_0) + A$.

Replace r_0 by $r_0 + L(L\cosh(r_0) + A) + A$, then $\gamma([T_1, T_2]) \subset B(\delta, r_0)$.

For general case, take a sequence of finite interval $I_1 \subset I_2 \subset \cdots \subset I$ and let δ_i be the corresponding geodesic arc.

Notice that δ_i are all L-Lipschitz.

By Ascoli-Arzela theorem and diagonal argument, δ_i converges to a geodesic δ as $i \to \infty$

Hence as $i \to \infty$, $\gamma(I) \subset B(\delta, r_0)$. □

Def 2.1.3. A map $f : \mathbb{H}^n \to \mathbb{H}^n$ is called (L, A)-quasi-isometry if

$$\frac{d(x,y)}{L} - A \leqslant d(f(x), f(y)) \leqslant Ld(x,y) + A,$$

Thm 2.1.2. *$f : \mathbb{H}^n \to \mathbb{H}^n$ is a quasi-isometry then f extend continuous to a quasiconformal map $\partial f : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$.*

We can define ∂f directly by Morse lemma, and to prove this theorem, we need some deeper properties of quasi-isometry maps. For brevity, we have placed the proof in chapter A.

2.2 M-condition

Def 2.2.1. Let $h : \mathbb{R} \to \mathbb{R}$ be a homeomorphism, we say that h satisfies the M-condition if

$$\frac{1}{M} \leqslant \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leqslant M.$$

Such h is also called quasisymmetric.

Lemma 2.2.1. *Let $f : \mathbb{H}^2 \to \mathbb{H}^2$ be k-quasiconformal such that $f(\infty) = \infty$, then $h(x) = f(x, 0)$ is an increasing homeomorphism which satisfies the M-condition for same $M = M(k)$.*

Proof. For $t \in \mathbb{R}$, By choosing affine map A, B, let $g = B \circ f \circ A$ such that

$$g(-1) = -1, g(0) = 0, g(1) = \frac{h(x+t) - h(x)}{h(x) - h(x-t)}.$$

Suppose there exists $f_m : \mathbb{H}^2 \to \mathbb{H}^2$ such that

$$\frac{h_m(x_m + t_m) - h_m(x_m)}{h_m(x_m) - h_m(x_m - t_m)} \to 0 \text{ or } \infty$$

Then we can rescaled f_n to get g_n such that

$$g_n(-1) = -1, g_n(0) = 0, g_n(1) \to 0 \text{ or } \infty.$$

And since g_n fixed $0, -1, \infty$.

So g_n converge to a k-quasiconformal $g_n \to g$.

But $g(1) = \lim_{n\to\infty} g_n(1) = 0$ or ∞, contradiction! □

Lemma 2.2.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be a homeomorphism satisfying M-condition with $h(0) = 0$, $h(1) = 1$, then*

$$\frac{1}{M+1} \leqslant \int_0^1 h(x)\mathrm{d}x \leqslant \frac{M}{M+1}.$$

Proof. Let $F(x) = \sup h_0(x)$ for all quasi-isometry h_0 with $h_0(0) = 0, h_0(1) = 1$.

Then for $0 < t < 1$,

$$\frac{h\left(\frac{t}{2}\right)}{h(t)} \leqslant F\left(\frac{1}{2}\right)$$

So

$$F\left(\frac{t}{2}\right) \leqslant F\left(\frac{1}{2}\right) F(t).$$

And similarly, we have

$$\frac{h\left(\frac{1-t}{2} + t\right) - h(t)}{1 - h(t)} \leqslant F\left(\frac{1}{2}\right).$$

Therefore

$$F\left(\frac{1+t}{2}\right) \leqslant F\left(\frac{1}{2}\right) + \left(1 - F\left(\frac{1}{2}\right)\right) F(t).$$

Thus we obtain

$$F\left(\frac{t}{2}\right) + F\left(\frac{1+t}{2}\right) \leqslant F\left(\frac{1}{2}\right) + F(t).$$

Hence

$$\begin{aligned}
\int_0^1 h(t)\mathrm{d}t \leqslant \int_0^1 F(t)\mathrm{d}t &= \frac{1}{2}\int_0^2 F\left(\frac{t}{2}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_0^1 \left(F\left(\frac{t}{2}\right) + F\left(\frac{1+t}{2}\right)\right)\mathrm{d}t \\
&\leqslant \frac{1}{2}F\left(\frac{1}{2}\right) + \frac{1}{2}\int_0^1 F(t)\mathrm{d}t \\
&\leqslant F\left(\frac{1}{2}\right) \leqslant \frac{M}{M+1}
\end{aligned}$$

□

Lemma 2.2.3 (Gauge rule)**.** *Let $f : \mathbb{H}^2 \to \mathbb{H}^2$ be k-quasiconformal with $f(\infty) = \infty$, then for any affine map A, B, take $h(x) = f(x, 0), h_1(x) = (B \circ f \circ A)(x, 0)$, prove that*

$$h_1 = B \circ h \circ A.$$

Proof. Let $A = az + b, B = cz + d$ with $a, c > 0$ and $b, d \in \mathbb{R}$, then

$$(B \circ f \circ A)(x, 0) = cf(ax + b, 0) + d = ch(ax + b) + d = B \circ h \circ A.$$

□

Thm 2.2.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a homeomorphism satisfying M-condition, then there exists a map $\phi = u + iv$ which is k-quasiconformal for $k = 2M(M + 1)$ and $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ extending h.*

Proof. Take

$$u(x, y) = \frac{1}{2y}\int_{-y}^{y} h(x + t)\mathrm{d}t, v(x, y) = \frac{1}{2y}\int_0^y (h(x + t) - h(x - t))\mathrm{d}t.$$

Then $v(x, y) \geqslant 0$ and $v(x, y) \to 0$ as $y \to 0$, *i.e.* ϕ is well-defined and $\phi(x, 0) = u(x, 0) = h(x)$.

And since we have

$$u_x = \frac{1}{2y}(h(x + y) - h(x - y)),$$

$$u_y = -\frac{1}{2y}\int_{x-y}^{x+y} h\mathrm{d}t + \frac{1}{2y}(h(x+y) + h(x-y)),$$

$$v_x = \frac{1}{2y}(h(x+y) - 2h(x) + h(x-y)),$$

$$v_y = -\frac{1}{2y^2}\left(\int_x^{x+y} h\mathrm{d}t - \int_{x-y}^{x} h\mathrm{d}t\right) + \frac{1}{2y}(h(x+y) - h(x-y)).$$

By gauge rule, we can assume $h(0) = 0, h(1) = 1$ and we only need to compute dilatation at $z = i$, then

$$u_x = \frac{1}{2}(1 - h(-1)), u_y = -\frac{1}{2}\int_{-1}^{1} h\mathrm{d}t + \frac{1 + h(-1)}{2},$$

$$v_x = \frac{1 + h(-1)}{2}, v_y = -\frac{1}{2}\left(\int_0^1 h\mathrm{d}t - \int_{-1}^0 h\mathrm{d}t\right) + \frac{1}{2}(1 - h(-1)).$$

Let

$$\xi = 1 - \int_0^1 h\mathrm{d}t, \beta = -h(-1), \eta\beta = -h(-1) + \int_{-1}^0 h\mathrm{d}t.$$

So we obtain

$$u_x = \frac{1+\beta}{2}, v_x = \frac{1-\beta}{2}, u_y = \frac{\xi - \eta\beta}{2}, v_y = \frac{1}{2}(\xi + \eta\beta).$$

$$\begin{aligned} d^2 &= \left|\frac{((1-\xi) + \beta(1-\eta)) + i((1+\xi) - \beta(1+\eta))}{((1+\xi) + \beta(1+\eta)) + i((1-\xi) - \beta(1-\eta))}\right|^2 \\ &= \frac{1 + \xi^2 + \beta^2(1+\eta^2) - 2\beta(\xi+\eta)}{1 + \xi^2 + \beta^2(1+\eta^2) + 2\beta(\xi+\eta)} \end{aligned}$$

Since we have estimates

$$M^{-1} \leqslant \beta \leqslant M, \frac{1}{M+1} \leqslant \xi, \eta \leqslant \frac{M}{M+1}.$$

Therefore

$$D = \frac{1+d}{1-d} \leqslant 2\frac{1+d^2}{1-d^2} < 2M(M+1).$$

Moreover, we must show that $\phi(z) \to \infty$ as $z \to \infty$, this is because

$$u^2 + v^2 = \frac{1}{2y}\left(\left(\int_x^{x+y} h\mathrm{d}t\right)^2 + \left(\int_{x-y}^{x} h\mathrm{d}t\right)^2\right).$$

Hence by monodromy theorem, ϕ is homeomorphic, *i.e.* ϕ is k-quasiconformal. □

Prop 2.2.1. *X and Y are Riemann surface diffeomorphic to Σ_g, then they are biholomorphic iff they are isometric w.r.t. hyperbolic metrics.*

Prop 2.2.2. *X and Y are Riemann surface diffeomorphic to Σ_g but not biholomorphic, let $f : X \to Y$ be a diffeomorphism and $g : \mathbb{H}^2 \to \mathbb{H}^2$ is the lift of f, then g is not differentiable anywhere with non-zero derivative on the boundary.*

Proof. Let Γ_X, Γ_Y be the Fuchian groups.

Then for $A \in \Gamma_X$, there exists $B \in \Gamma_Y$ such that $g \circ A = B \circ g$.

WLOG, assume $g(0) = 0$ and suppose g is differentiable at 0 with non-zero derivative.

Let $A_n(z) = \frac{z}{n}, g_n = A_n^{-1} \circ g \circ A_n$.

Since $g'(0) \neq 0$.

Take arbitrary $z_0 \in \mathbb{H}^2$ and a fundamental domain F of X.

Then there exists $\gamma_n \in \Gamma_X$ such that $\gamma_n \circ A_n(z_0) \in F$.
So by Ascoli-Arzela theorem, there exists $\{n_i\}$ such that $\gamma_{n_i} \circ A_{n_i}$ converges to $\sigma_1 : \mathbb{H}^2 \to \mathbb{H}^2$.
Therefore $A_{n_i}^{-1} \circ \gamma_{n_i}^{-1} \circ \gamma \circ \gamma_{n_i} \circ A_{n_i}$ converges to some $\sigma_1^{-1}\gamma\sigma_1$ for every $\gamma \in \Gamma_X$.
And consider $\delta_n \in \Gamma_Y$ such that $g \circ \gamma_n = \delta_n \circ g$.
Then $\delta_n A_n \left(ng\left(\frac{z_0}{n}\right)\right) \in g(F)$.
WLOG, we assume $\delta_{n_i} \circ A_{n_i}$ converges to $\sigma_2 : \mathbb{H}^2 \to \mathbb{H}^2$.
Let g_n converges to $h : \mathbb{H}^2 \to \mathbb{H}^2$ and $a = g'(0)$, then

$$\begin{aligned} h \circ \sigma_1^{-1} \circ \gamma \circ \sigma_1 &= \lim_{i\to\infty} g_{n_i} \circ A_{n_i}^{-1} \circ \gamma_{n_i}^{-1} \circ \gamma \circ \gamma_{n_i} \circ A_{n_i} \\ &= \lim_{i\to\infty} A_n^{-1} \circ g \circ \gamma_{n_i}^{-1} \circ \gamma \circ \gamma_{n_i} \circ A_{n_i} \\ &= \lim_{i\to\infty} A_n^{-1} \circ \delta_{n_i}^{-1} \circ \delta \circ \delta_{n_i} \circ g \circ A_{n_i} \\ &= \sigma_2^{-1} \circ \delta \circ \sigma_2 \circ h \end{aligned}$$

where $g \circ \gamma = \delta \circ g$ for $\gamma \in \Gamma_X$, $\delta \in \Gamma_Y$.
Notice that for $x \in \mathbb{R}$,

$$h(x) = \lim_{n\to\infty} \frac{g\left(\frac{x}{n}\right) - g(0)}{\frac{1}{n}} = ax$$

Hence we can get $h : X \to Y, x \mapsto ax$ which is linear and biholomorphic, contradiction! □

Coro 2.2.1. *There are quasisymmetric map $h : \mathbb{R} \to \mathbb{R}$ which are not absolutely continuous.*

Proof. Consider the biholomorphic map $g; \mathbb{H}^2 \to \mathbb{H}^2$ in proposition 2.2.2.
Then ∂g is not absolutely continuous. □

Using similar method, we can prove the Mostow rigidity theorem. Since the proof is so long and is not quite related to the course, it can be found in chapter A.

Thm 2.2.2 (Mostow rigidity)**.** *M, N are closed hyperbolic n-manifold with $n \geqslant 3$, then every homotopy equivalent $f : M \to N$ must be homeomorphism to an isometry.*

Lemma 2.2.4. *Let X be a closed Riemann surface with genus $g \geqslant 2$, then there is a unique closed geodesic in a free homotopy type w.r.t. parametrization.*

Proof. Consider the covering map $\pi : \mathbb{H}^2 \to X$, a free homotopy type in X corresponds to a deck transformation $F : \mathbb{H}^2 \to \mathbb{H}^2$.
For a closed geodesic γ in the given free homotopy type with $\gamma(1) = \gamma(0)$.
Consider the lifting $\tilde{\gamma} : \mathbb{R} \to \mathbb{H}^2$.
Then $\tilde{\gamma}(t + 1) = F(\tilde{\gamma}(t))$, *i.e.* $\tilde{\gamma}$ is the axis of F.
Hence γ is unique w.r.t. parametrization. □

Lemma 2.2.5. *Let X be a closed Riemann surface with genus $g \geqslant 2$ and f be an automorphism that homotopies to* Id*, then $f =$* Id*.*

Proof. Consider a figure-eight closed geodesic γ that self-intersects at p.
Since $f \simeq \mathrm{Id}$.
So $f(\gamma)$ and γ are in the same free homotopy type and are both closed geodesic.
By lemma 2.2.4, $f(\gamma) = \gamma$, in particular, $f(p) = p$.
Denote the two parts of γ divided by p by γ_1, γ_2, which are geodesics.
We only need to prove that $f(\gamma_i) = \gamma_i$ since f is an isometry.
Suppose $f(\gamma_1) = \gamma_2, f(\gamma_2) = \gamma_1$.
Then $[\gamma_2] \cdot [\gamma_1] = [f(\gamma)] = [\gamma] = [\gamma_1] \cdot [\gamma_2]$.
By Preissmann theorem, $[\gamma_1] = [a]^p$ and $[\gamma_2] = [a]^q$ for some $[a] \in \pi_1(X)$ and $p, q \in \mathbb{Z}$.
So $[\gamma] = [a]^{p+q}$, in this case, γ must be $(p + q)$-times iteration of $[a]$, contradiction! □

Prop 2.2.3. *Fix an closed Riemann surface X and let G be the group of all conformal automorphism of X, then $|G| < +\infty$.*

Proof. Suppose $|G| = \infty$, let $\{f_n\}$ be a sequence of automorphism.

Then $f_n \to f \in G$.

Let $g_n = f_n^{-1} \circ f$.

Then $g_n \to \mathrm{Id}$.

So for sufficiently large n, g_n is homotopic to Id, contradiction with lemma 2.2.5! □

Def 2.2.2. The restriction $f : \mathbb{S}^1 \to f(\mathbb{S}^1)$ is called quasisymmetric map and $f(\mathbb{S}^1)$ is called quasicircle.

Chapter 3

Beltrami differential

3.1 Beltrami differential

Def 3.1.1. $f : \Omega \to f(\Omega)$ is a diffeomorphism, its Beltrami differential is defined as

$$\mu_f(z) = \mathrm{Belt}(f)(z) = \left(\frac{f_{\bar{z}}}{f_z}\right)(z).$$

Prop 3.1.1.

$$D_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

Proof. Directly follows from proposition 1.1.2. □

Prop 3.1.2.

$$\mu_{h \circ f^{-1}} \circ f = \frac{f_z}{\overline{f_z}} \frac{\mu_h - \mu_f}{1 - \bar{\mu}_f \mu_h}.$$

Proof.

$$\begin{aligned}
\frac{(h \circ f^{-1})_{\bar{z}}}{(h \circ f^{-1})_z}(f(z)) &= \frac{h_z(z) f^{-1}_{\bar{z}}(f(z)) + h_{\bar{z}}(z)\overline{f^{-1}_z}(f(z))}{h_z(z) f^{-1}_z(f(z)) + h_{\bar{z}}(z)\overline{f^{-1}_{\bar{z}}}(f(z))} \\
&= \frac{-h_z(z)\mu_f(z) + h_{\bar{z}}(z)}{h_z(z) - h_{\bar{z}}(z)\bar{\mu}_f} \cdot \frac{f_z - f_{\bar{z}}\bar{\mu}_f}{\overline{f_z}(z) - \overline{f_{\bar{z}}}(z)\mu_f(z)} \\
&= \frac{\mu_h(z) - \mu_f(z)}{1 - \bar{\mu}_f \mu_h} \frac{f_z}{\overline{f_z}}
\end{aligned}$$

□

Coro 3.1.1. *If $\mu_f = \mu_h$ a.e., then $h \circ f^{-1}$ is conformal.*

Proof. $\mu_{h \circ f^{-1}} \circ f = 0$ a.e.
So $h \circ f^{-1}$ is conformal. □

Def 3.1.2. A Riemannian metric $\mathrm{d}s^2$ on a differentiable surface is given by

$$\mathrm{d}s^2 = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2$$

in (x, y) local coordinate. In other words,

$$\mathrm{d}s = \lambda|\mathrm{d}z + \mu\mathrm{d}\bar{z}|$$

with

$$\lambda^2 = \frac{1}{4}\left(E + G + 2\sqrt{EG - F^2}\right), \mu = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}$$

Remark 3.1.1.
$$|\mu|^2 = \frac{E + G - 2\sqrt{EG - F^2}}{E + G + 2\sqrt{EG - F^2}} < 1.$$

Prop 3.1.3. *Let $f : \Omega \to \Omega_1$ such that $\mu = \mu_f$, then there exists $\sigma(w)|\mathrm{d}w|$ on Ω_1, where $w = f(z)$, such that*
$$f^*(\sigma(w)|\mathrm{d}w|) = \mathrm{d}s.$$

Proof.
$$\begin{aligned} f^*\sigma &= \sigma(f(z))|\mathrm{d}f| \\ &= (\sigma \circ f)(z)|f_z \mathrm{d}z + f_{\bar{z}} \mathrm{d}\bar{z}| \\ &= (\sigma \circ f)(z)|f_z|(z) \cdot |\mathrm{d}z + \mu \mathrm{d}\bar{z}| = \mathrm{d}s \end{aligned}$$

So we take
$$\sigma = \frac{\lambda}{|f_z|} \circ f^{-1}.$$

□

3.2 Quasiconformal groups

Def 3.2.1. Suppose G is a group of quasiconformal maps of $\mathbb{S}^2$, we say that G is a quasiconformal group if there exists k such that all $f \in G$ is k-quasiconformal.

Exam 3.2.1. *Group of Mobius transformations M is a quasiconformal group with $k = 0$. Similarly, we can consider the conjugate group fMf^{-1} by some quasiconformal f, it is also a quasiconformal group.*

Prop 3.2.1.
$$\mu_{f \circ g}(z) = \frac{\mu_g(z) + \mu_f(g(z))\overline{\theta(z)}}{1 + \overline{\mu_g(z)}\mu_f(g(z))\overline{\theta(z)}}, \theta = \frac{g_z}{\overline{g_z}}.$$

We denote $T_g(\mu_f(g)) = \mu_{f \circ g}(z)$.

Proof. Let $h = f \circ g$.
By proposition 3.1.2,
$$\mu_f \circ g = \frac{g_z}{\overline{g_z}} \frac{\mu_h - \mu_g}{1 - \bar{\mu}_g \mu_h}$$

So we obtain
$$\mu_h(z) = \frac{\mu_f(g(z)) + \theta(z)\mu_g(z)}{\theta(z) + \mu_f(g(z))\overline{\mu_g(z)}} = \frac{\mu_g(z) + \mu_f(g(z))\overline{\theta(z)}}{1 + \overline{\mu_g(z)}\mu_f(g(z))\overline{\theta(z)}}.$$

□

We now want to prove that actually every group of quasiconformal maps must be conjugated to a group of Mobius transformations. And so we first show the following lemma.

Lemma 3.2.1. *Given a compact set $X \in \mathbb{D}$, there exists a barycenter $b(X) \in \mathbb{D}$ such that $A(b(X)) = b(A(X))$ for any $A \in \mathrm{Aut}(\mathbb{D})$. More explicitly, $b(X)$ can be given by the unique point such that*
$$\int_X \frac{z - b(X)}{1 - \overline{b(X)}z} \frac{\frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 - |z|^2)^2} = 0.$$

To proof this lemma, we need some properties from analytic.

Prop 3.2.2. *The function*

$$G_X(z) = -\int_X \log \frac{(1-|z|^2)(1-|w|^2)}{|z-w|^2 + (1-|z|^2)(1-|w|^2)} \frac{\frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2}$$

has a unique minimum in $\mathbb{D}$.

Proof.

$$\frac{(1-|z|^2)(1-|w|^2)}{|z-w|^2 + (1-|z|^2)(1-|w|^2)} = 1 - \frac{|z-w|^2}{|1-\bar{z}w|^2} = 1 - \tanh^2 d(z,w) = \operatorname{sech}^2 d(z,w).$$

$$G_X(z) = \int_X \log\cosh^2 d(z,w) \frac{\frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2}$$

Notice that $d(z,w)$ is convex function of z for fixed w since $\mathbb{D}$ is hyperbolic.
And $\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log\cosh^2 t = \operatorname{sech}^2 t > 0$.
So G_X is convex.
Since $G_X(z)$ tends to $+\infty$ as $|z| \to 1$.
Hence G has a unique minimum in $\mathbb{D}$. □

Proof of lemma 3.2.1. By proposition 3.2.2, we take the minimum $b \in \mathbb{D}$ of G_X and let

$$T_b(z) = \frac{z-b}{1-\bar{b}z}, G(z) = G_X(T_b(z)).$$

Then we have

$$\begin{aligned} G(z) &= \int_A \log\cosh^2(d(T_b(z),w)) \frac{\frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2} \\ &= \int_A \log\cosh^2(d(z,T_b(w))) \frac{\frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2} \end{aligned}$$

So

$$\nabla G(z) = \int_A \left(\frac{2z}{1-|z|^2} + \frac{2z|T_b(w)|^2 - 2T_b(w)}{|z-w|^2 + (1-|z|^2)(1-|w|^2)} \right) \frac{\frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2}$$

In particular, $\nabla G(0) = 0$, *i.e.*

$$-2\int_X T_b(w) \frac{\frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2} = 0$$

since $z = 0$ is the minimum of G.
Take $b(X) = b$, we now prove that it is conformally invariant.

$$\begin{aligned} \int_{A(X)} T_{A(b(X))}(\zeta) \frac{\frac{i}{2}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}}{(1-|\zeta|^2)^2} &= \int_X T_{A(b(X))}(A(w)) \frac{\frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2} \\ &= \int_X e^{i\theta} T_{b(X)}(w) \frac{\frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2} \\ &= 0 \end{aligned}$$

where $\zeta = A(w)$ and $\theta \in [0, 2\pi)$.
Hence $A(b(X))$ is the unique minimum of $G_{A(X)}(z)$, *i.e.* $A(b(X)) = b(A(X))$. □

Thm 3.2.1. *Every quasiconformal group G is conjugated (by a quasiconformal map) to a subgroup of* $\mathrm{Aut}(\mathbb{S}^2)$.

Proof. Consider the sets $M_z = \{\mu_f(z) | f \in G\}$ for $z \in \mathbb{S}^2$.

Then for $g \in G$, $T_g(M_{g(z)}) = \{T_g(\mu_f(g(z))) | f \in G\} = \{\mu_{f\circ g}(z) | f \in G\} = M_z$.

Let $\mu(z) = P(M_z)$ and solve Beltrami equation

$$\frac{h_{\bar{z}}}{h_z} = \mu.$$

Then we have

$$\begin{aligned} \mu_{h\circ g}(z) &= T_g(\mu_h(g(z))) = T_g(P(M_{g(z)})) \\ &= P(T_g(M_{g(z)})) = P(M_z) = \mu_h(z) \end{aligned}$$

So $h \circ g \circ h^{-1}$ is conformal for every $g \in G$. □

In the proof we actually use the fact that the Beltrami equation can be solved, this will be proved later.

3.3 Holomorphic motions

Def 3.3.1. Let X be a connected complex manifold and E be a subset of $\bar{\mathbb{C}}$, we say a map $f : E \times X \to \mathbb{S}^2$ is a holomorphic motion of E over X if

(1) For fixed $\lambda \in X$, $f(-, \lambda) : E \to \mathbb{S}^2$ homeomorphic to its image.

(2) For fixed $z \in E$, $f(z, -) : X \to \mathbb{S}^2$ is holomorphic.

(3) For any $z \in E$, $f(z, \lambda_0) = z$ for some λ_0.

Thm 3.3.1 (Schottky)**.** *If h is analytic on $\mathbb{D}$ and not equal to 0 or 1, then*

$$|h(z)| \leqslant \Phi\left(|h(0)|, \frac{1 + |z|}{1 - |z|}\right)$$

for $|z| < 1$ where $\Phi(x, y)$ is a universal strictly increasing continuous function for $x \geqslant 0$ with $\Phi(0, y) = 0$.

Lemma 3.3.1 (λ-lemma)**.** *A holomorphic motion $f(z, \lambda)$ of $E = \mathbb{D}$ over $\mathbb{D}$ can be extended to a quasiconformal map from $\bar{\mathbb{D}}$ into a closed quasidisk in $\mathbb{C}$ in the z variable, and $f(z, -)$ is holomorphic for any $z \in \bar{\mathbb{D}}$.*

Proof. Let z_1, z_2, z_3 be three points in $\mathbb{D}$ and

$$h(\lambda) = \frac{f(z_1, \lambda) - f(z_2, \lambda)}{f(z_3, \lambda) - f(z_2, \lambda)}.$$

WLOG, we assume $\lambda_0 = 0$.

Then by Schottky theorem,

$$\left|\frac{f(z_1, \lambda) - f(z_2, \lambda)}{f(z_3, \lambda) - f(z_2, \lambda)}\right| \leqslant \Phi\left(\left|\frac{z_1 - z_2}{z_3 - z_2}\right|, \frac{1 + |\lambda|}{1 - |\lambda|}\right).$$

Fixing z_2, z_3, $f(t, z)$ is bounded as $z \in \mathbb{D}$ for a fixed λ.

And as $z_1 \to z_2$, $f(z_1, \lambda)$ tends to $f(z_2, \lambda)$ since $\Phi(0, \lambda) = 0$.

So $f(-, \lambda)$ is uniformly continuous, *i.e.* it can be continuously extended to $\bar{\mathbb{D}}$.

And by definition, $f(-, \lambda)$ is quasisymetric on $\partial\mathbb{D}$, *i.e.* $f(-, \lambda)$ is quasiconformal on $\bar{\mathbb{D}}$.

On the other hand, $f(z, -)$ is the uniform limit of some holomorphic maps for $z \in \partial\mathbb{D}$.

Hence $f(z, -)$ is holomorphic for any $z \in \bar{\mathbb{D}}$. □

Thm 3.3.2. *Suppose $f(z,\lambda):\mathbb{S}^2\times\mathbb{D}\to\mathbb{S}^2$ is a holomorphic motion, then $f(-,\lambda)$ is $\frac{1+|\lambda|}{1-|\lambda|}$-quasiconformal.*

Proof. We restrict f on $B_r\times\mathbb{D}$ and denote it as f_r, where $B_r=\{z||z|\leqslant R\}$.
By λ-lemma, $f_r(-,\lambda)$ is k-quasiconformal where k does not depend on r by lemma 2.2.1.
So as $r\to\infty$, we obtain that $f(-,\lambda)$ is k-quasiconformal.
Let $\mu(z,\lambda)$ be the Beltrami differential of $f(-,\lambda)$.
Then $\mu(z,-)$ is holomorphic for any $z\in\mathbb{S}^2$.
So by Schwarz lemma, $|\mu(z,\lambda)|\leqslant|\lambda|$.
Hence $f(-,\lambda)$ is $\frac{1+|\lambda|}{1-|\lambda|}$-quasiconformal. □

Def 3.3.2. Cross ratio of $z_1,z_2,z_3,z_4\in\mathbb{S}^2$ is

$$\operatorname{Cr}(z_1,z_2,z_3,z_4)=\frac{z_4-z_1}{z_4-z_3}\cdot\frac{z_2-z_3}{z_2-z_1}.$$

Prop 3.3.1. *Let f be a k-quasiconformal map, giving a compact set $S\subset\mathbb{S}^2\backslash\{0,1,\infty\}$, then*

$$\{\operatorname{Cr}(f(z_1),f(z_2),f(z_3),f(z_4))|\operatorname{Cr}(z_1,z_2,z_3,z_4)\in S\}$$

is compact.

Proof. Suppose the conclusion is wrong.
Then there exists $(z_1^n,z_2^n,z_3^n,z_4^n)$ such that $\operatorname{Cr}(z_1^n,z_2^n,z_3^n,z_4^n)\in E$ but

$$\operatorname{Cr}(f(z_1^n),f(z_2^n),f(z_3^n),f(z_4^n))\to x\in\{0,1,\infty\}.$$

Let A_n be the Mobius transformation that maps z_1^n,z_2^n,z_3^n to $0,1,\infty$ resp.
Then $\operatorname{Cr}(z_1^n,z_2^n,z_3^n,z_4^n)=A_n(z_4^n)$.
WLOG, we assume $A_n(z_4^n)\to z\in E$ by passing to a subsequence.
Similarly, consider B_n that maps $f(z_1^n),f(z_2^n),f(z_3^n)$ to $0,1,\infty$ resp.
Then $\operatorname{Cr}(f(z_1^n),f(z_2^n),f(z_3^n),f(z_4^n))=B_n(f(z_4^n))$.
Let $g_n=B_n\circ f\circ A_n^{-1}$.
Then g_n fixes $0,1$ and ∞.
By lemma 1.7.1, we assume $g_n\to g$ uniformly by passing to subsequence.
So $\lim\limits_{n\to\infty}g_n(A_n(z_4^n))=g(z)=x\in\{0,1,\infty\}$, contradiction! □

We now proof the converse statement.

Thm 3.3.3. *Let $f:\mathbb{C}\to\mathbb{C}$ be a homeomorphism and take a fixed point $z_0\in\mathbb{C}\backslash\mathbb{R}$, if $N_f(z_0)$ is compact in $\hat{\mathbb{C}}\backslash\{0,1,\infty\}$, then f is quasiconformal.*

Proof. Suppose f is not quasiconformal, *i.e.* there exists z_n,x_n,y_n with $x_n,y_n\in\partial B_{r_n}(z_n)$ such that

$$\frac{|f(y_n)-f(z_n)|}{|f(x_n)-f(z_n)|}\to\infty.$$

Take Mobius transformations

$$A_n(z)=z_n+(x_n-z_n)z,B_n(z)=\frac{z-f(z_n)}{f(x_n)-f(z_n)}$$

Then $A_n(0)=z_n,A_n(1)=x_n,B_n(f(z_n))=0,B_n(f(x_n))=1,A_n(\infty)=B_n(\infty)=\infty$.
So $g_n=B_n\circ f\circ A_n\in N_f$, *i.e.* $\{g_n(z_0)\}$ is compact in $\hat{\mathbb{C}}\backslash\{0,1,\infty\}$.
Around z_n, we consider $f\circ A_n$ locally as an $\mathbb{R}$-linear map D_n.

WLOG, we choose x_n, y_n to be the short and long axis of ellipse $D_n(z_0)$ resp. , then

$$|g_n(z_0)| = \frac{|f(A_n(z_0)) - f(z_n)|}{|f(x_n) - f(z_n)|} \approx \frac{|x_0 D_n(1) + y_0 D_n(i)|}{|D_n(1)|} \to \infty,$$

since $y_0 \neq 0$ and $\frac{|D_n(i)|}{|D_n(1)|} \to \infty$.
Contradiction!
Hence f is quasiconformal. □

Remark 3.3.1. Actually, I don't know how to write the proof restrictly. This is just a sketch and I think it can work after some deeper discussions.

Exam 3.3.1. *When $z_0 \in \mathbb{R}\backslash\{0,1\}$, f may not be quasiconformal.*

Take a quasisymmetric $h : \mathbb{R} \to \mathbb{C}$ with zero derivative almost everywhere and set $f = x + ih(y)$.

Then we can easily check that f is quasisymmetric on every line and circle, while four points with real cross ratio must contained in the same line or circle.

So $N_f(z_0) \subset \mathbb{S}^2\{0, 1, \infty\}$ is compact, but f is not quasiconformal, since it is obviously not absolutely continuous on lines.

Thm 3.3.4 (Slodkowski)**.** *Every holomorphic motion $f : E \times \mathbb{D} \to \mathbb{S}^2$ can be extended to holomorphic motion $f : \mathbb{S}^2 \times \mathbb{D} \to \mathbb{S}^2$.*

3.4 Two integral operators

We now start to solve the Beltrami equation.

Def 3.4.1. Let $h \in L^p(\mathbb{C})$ with $p > 2$, the Cauchy operator is

$$Ph(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}} h(z) \left(\frac{1}{z - \zeta} - \frac{1}{z} \right) \mathrm{d}x\mathrm{d}y.$$

Lemma 3.4.1. Ph *is a continuous function with is Hölder with exponent* $1 - \frac{2}{p}$.

Proof. By Hölder inequality,

$$\begin{aligned} |Ph(\zeta)| &= \left| \frac{\zeta}{\pi} \int_{\mathbb{C}} h(z) \cdot \frac{1}{z(z-\zeta)} \mathrm{d}x\mathrm{d}y \right| \\ &\leqslant \frac{|\zeta|}{\pi} \|h\|_p \left\| \frac{1}{z(z-\zeta)} \right\|_q \end{aligned}$$

And since

$$\int_{\mathbb{C}} |z(z-\zeta)|^{-q} \mathrm{d}x\mathrm{d}y = |\zeta|^{2-2q} \int |z(z-1)|^{-q} \mathrm{d}x\mathrm{d}y$$

So

$$|Ph(\zeta)| \leqslant \frac{1}{\pi} \left\| \frac{1}{z(z-1)} \right\|_q \|h\|_p |\zeta|^{1-\frac{2}{p}}.$$

Set $h_1(z) = h(z + \zeta_1)$, then

$$\begin{aligned} |Ph(\zeta_2) - Ph(\zeta_1)| &= |Ph_1(\zeta_2 - \zeta_1)| \\ &\leqslant K_p \|h\|_p |\zeta_1 - \zeta_2|^{1-\frac{2}{p}} \end{aligned}$$

□

Def 3.4.2. For $h \in C_0^2(\mathbb{C})$, we define

$$Th(\zeta) = \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{|z-\zeta|>\varepsilon} \frac{h(z)}{(z-\zeta)^2} \mathrm{d}x\mathrm{d}y$$

Lemma 3.4.2. *If* $h \in C_0^2$, *then* Th *is* C^1 *and*

$$(Ph)_z = Th, (Ph)_{\bar{z}} = h, \int |Th|^2 \mathrm{d}x\mathrm{d}y = \int |h|^2 \mathrm{d}x\mathrm{d}y.$$

Proof. Notice that

$$\begin{aligned} (Ph)_{\bar{\zeta}} &= -\frac{1}{\pi} \int \frac{h_{\bar{z}}}{z-\zeta} \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2\pi i} \int \frac{h_{\bar{z}}}{z-\zeta} \mathrm{d}z\mathrm{d}\bar{z} \\ &= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z-\zeta|>\varepsilon} \frac{h_{\bar{z}}}{z-\zeta} \mathrm{d}z\mathrm{d}\bar{z} \\ &= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z-\zeta|=\varepsilon} \frac{h\mathrm{d}z}{z-\zeta} = h(\zeta) \end{aligned}$$

$$\begin{aligned} \int |Th|^2 \mathrm{d}x\mathrm{d}y &= -\frac{1}{2i} \int (Ph)_z \overline{(Ph)_z} \mathrm{d}z\mathrm{d}\bar{z} \\ &= \frac{1}{2i} \int Ph \left(\overline{Ph}\right)_{\bar{z}z} \mathrm{d}z\mathrm{d}\bar{z} \\ &= \frac{1}{2i} \int Ph \bar{h}_{\bar{z}} \mathrm{d}z\mathrm{d}\bar{z} \\ &= -\frac{1}{2i} \int (Ph)_{\bar{z}} |h|^2 \\ &= \int |h|^2 \mathrm{d}x\mathrm{d}y \end{aligned}$$

□

Remark 3.4.1. This implies that Ph actually gives an solution of the $\bar{\partial}$ problem.

Thm 3.4.1 (Calderon-Zygmund)**.** *Let* $h \in L^p(\mathbb{C})$ *with* $p > 1$, *then* $\|Th\|_p \leqslant C_p \|h\|_p$ *where* $c_p \to 1$ *as* $p \to 2$.

Conj 3.4.1. $\|T\|_p = p^* - 1$ *where* $p^* = \max\left\{p, \frac{p}{p-1}\right\}$.

Remark 3.4.2. The inequality $\|T\|_p \geqslant p * -1$ is known.

3.5 Beltrami equations

Thm 3.5.1. *Suppose* μ *has compact support and* $\|\mu\|_\infty \leqslant k < 1$, *fix* $p > 2$ *such that* $kC_p < 1$, *then there exists a unique solution of* $f_{\bar{z}} = \mu f_z$ *such that* $f(0) = 0$ *and* $f_z - 1 \in L^p(\mathbb{C})$.

Proof. We first proof the uniqueness.
Since $f_{\bar{z}} = f_z \mu$.
So $f_{\bar{z}} \in L^p$ and $P(f_{\bar{z}})$ is well-defined.
Take $F = f - P(f_{\bar{z}})$.
Then $F_{\bar{z}} = 0$ a.e., *i.e.* F is holomorphic.
Since $F' = F_z = f_z - T(f_{\bar{z}})$.
Therefore $F' - 1 \in L^p(\mathbb{C})$, *i.e.* $F' - 1 = 0$.

Thus $f(z) = z + P(f_{\bar{z}})$.
Suppose f, g is the solution of $f_{\bar{z}} = \mu f_z$, then

$$f_z - g_z = T(f_{\bar{z}} - g_{\bar{z}}) = T(\mu(f_z - g_z))$$

So by Calderon-Zygmund,

$$\|f_z - g_z\|_p \leqslant c_p \|\mu(f_z - g_z)\| \leqslant kc_p \|f_z - g_z\|,$$

contradiction!
Define $h = T\mu + T\mu T\mu + \cdots$.
Since linear operator $h \mapsto T(\mu h)$ has norm $kC_q < 1$.
So h is well-defined.
Take $f = P(\mu(h+1)) + z$.
Then $f_z = T(\mu(h+1)) + 1 = h + 1, f_{\bar{z}} = \mu(h+1)$. □

Coro 3.5.1. *Suppose $\mu_k \to \mu$ pointwisely a.e., $\|\mu_k\|, \|\mu\| < k$ and* supp μ_k, supp $\mu \subset B(R)$, *then $f_n \rightrightarrows f$.*

Proof.
$$\begin{aligned}\|(f_n)_z - f_z\|_p &\leqslant \|T\mu((f_n)_z - f_z)\|_p + \|T(\mu - \mu_n)(f_n)_z\|_p \\ &\leqslant kC_p\|(f_n)_z - f_z\| + C_p\|(\mu_n - \mu)f_z\|_p\end{aligned}$$

As $n \to \infty$, $\|\mu_n - \mu\|$ tends to 0.
So $\|(f_n)_z - f_z\|_p \to 0$, *i.e.* $f_n \rightrightarrows f$. □

Lemma 3.5.1. *If μ is smooth with compact support, then f is a topological mapping.*

Proof. Let $f_z = \lambda$ and so $\mu\lambda = f_{\bar{z}}$, then

$$\lambda_{\bar{z}} = (\mu\lambda)_z = \lambda_z\mu + \lambda\mu_z.$$

$$(\log\lambda)_{\bar{z}} = \mu(\log\lambda)_z + \mu_z.$$

Take $\sigma = \log\lambda$, then
$$\sigma_{\bar{z}} = \mu\sigma_z + \mu_z.$$

Let $q \in L^p(\mathbb{C})$ such that
$$q = T(\mu q) + T\mu_z.$$

If such q exists, then we can let

$$\sigma = P(\mu q + \mu_z) + C.$$

To find q, we consider operator $T : L^p \to L^p, q \mapsto T(\mu q) + T\mu_z$.
By Schauder fixed point theorem, q exists.
So $|f_z| = |e^\sigma| > 0$, *i.e.* f is a topological mapping. □

Thm 3.5.2. *If μ with compact support and $\|\mu\|_\infty \leqslant k < 1$, then the solution of Beltrami equation is quasiconformal homeomorphism.*

Thm 3.5.3. *For any measurable function μ on $\mathbb{C}$ with $\|\mu\|_\infty \leqslant k < 1$, there exists a unique quasiconformal map f^μ such that $f^\mu_{\bar{z}} = \mu f^\mu_z$ a.e., and $f^\mu(0) = 0, f^\mu(1) = 1, f^\mu(\infty) = \infty$.*

Proof. We first consider the special case that $\mu = 0$ near 0.

Then take

$$\hat{\mu}(z) = \mu\left(\frac{1}{z}\right)\frac{z^2}{\bar{z}^2}.$$

So $\hat{\mu}$ has compact support and

$$f^{\mu}(z) = \frac{1}{f^{\hat{\mu}}\left(\frac{1}{z}\right)}.$$

For general case, take $\mu = \mu_1 + \mu_2$ where μ_1 has compact support and $\mu_2 = 0$ near 0.

Take

$$\lambda = \left(\frac{\mu - \mu_2}{1 - \mu\bar{\mu}_2}\frac{f_z^{\mu_2}}{\overline{f_z^{\mu_2}}}\right) \circ (f^{\mu_2})^{-1}.$$

Then λ has compact support and

$$f^{\mu} = f^{\lambda} \circ f^{\mu_2}.$$

□

Thm 3.5.4. *Suppose X is a Riemann surface, and let μ be a $(-1,1)$ form on X with $\|\mu\|_{\infty} < 1$, then there exists a new Riemann surface X^{μ} and a quasiconformal map $f^{\mu} : X \to X^{\mu}$ such that*

$$\bar{\partial} f^{\mu} = \mu \partial f^{\mu}.$$

Proof. Let $X = \mathbb{H}/\Gamma$ and lift μ to $\tilde{\mu}$ on $\mathbb{H}$ satisfying that

$$(\tilde{\mu} \circ A)\frac{\overline{A'}}{A'} = \tilde{\mu},$$

for any $A \in \Gamma$.

Extend $\tilde{\mu}$ to $\mathbb{C}$ by taking $\overline{\tilde{\mu}(\bar{z})}$ under the real-axis and take $f = f^{\tilde{\mu}}$, then

$$\begin{aligned}
\mathrm{Belt}(f \circ A) &= (\mathrm{Belt}(f) \circ A)\frac{\overline{A'}}{A'} \\
&= (\tilde{\mu} \circ A)\frac{\overline{A'}}{A'} \\
&= \bar{\mu} = \mathrm{Belt}(f)
\end{aligned}$$

So $f \circ A = B \circ f$ for some Mobius transformation B.

Hence $\Gamma_{\mu} = f \circ \Gamma \circ f^{-1}$ is a Fuchsian group and we can take $X^{\mu} = \mathbb{H}/\Gamma_{\mu}$. □

Remark 3.5.1. We can extend μ arbitrarily and use Riemann mapping theorem to let f maps $\mathbb{D}$ to $\mathbb{D}$.

Exam 3.5.1. *Take $f(x,y) = (2x, y)$ in $0 \leqslant x \leqslant 1$ and $f(x,y) = \left(\frac{1}{2}x + \frac{3}{2}, y\right)$.*

Then we extend f to $\mathbb{C}$ by $f(x + 3n, y) = f(x,y) + 3n$ and so

$$\mathrm{Belt}(f)(x,y) = \begin{cases} \frac{1}{3} & 3n \leqslant x \leqslant 3n+1 \\ -\frac{1}{3} & 3n+1 \leqslant x \leqslant 3n+3 \end{cases}$$

Let $f_n(z) = \frac{1}{n} f(nz)$.

Then $f_n(z) \to z$ as $n \to \infty$, but $\mu_n \not\to 0$ in distribution sense.

Moreover, $\mu_u \to -\frac{1}{9}$ in weak topology.*

3.6 Decomposition of quasiconformal maps

Thm 3.6.1. *Let f be a k-quasiconformal map and $0 < t < 1$, then $f = f_2 \circ f_1$ where f_1 is K^t-quasiconformal and f_2 is K^{1-t}-quasiconformal.*

Proof. Let $\frac{\mu_1(z)}{\mu(z)} \in \mathbb{C}$ with $d(0, \mu_1(z)) = td(0, \mu(z))$, then

$$\frac{1 + |\mu_1(z)|}{1 - |\mu_1(z)|} = \left(\frac{1 + |\mu(z)|}{1 - |\mu(z)|}\right)^t.$$

So $f_1 = f^{\mu_1}$ is K^t-quasiconformal.

Take $f_2 = f \circ f_1^{-1}$ and $\mu_2 = \mathrm{Belt}(f_2)$, then

$$|\mu_2(f_1(z))| = \left|\frac{\mu - \mu_1}{1 - \bar{\mu}_1 \mu}\right|$$

Therefore we obtain

$$\log\left(\frac{1 + |\mu_2(f_1(z))|}{1 - |\mu_2(f_1(z))|}\right) = 2d(\mu_1(z), \mu(z)) = (1 - t)\log\left(\frac{1 + |\mu(z)|}{1 - |\mu(z)|}\right).$$

Hence f_2 is K^{1-t} quasiconformal. □

Coro 3.6.1. *Let $\varepsilon > 0$ and f be a k-quasiconformal map, then $f = f_1 \circ \cdots \circ f_n$ where $(1+\varepsilon)$-quasiconformal maps $f_1, \ldots, f_n$ for sufficiently large n.*

Proof. Take n such that $(1 + \varepsilon)^n > k$ and $g_1 = f$.

By the above theorem, we construct $g_i = f_i \circ g_{i+1}$ where f_i is $K^{\frac{1}{n}}$-quasiconformal inductively. □

Conj 3.6.1. *Let $\varepsilon > 0$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a k-quasiconformal map, are there $(1+\varepsilon)$-quasiconformal maps $f_1, \ldots, f_k$, such that $f = f_1 \circ \cdots \circ f_k$.*

Conj 3.6.2. *Let $\varepsilon > 0$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a L-biLipschitz map, are there $f_1, \ldots, f_r$ which are $(1+\varepsilon)$-biLipschitz such that $f = f_1 \circ \cdots \circ f_k$.*

3.7 Dependence on parameter

Def 3.7.1. Let ν be an essentially bounded measurable function in $\mathbb{C}$, for small $t > 0$ with $\|t\nu\|_\infty < 1$, we denote

$$f^{t\nu}(z) = z + t\dot{f}[\nu](z) + o(t), i.e.\ \dot{f}[\nu](z) = \lim_{t\to 0} \frac{f^{t\nu}(z) - z}{t}$$

Prop 3.7.1. *Suppose ν is compact support and Let $f^{t\nu}$ be the unique solution such that $f(0) = 0$ and $f_z - 1 \in L^p$, then*

$$\dot{f}[\nu](z) = -\frac{1}{\pi}\int_{\mathbb{C}} \nu(w)\left(\frac{1}{w - z} - \frac{1}{w}\right) \mathrm{d}u\mathrm{d}v$$

Proof.

$$f^{t\nu} = z + P\left(t\nu + t\nu T(t\nu) + \cdots\right).$$

So we obtain

$$\dot{f}[\nu](z) = P(\nu) = -\frac{1}{\pi}\int_{\mathbb{C}} \nu(w)\left(\frac{1}{w - z} - \frac{1}{w}\right) \mathrm{d}u\mathrm{d}v$$

□

Thm 3.7.1. *Let $f^{t\nu}$ be the unique solution fixing $0, 1$ and ∞, which we call the normalized quasiconformal map, then*

$$\dot{f}[\nu](z) = -\frac{1}{\pi}\int_{\mathbb{C}} \nu(w)\frac{z(z-1)}{w(w-1)(w-z)}\mathrm{d}u\mathrm{d}v$$

Coro 3.7.1. *Let $\{\mu(t)\}$ be an family of Beltrami coefficients depending on a real or complex parameter t such that $\mu(t)$ is differentiable at $t = 0$, that is,*

$$\mu(t)(z) = \mu(z) + t\nu(z) + t\varepsilon(t)(z)$$

for $z \in \mathbb{C}$ and $\nu, \varepsilon(t) \in L^\infty(\mathbb{C})$ such that $\|\varepsilon(t)\|_\infty \to 0$ as $t \to 0$. Then for normalized quasiconformal maps $f^{\mu(t)}$,

$$\dot{f}[\nu](z) = -\frac{1}{\pi}\int_{\mathbb{C}} \nu(w)\frac{f^\mu(z)(f^\mu(z)-1)\left((f^\mu)_z(w)\right)^2}{f^\mu(w)(f^\mu(w)-1)(f^\mu(w)-f^\mu(z))}\mathrm{d}u\mathrm{d}v$$

Chapter 4

Teichmuller space

4.1 Integrable holomorphic quadratic differential

Def 4.1.1. X is a Riemann surface, a holomorphic quadratic differential ϕ on X is a holomorphic $(2,0)$-form.

Prop 4.1.1. *Let $\phi = \phi(z)\mathrm{d}z^2$ be a holomorphic quadratic differential, then $|\phi|$ is a volume form on X.*

Proof. Under holomorphic change of coordiantes, $\phi(A(w))A'(w)^2 = \tilde{\phi}(w)$. □

Def 4.1.2. Denote the integrable quadratic differentials by $\mathrm{MQD}^1(X)$ and integrable holomorphic quadratic differentials by $\mathrm{QD}^1(X)$.

We define $B^\infty(X) = \{\mu|\mu \text{ is } (-1,1) \text{ measurable form with } \|\mu\|_\infty < +\infty\}$.

Prop 4.1.2. *Take $\phi \in \mathrm{QD}^1(X), \mu \in B^\infty(X)$, then $\phi\mu$ is a volume form.*

Proof. Under holomorphic change of coordiantes,

$$\tilde{\phi}(w) = (\phi\mu)(A(w))\frac{\overline{A'(w)}}{A'(w)}A'(w)^2 = (\phi\mu)(A(w))A'(w)\overline{A'(w)}$$

□

Prop 4.1.3. *Suppose $\lambda : \mathrm{QD}^1(X) \to \mathbb{C}$ is a bounded linear function, then it can extend to $\lambda : \mathrm{MQD}^1(X) \to \mathbb{C}$ with the same norm and there exits $\mu \in B^\infty(X)$ such that for any $\phi \in \mathrm{QD}^1(X)$, we have*

$$\lambda(\phi) = \int_X \mu\phi, \|\mu\|_\infty = \|\lambda\|.$$

Moreover, μ is unique if $\dim \mathrm{QD}^1(X) < \infty$.

Proof. By Hahn-Banach theorem, we can extend λ to $\lambda : \mathrm{MQD}^1(X) \to \mathbb{C}$ with the same norm.

And by Riesz representation theorem, such μ exists.

For the uniqueness, suppose $\|\lambda\| = 1$ and $\dim \mathrm{QD}^1(X) < \infty$.

Then there exists $\phi_0 \in \mathrm{QD}^1(X)$ such that $|\lambda(\phi_0)| = \|\phi_0\|_1$, so

$$\|\phi_0\|_1 = \int_x \mu\phi_0 \leqslant \|\mu\|_\infty\|\phi_0\|_1.$$

Hence the only possibility is that $\mu = \frac{|\phi_0|}{\phi_0}$. □

Exam 4.1.1. *Suppose $X = \{z | 0 < \operatorname{Im} z < 1\}$ is the strip, $\lambda(\phi) = \int_X \phi$ with $\phi \mathrm{d}z^2 \in \mathrm{QD}^1(X)$.*
Take $\phi_n = \frac{1}{n+z^2}$, then

$$\int_X \phi_n = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{n+x^2}, \int_X |\phi_n| = \int_X \frac{1}{n\left|1+\frac{z^2}{n}\right|} = \frac{\sqrt{n}}{n} \int_0^1 \mathrm{d}y \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\left|1+\left(x+\frac{y}{\sqrt{n}}\right)^2\right|}$$

So

$$\lim_{n\to\infty} \frac{|\lambda(\phi_n)|}{\|\phi_n\|_1} = 1.$$

Hence $\|\lambda\| = 1$.

Exam 4.1.2. *Suppose $X = \mathbb{H}$, $\lambda(\phi) = \int_X \phi$ with $\phi \mathrm{d}z^2 \in \mathrm{QD}^1(X)$.*
Then by Cauchy theorem, $\lambda(\phi) = 0$ since ϕ is integrable.

Lemma 4.1.1. *Meromorphic $(2,0)$-forms with first order poles on $E = \partial\mathbb{D}$ are dense in $\mathrm{QD}^1(\mathbb{D})$*

Proof. Let $L \subset \mathrm{QD}^1(\mathbb{D})$ be the closure of such meromorphic functions.
Then we want to show $L = \mathrm{QD}^1(\mathbb{D})$, suppose not.
So there exists a nonzero bounded linear functional $\lambda : \mathrm{QD}^1(\mathbb{D}) \to \mathbb{C}$ such that $\lambda(L) = 0$.
By proposition 4.1.3, we take such unique $\mu \in B^\infty(\mathbb{D})$.
Let

$$Q(z) = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{\mu(w)}{w-z} \mathrm{d}u\mathrm{d}v.$$

Then $Q_{\bar{z}}(z) = \mu(z)$, $Q(z) = 0$ for $|z| = 1$ and Q is continuous on $\mathbb{C}$.
Suppose $\phi \mathrm{d}z^2 \in \mathrm{QD}^1(\mathbb{D})$ is holomorphic in some neighborhood of $\mathbb{D}$, then

$$\int_{\mathbb{D}} \mu(z)\phi(z)\mathrm{d}z\mathrm{d}\bar{z} = \int_{\mathbb{D}} Q_{\bar{z}}(z)\phi(z)\mathrm{d}z\mathrm{d}\bar{z} = \int_{\partial\mathbb{D}} Q(z)\phi(z)\mathrm{d}z = 0.$$

Now for any $\phi \mathrm{d}z^2 \in \mathrm{QD}^1(\mathbb{D})$, we let $\phi_r(z) = \phi(rz)$ for $r < 1$, then

$$\int \mu(z)\phi_r(z)\mathrm{d}z\mathrm{d}\bar{z} = 0$$

So $\int \mu\phi = 0$ since $\phi_r \xrightarrow{L^1} \phi$ as $r \to 1$, contradiction! □

Def 4.1.3. Suppose $X = \mathbb{D}/\Gamma$ is a Riemann surface, we define the Poincaré series operator as

$$\begin{aligned} \theta \colon \mathrm{QD}^1(\mathbb{D}) &\longrightarrow \mathrm{QD}^1(X) \\ \phi \mathrm{d}z^2 &\longmapsto \sum_{A\in\Gamma} (\phi \circ A)(A')^2 \mathrm{d}z^2 \end{aligned}$$

Prop 4.1.4. *$\theta : QD^1(\mathbb{D}) \to QD^1(X)$ is well-defined and $\|\theta\| \leqslant 1$.*

Proof. For $A \in \Gamma$ and $w = A(z)$,

$$\begin{aligned} \theta\left(\phi \mathrm{d}z^2\right)(w) &= \sum_{B\in\Gamma} \phi(B(w)) \cdot (B'(w))^2 \mathrm{d}w^2 \\ &= \sum_{B\in\Gamma} \phi(B \circ A(z)) \cdot \left(B'(A(z)) \cdot A'(z)\right) \mathrm{d}z^2 \\ &= \sum_{B\in\Gamma} \phi(B(z))(B'(z))^2 \mathrm{d}z^2 \end{aligned}$$

Moreover, let F be a fundamental domain of X, then

$$\begin{aligned}\left\|\theta\left(\phi \mathrm{d}z^2\right)\right\|_1 &= \int_F \left|\sum_{A\in\Gamma}(\phi\circ A)(A')^2\right| \mathrm{d}x\mathrm{d}y \\ &\leqslant \sum_{A\in\Gamma}\int_F |\phi(A(z))|\left|A'(z)\right|^2 \mathrm{d}x\mathrm{d}y \\ &= \sum_{A\in\Gamma}\int_{A(F)} |\phi(z)|\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{D}} |\phi(z)|\mathrm{d}x\mathrm{d}y = \|\phi\|_1\end{aligned}$$

□

Thm 4.1.1. *Given $\psi \in \mathrm{QD}^1(X)$, there exists $\phi \in \mathrm{QD}^1(\mathbb{D})$ such that $\theta(\phi) = \psi$ and $\|\phi\| \leqslant 3\|\psi\|$.*

Remark 4.1.1. We will prove this in next section.

Thm 4.1.2. *Let $\nu \in B^\infty(\mathbb{H}/\Gamma)$, then*

(1) $\dot{f}[\nu](z) = 0$ for any $z \in \partial\mathbb{H}$ iff $\int_X \nu\phi = \int_{\mathbb{H}/\Gamma}\nu\phi = 0$ for any $\phi \in QD^1(X) \cong QD^1(\mathbb{H}/\Gamma)$.

(2) Moreover, in this case there exists $\delta(t) \in \mathrm{Belt}(\mathbb{H}/\Gamma)$ such that

a) $\|\delta(t)\| \leqslant 12t^2$.
b) $f^{t\nu}(z) = f^{\delta(t)}(z)$ for $z \in \partial\mathbb{H}$.

Proof. (1) By theorem 3.7.1

$$\dot{f}[\nu](z) = -\frac{1}{\pi}\int_{\mathbb{C}} \nu(w)\left(\frac{1}{w-z} - \frac{z}{w-1} - \frac{1}{w}\right)$$

So $\left(\dot{f}[\nu]\right)_{\bar{z}} = \nu$.

For meromorphic map ϕ with simple pole in $\partial\mathbb{H}$,

$$\begin{aligned}\int_{\mathbb{H}} \nu\phi\mathrm{d}z\mathrm{d}y &= \frac{1}{2i}\int_{\mathbb{H}}\left(\dot{f}[\nu]\right)_{\bar{z}}\phi\mathrm{d}z\mathrm{d}\bar{z} \\ &= \frac{1}{2i}\int_{\partial\mathbb{H}}\dot{f}[\nu]\phi\mathrm{d}z = 0\end{aligned}$$

By lemma 4.1.1, $\int_{\mathbb{H}}\nu\phi = 0$ for any $\phi \in QD^1(\mathbb{H})$.

And since

$$\begin{aligned}\int_X \nu\theta\left(\phi\mathrm{d}z^2\right) &= \int_F \nu\sum_{A\in\Gamma}(\phi\circ A)(A')^2\mathrm{d}z^2 \\ &= \sum_{A\in\Gamma}\int_F \nu(\phi\circ A)(A')^2\mathrm{d}z^2 \\ &= \sum_{A\in\Gamma}\int_{A(F)}(\nu\circ A^{-1})\frac{\overline{A_z^{-1}}}{A_z}\phi\mathrm{d}A(z)^2 \\ &= \sum_{A\in\Gamma}\int_{A(F)}\nu\phi = \int_{\mathbb{H}}\nu\phi.\end{aligned}$$

Hence $\int_X \nu\phi = 0$ for any $\phi \in \mathrm{QD}^1(X)$.

The converse is trivial by taking

$$\phi(w) = \left(\frac{1}{w-z} - \frac{z}{w-1} - \frac{1}{w}\right)\mathrm{d}z^2.$$

(2) It is too technical to prove, so we will not give the proof.

□

4.2 Bergman projection

We denote the hyperbolic metric by $\rho(z) = \frac{2}{1-|z|^2}$.

Def 4.2.1. Let $A_2(\mathbb{D})$ be the space of holomorphic functions φ on $\mathbb{D}$ such that

$$\|\phi\|^2 = \int_{\mathbb{D}} \rho(z)^{-2}|\phi(z)|^2\mathrm{d}x\mathrm{d}y < \infty.$$

We define the Petersson scalar product on $A_2(\mathbb{D})$ by

$$\langle\phi,\psi\rangle_{\mathbb{D}} = \int_{\mathbb{D}} \rho(z)^{-2}\phi(z)\overline{\psi(z)}\mathrm{d}x\mathrm{d}y.$$

Then $A_2(\mathbb{D})$ becomes a separable Hilbert space with this scalar product.

We take

$$\phi_n(z) = \sqrt{\frac{2}{\pi}(n+1)(n+2)(n+3)}z^n.$$

Then $\{\phi_n\}_{n=0}^{\infty}$ is a complete orthonormal basis for $A_2(\mathbb{D})$.

Def 4.2.2. The Bergman kernel is the reproducing kernel for $A_2(\Delta)$, that is,

$$K(z,w) = \sum_{n=0}^{\infty}\phi_n(z)\overline{\phi_n(w)} = \frac{12}{\pi(1-z\bar{w})^4}.$$

Prop 4.2.1 (reproducing formula)**.** *For $\phi \in A_2(\mathbb{D})$,*

$$\phi(z) = \langle\phi, K(\cdot,z)\rangle_{\mathbb{D}}.$$

Proof.

$$\langle\phi,K(\cdot,z)\rangle_{\mathbb{D}} = \sum_{n=0}^{\infty}\langle\phi,\phi_n\cdot\overline{\phi_n(z)}\rangle_{\mathbb{D}} = \sum_{n=0}^{\infty}\phi_n(z)\langle\phi,\phi_n\rangle_{\mathbb{D}} = \phi(z)$$

□

Thm 4.2.1. *(1) $K(z,w) = \overline{K(w,z)}$*

(2) For $A \in \mathrm{Aut}(\mathbb{D})$, $K(Az,Aw)A'(z)^2\overline{A'(w)}^2 = K(z,w)$

(3) $\int_{\mathbb{D}}|K(z,w)|\mathrm{d}x\mathrm{d}y = 3\rho^2(w)$

(4) If $\left|\rho^{-2}(z)\phi(z)\right| \leqslant C$ for $z \in \mathbb{D}$ and holomorphic ϕ, then

$$\phi(z) = \int_{\mathbb{D}}\rho^{-2}(w)\phi(w)K(z,w)\mathrm{d}u\mathrm{d}v$$

Proof. (1) Trivial.

(2) Since

$$\begin{aligned}
&\left\langle(\phi\circ A)(A')^2,(\psi\circ A)(A')^2\right\rangle_{\mathbb{D}}\\
&=\int_{\mathbb{D}}\rho(z)^{-4}\phi(A(z))\overline{\psi(A(z))}A'(z)^2\overline{A'(z)}^2\mathrm{d}A_{\mathbb{D}}(z)\\
&=\int_{\mathbb{D}}\left(\rho(z)^{-1}\left|A'\right|\right)^4\phi(A(z))\overline{\psi(A(z))}\mathrm{d}A_{\mathbb{D}}(A(z))\\
&=\int_{A(\mathbb{D})}\rho(z)^{-4}\phi(z)\overline{\psi(z)}\mathrm{d}A_{\mathbb{D}}(z)=\langle\phi,\psi\rangle_{\mathbb{D}}
\end{aligned}$$

Here

$$\mathrm{d}A_{\mathbb{D}}(z)=\frac{4\mathrm{d}x\mathrm{d}y}{\left(1-|z|^2\right)^2}$$

is the hyperbolic area.

So $\left\{(\phi_n\circ A)(A')^2\right\}$ is also a complete orthonormal basis for $A_2(\mathbb{D})$.

Hence the reproducing kernel is the same, *i.e.*

$$K(Az,Aw)A'(z)^2\overline{A'(w)}^2=K(z,w)$$

(3)

$$\begin{aligned}
\int_{\mathbb{D}}|K(z,w)|\mathrm{d}x\mathrm{d}y&=\frac{12}{\pi}\int_0^1\int_0^{2\pi}\frac{r}{(1-2sr\cos\theta+s^2r^2)^2}\mathrm{d}\theta\mathrm{d}r\\
&=\frac{12}{\pi}\int_0^1\frac{2\pi(1+s^2r^2)}{(1-s^2r^2)^3}\mathrm{d}r\\
&=24\int_1^{1-s^2}\frac{2-t}{t^3}\cdot\frac{-\mathrm{d}t}{2s^2}\\
&=\frac{12}{s^2}\int_{1-s^2}^1\frac{2-t}{t^3}\mathrm{d}t\\
&=\frac{12}{s^2}\left(\frac{1}{(1-s^2)^2}-\frac{1}{1-s^2}\right)\\
&=\frac{12}{(1-s^2)^2}=3\rho^2(w)
\end{aligned}$$

Here $z=re^{i\theta},w=se^{i\varphi},t=1-r^2s^2$

(4) Since $\left|\rho^{-2}(z)\phi(z)\right|\leqslant C$.

So the integral is well-defined for all z.

For $z=0$,

$$\begin{aligned}
\phi(0)&=6\phi(0)\int_0^1(1-r^2)^2r\mathrm{d}r\\
&=\frac{3}{\pi}\int_{\mathbb{D}}(1-|w|^2)^2\phi(w)\mathrm{d}u\mathrm{d}v\\
&=\int_{\mathbb{D}}\rho(w)^{-2}\phi(w)K(0,w)\mathrm{d}u\mathrm{d}v
\end{aligned}$$

Now for any $z\in\mathbb{D}$, take $A\in\mathrm{Aut}(\mathbb{D})$ such that $A(0)=z$ and $\psi=(\phi\circ A)(A')^2$.

Then we obtain that

$$\begin{aligned}\psi(0) &= \int_{\mathbb{D}} \rho(\zeta)^{-2}\psi(\zeta)K(0,\zeta)\mathrm{d}\xi\mathrm{d}\eta \\ &= \int_{\mathbb{D}} (\rho(A(\zeta))|A(\zeta)|)^{-2}\,\phi(A(\zeta))A'(\zeta)^2K(A(0),A(\zeta))A'(0)^2\overline{A'(\zeta)}^2|A(\zeta)|^{-2}\mathrm{d}u\mathrm{d}v \\ &= A'(0)^2\int_{\mathbb{D}} \rho(w)^{-2}\phi(w)K(z,w)\mathrm{d}u\mathrm{d}v\end{aligned}$$

Hence this conclude the desired formula.

□

Def 4.2.3 (Bergman projection)**.**

$$P: MQD^1(\mathbb{D}) \to QD^1(\mathbb{D}), P(\phi)(z) = \int_{\mathbb{D}} \rho^{-2}(w)\phi(w)K(z,w)\mathrm{d}u\mathrm{d}v.$$

Proof of theorem 4.1.1. Let F be a fundamental domain of X and take

$$\psi(z) = P(\chi_F\phi)(z) = \int_{\mathbb{H}} \rho^{-2}(w)(\chi_F(w)\phi(w))K(z,w)\mathrm{d}u\mathrm{d}v.$$

Then $\theta\psi = \phi$ and

$$\begin{aligned}\int_{\mathbb{H}} |\psi(z)|\mathrm{d}x\mathrm{d}y &\leqslant \int_{\mathbb{H}} \left|\rho^{-2}\chi_F\phi\right|(w)\left(\int_{\mathbb{H}} |K(z,w)|\mathrm{d}x\mathrm{d}y\right)\mathrm{d}u\mathrm{d}v \\ &\leqslant \int_{\mathbb{H}} \left|\rho^{-2}\chi_F\phi\right|(w)\cdot 3\rho^2(w)\mathrm{d}u\mathrm{d}v \\ &= 3\int_F |\phi(w)|\mathrm{d}u\mathrm{d}v = 3\|\phi\|_1\end{aligned}$$

□

4.3 Teichmuller spaces

Def 4.3.1. Let S_0 be a Riemann Surface, we let $T(S_0) = \{(S,f)|f: S_0 \to S \text{ quasiconformal}\}/\sim$ and $(S_1,f_1) \sim (S_2,f_2)$ if $f_2 \circ f_1^{-1}: S_1 \to S_2$ is homotopic to a conformal map.

Belt(S_0) is the unit ball of $(-1,1)$-form on S_0 w.r.t. $\|\cdot\|_\infty$.

Remark 4.3.1. $T(S_0) = \text{Belt}(S_0)/\sim$ where $\mu \sim \nu$ iff $f^\nu \circ (f^\mu)^{-1}$ is homotopic to conformal map.

Def 4.3.2. $f_\mu: \mathbb{C} \to \mathbb{C}$ is the unique normalized quasiconformal map with Beltrami differential

$$\begin{cases}\mu(z) & z \in \mathbb{H} \\ 0 & z \in \mathbb{R} \cup \mathbb{H}^-\end{cases}$$

Prop 4.3.1. *$f^\mu = f^\nu$ on $\mathbb{R}$ iff $f_\mu = f_\nu$ on $\mathbb{R}$.*

Proof. If $f_\mu = f_\nu$ on $\mathbb{R}$, then $f_\mu \circ (f^\mu)^{-1}, f_\gamma \circ (f^\gamma)^{-1}$ are conformal maps which map $\mathbb{H}$ to Ω, whose boundary is $f_\mu(\mathbb{R})$, and fix $0, 1$ and ∞.

Then $f_\mu \circ (f^\nu)^{-1} = f_\gamma \circ (f^\gamma)^{-1}$ on $\overline{\mathbb{H}}$.

And since $f_\mu = f_\nu$ on $\mathbb{R}$.

So $f^\mu = f^\nu$ on $\mathbb{R}$.

Conversely, we take

$$g(z)=\begin{cases}\left((f^{\nu})^{-1}\circ f^{\mu}\right)(z) & z\in\mathbb{H}\\ z & z\in\mathbb{R}\cup\mathbb{H}^{-}\end{cases}$$

And let $A=f_{\gamma}\circ g\circ(f_{\mu})^{-1}$.
Then A is conformal on $\Omega=f_{\mu}(\mathbb{H})$ since $A=f_{\gamma}\circ(f^{\gamma})^{-1}\circ f^{\mu}\circ(f_{\mu})^{-1}$ on Ω.
And similarly, A is conformal on Ω^* and A is quasiconformal on $\mathbb{C}$.
So A is conformal on $\mathbb{C}$ and fix $0,1$ and ∞, *i.e.* $A=\mathrm{Id}$.
Hence $f_{\mu}=f_{\nu}$ on $\mathbb{R}$. □

Def 4.3.3. We define the Teichmuller metric on Teichmuller space as

$$d((S_1,f_1),(S_2,f_2))=\frac{1}{2}\inf_{g\sim f_2\circ f_1^{-1}}\log K_g$$

Lemma 4.3.1. *Let $f:S_0\to S$ be quasiconformal, then the set of all quasiconformal maps $g:S_0\to S$ which are homotopic to f contains an extremal map with the smallest maximal dilatation.*

Proof. The lemma is trivial if the universal covering is $\hat{\mathbb{C}}$ or $\mathbb{C}$ and M is not compact.
The torus case is similar as below.
Now suppose $\mathbb{H}$ is the universal covering of M.
Then lift g to a map $\tilde{g}:\mathbb{H}\to\mathbb{H}$ such that all maps $\tilde{g}$ agrees on boundary.
So the set of $\tilde{g}$ contains its limits, *i.e.* there exists a map $\tilde{g}_0$ with smallest maximal dilatation.
Hence g_0 is the required extremal map. □

Prop 4.3.2. *Teichmuller metric is a metric.*

Proof. Take $(S_1,f_1),(S_2,f_2),(S_3,f_3)$, then

$$\begin{aligned}d([S_1,f_1],[S_2,f_2])&=\frac{1}{2}\inf_{g\sim f_2\circ f_1^{-1}}\log K_g\\&\leqslant\frac{1}{2}\inf_{\substack{g_1\sim f_2\circ f_3^{-1}\\ g_2\sim f_3\circ f_1^{-1}}}\log K_{g_1}K_{g_2}\\&=d([S_1,f_1],[S_3,f_3])+d([S_2,f_2],[S_3,f_3])\end{aligned}$$

Suppose $d([S_1,f_1],[S_2,f_2])=0$.
Then there exists extremal map $g\sim f_2\circ f_1^{-1}$ such that $K_g=1$, *i.e.* g is conformal.
So $[S_1,f_1]=[S_2,f_2]$. □

Remark 4.3.2. In terms of Beltrami differentials,

$$d_T([\mu_0],[\nu_0])=\frac{1}{2}\inf_{\mu\in[\mu_0],\nu\in[\nu_0]}\log\frac{1+\left\|\frac{\mu-\nu}{1-\bar{\mu}\nu}\right\|_\infty}{1-\left\|\frac{\mu-\nu}{1-\bar{\mu}\nu}\right\|_\infty}$$

Def 4.3.4. A Beltrami differential $\mu\in[\mu_0]$ is called extremal if $\|\mu\|_\infty\leqslant\|\nu\|_\infty$ for any $\nu\in[\mu_0]$.

Prop 4.3.3. *Let $[\mu_1]\in T(S_0)$, if μ is an extremal Beltrami differential in the Teichmuller class $[\mu_1]$, then*

$$\mu_t=\frac{(1+|\mu|)^t-(1-|\mu|)^t}{(1+|\mu|)^t+(1-|\mu|)^t}\cdot\frac{\mu}{|\mu|}$$

for $0\leqslant t\leqslant 1$ is extremal for $[\mu_t]\in T(S_0)$. The arc $t\to[\mu_t]$ is a geodesic from 0 to $[\mu_1]$, and $d_T([\mu_t],0)=td_T([\mu],0)$.

Proof. Since μ represents a point in $T(S_0)$.

So $\mu = (\mu \circ A)\frac{\overline{A'}}{A'}$ for every deck transformation A.

Therefore μ_t also satisfies this condition, *i.e.* $[\mu_t] \in T(S_0)$ is well-defined.

For $g \in [\mu_t]$, let g_0 be the conformal map which is homotopic to $f^{\mu_t} \circ g^{-1}$ and

$$h = f^{\mu} \circ (f^{\mu_t})^{-1} \circ g_0 \circ g.$$

Then h is homotopic to f^{μ}, *i.e.* $h \in [\mu_1]$.

By theorem 3.6.1, $K_1 \leqslant K_h \leqslant K_1^{1-t} K_g$, *i.e.* $K_g \geqslant K_1^t$.

Therefore μ_t is extremal for $[\mu_t]$.

And since $d_T([\mu_t], [\mu_1]) \leqslant \frac{1}{2} \log K_{f^{\mu} \circ (f^{\mu_t})^{-1}} = (1-t) d_T([\mu], 0)$.

So $d_T(0, [\mu_t]) + d_T([\mu_t], [\mu_1]) = d_T(0, [\mu_1])$ for every $0 \leqslant t \leqslant 1$.

Hence $t \to [\mu_t]$ is a geodesic. □

Coro 4.3.1. *$(T(S_0), d_T)$ is path connected.*

Proof. For $[\mu] \in T(S_0)$, the arc $t \to [\mu_t]$ is a path from 0 to $[\mu_1]$. □

Thm 4.3.1. *$(T(S_0), d_T)$ is complete.*

Proof. Take a Cauchy sequence in $(T(S_0), d_T)$ whose points are represented by Beltrami differential μ_n and $f_n \in [\mu_n]$.

Fix a map $f_i \in [\mu_i]$ such that for $j \geqslant 1$,

$$\inf_{g_{i+j} \in [\mu_{i+j}]} \log K_{g_{i+j} \circ f_i^{-1}} < \frac{1}{2}.$$

Suppose $f_i = f_1$ by passing to a subsequence and take $f_n \in [\mu_n]$ such that

$$\log K_{f_n \circ f_1^{-1}} < \frac{1}{2}.$$

Choose f_k such that for $j \geqslant 1$,

$$\inf_{g_{k+j} \in [\mu_{k+j}]} \log K_{g_{k+j} \circ f_k^{-1}} < \frac{1}{4}.$$

Suppose $f_k = f_2$ by passing to a subsequence and take $f_n \in [\mu_n]$ such that

$$\log K_{f_n \circ f_2^{-1}} < \frac{1}{4}.$$

Repeating this procedure gives a sequence $\{f_n\}$ such that $\{[f_n]\}$ is a subsequence of the Cauchy sequence and

$$\log K_{f_{n+1} \circ f_n^{-1}} < 2^{-n}.$$

So

$$\log K_{f_{n+j} \circ f_n^{-1}} \leqslant \sum_{i=1}^{j} 2^{-(n+i-1)} < 2^{-n+1}.$$

And the dilatation μ_n of f_n satisfy that

$$\|\mu_{n+j} - \mu_n\|_\infty \leqslant 2 \left\| \frac{\mu_{n+j} - \mu_n}{1 - \overline{\mu_n}\mu_{n+j}} \right\|_\infty = 2\tanh\left(\frac{1}{2}\log K_{f_{n+j} \circ f_n^{-1}}\right) < 2\tanh 2^{-n}.$$

Therefore $\{\mu_n\}$ is a Cauchy sequence in $B^\infty(S_0)$, *i.e.* $\mu_n \to \mu$ as $n \to \infty$ for some $\mu \in B^\infty(S_0)$. □

Prop 4.3.4. *Suppose $g : S_0 \to X_0$ is quasiconformal, then $T(S_0) \cong T(X_0)$.*

Proof. Consider $\phi : T(S_0) \to T(X_0), [S, f] \mapsto [S, f \circ g^{-1}]$.
Then $d([S_1, f_1], [S_2, f_2]) = d([S_1, f_1 \circ g^{-1}], [S_2, f_2 \circ g^{-1}])$. □

Def 4.3.5. $\mathrm{Mod}_g = \{g : \Sigma_g \to \Sigma_g | g \text{ is diffeomorphism}\}/\sim$ and $g_1 \sim g_2$ if g_1 homotopic to g_2.

Prop 4.3.5. $\mathrm{Isom}(T_g) \cong \mathrm{Mod}_g$.

Proof. By proposition 4.3.4, every isometry $T_g \to T_g$ corresponds to quasiconformal map $\Sigma_g \to \Sigma_g$ and two maps f_1, f_2 correspond to the same isometry iff $[f_1] = [f_2] \in T_g$, *i.e.* $f_1 \simeq f_2$. □

4.4 Douady-Earle extension

Thm 4.4.1 (Schoen conjecture)**.** *If $f : \mathbb{S}^1 \to \mathbb{S}^1$ is quasi-symmetric, then there exists extension $h : \mathbb{D} \to \mathbb{D}$ which is harmonic w.r.t. hyperbolic metric on $\mathbb{D}$.*

Thm 4.4.2. *If $f : \mathbb{S}^1 \to \mathbb{S}^1$ is homeomorphism, then there exists a homeomorphic extension $\phi_f : \mathbb{D} \to \mathbb{D}$ such that*

(1) (Conformally natural) $A \circ \phi_f \circ B = \phi_{A\circ f\circ B}$ for $A, B \in \mathrm{Aut}(\mathbb{D})$

(2) If φ is quasi-symmetric, then $\tilde{\varphi}$ is quasiconformal in $\mathbb{D}$.

Moreover, it is uniquely determined with the condition that

$$\int_{\mathbb{S}^1} f(\zeta)|\mathrm{d}\zeta| = 0 \Leftrightarrow \phi_f(0) = 0$$

Remark 4.4.1. We take $w = \phi_f(z)$ as the unique $w \in \mathbb{D}$ such that

$$F(z, w) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \left(\frac{f(\zeta) - w}{1 - \bar{w} f(\zeta)} \right) \left(\frac{1 - |z|^2}{|z - \zeta|^2} \right) |\mathrm{d}\zeta| = 0.$$

It needs many technique of analysis to prove that the solution of $F(z, w) = 0$ is unique for any $z \in \mathbb{D}$ and such ϕ_f satisfies the required statements, so we skip the detail.

Lemma 4.4.1. *Let $S_0 = \mathbb{D}/\Gamma$, consider the map*

$$\begin{aligned} \sigma\colon \mathrm{Belt}(\mathbb{D}) &\longrightarrow \mathrm{Belt}(\mathbb{D}) \\ \mu &\longmapsto \mu_{\phi_{\varphi^\mu}} \end{aligned}$$

where $\varphi^\mu = f^\mu|_{\mathbb{S}^1}$, then

(1) σ maps $\mathrm{Belt}(S_0)$ to itself

(2) there exists a continuous map $s : T(S_0) \to \mathrm{Belt}(S_0)$ such that $s \circ \pi = \sigma$

(3) $\pi \circ \sigma = \pi$.

Proof. (1) Let $B \circ f^\mu \circ A = f^\mu$ for some $B \in \mathrm{Aut}(\mathbb{D}), A \in \Gamma$.

Then $B \circ \varphi^\mu \circ A$.

And since ϕ is conformmally natural.

So $B \circ \phi_{\varphi^\mu} \circ A = \phi_{\varphi^\mu}$, *i.e.* $\sigma(\mu) \in \mathrm{Belt}(S_0)$.

(2) Take $s([\mu]) = \sigma(\mu)$.
Suppose $[\mu_1] = [\mu_2]$, *i.e.* $f^{\mu_1} \simeq f^{\mu_2}$.
Then $\varphi^{\mu_1} = \varphi^{\mu_2}$
So $\phi_{\varphi^{\mu_1}} = \phi_{\varphi^{\mu_2}}$, *i.e.* $\sigma(\mu_1) = \sigma(\mu_2)$.
Hence s is well-defined.

(3) Since $\phi_{\varphi^\mu}\big|_{\mathbb{S}^1} = \varphi^\mu$.
So $\phi_{\varphi^\mu} \simeq f^\mu$.
Hence $\pi \circ \sigma(\mu) = [\mu]$.

□

Thm 4.4.3. *Let $S_0 = \mathbb{D}/\Gamma$, then $T(S_0)$ is contractible.*

Proof. By the above lemma, since $\pi \circ s \circ \pi = \pi \circ \sigma = \pi$.
So $\pi \circ s = \mathrm{Id}$.
Take $H : T(S_0) \times [0,1] \to T(S_0), ([\mu], t) = [(1-t)s([\mu])]$.
Then $H(-,0) = \mathrm{Id}$ and $H(-,1) = 0$.
Hence $T(S_0)$ is contractible. □

4.5 Teichmuller space of torus

Def 4.5.1. Take $T_\tau(z) = z + \tau$ and $G_\tau = \langle T_1, T_\tau \rangle$, then a torus is given by $S_\tau = \mathbb{C}/G_\tau$.

Prop 4.5.1. *$S_\tau = S_{\tau'}$ iff $\tau' = A(\tau)$ for some $A \in \mathrm{PSL}(2,\mathbb{Z})$.*

Lemma 4.5.1. *Let $\theta : G_\tau \to G_{\tau'}$ be an isomorphism, then there exists a homeomorphism $f : S_\tau \to S_{\tau'}$ inducing θ.*

Proof. Let $T_{\omega_1} = \theta(T_1), T_{\omega_2} = \theta(T_\tau)$.
Then consider the affine map f fixing 0 and mapping $1, \tau$ to ω_1, ω_2 resp.
Since $\frac{\omega_2}{\omega_1} = A(\tau')$ for some $A \in \mathrm{PSL}(2,\mathbb{Z})$.
So f projects to the map we desired. □

Lemma 4.5.2. *Let $\tau, \tau' \in \mathbb{H}$, S_τ and $S_{\tau'}$ are conformally equivalent iff $\tau' = A(\tau)$ for some $A \in \mathrm{SL}(2,\mathbb{Z})$.*

Proof. Suppose S_τ and $S_{\tau'}$ are conformally equivalent and the conformal map lifts to $f : \mathbb{C} \to \mathbb{C}$.
Then $f(0) = 0$, *i.e.* $f(z) = \alpha z$.
So $(\alpha, \alpha\tau)$ is a base of $S_{\tau'}$, *i.e.* $\tau' = A(\tau)$ for some $A \in \mathrm{SL}(2,\mathbb{Z})$.
Conversely, by proposition 4.5.1, $S_\tau = S_{\tau'}$. □

Thm 4.5.1. *In each homotopy class of sense-preserving homeomorphisms between tori, the extremal map can be lifted to an affine map.*

Proof. Suppose $f : S_\tau \to S_{\tau'}$ and the lift $\tilde{f} : \mathbb{C} \to \mathbb{C}$ satisfies that $f(0) = 0$ and

$$\tilde{f}(z + m + n\tau) = \tilde{f}(z) + m\omega_1 + n\omega_2,$$

we denote the set of such $\tilde{f}$ by F.
Then there exists a unique affine map

$$g(z) = \frac{\omega_2 - \omega_1\bar{\tau}}{\tau - \bar{\tau}} z + \frac{\omega_1\tau - \omega_2}{\tau - \bar{\tau}} \bar{z}$$

in F.
Suppose f is K-quasiconformal, let $f_k(z) = \frac{\tilde{f}(kz)}{k}$
Then f_k converges to an affine map in F, *i.e.* $f_k \to g$.
So $K_g \leqslant K$.
Now suppose that $K = K_g$, assume $g(x+iy) = Kx + iy$ WLOG.
Since $\tilde{f}, g \in F$, there exists M such that

$$\left|\tilde{f}(z) - g(z)\right| \leqslant M.$$

So we obtain

$$\int_0^r \left|\tilde{f}_x(x+iy)\right| \mathrm{d}x \geqslant \left|\int_0^r \tilde{f}_x(x+iy)\mathrm{d}x\right| \geqslant Kr - 2L$$

Consider $R = [0,r] \times [0,r]$, then

$$\int_R \left|\tilde{f}_x\right| \mathrm{d}x\mathrm{d}y \geqslant Kr^2 - 2Lr.$$

As $r \to \infty$, the number of period parallelograms P meeting R is about $\frac{r^2(1+o(1))}{A(P)}$, and so

$$\int_P \left|\tilde{f}_x\right| \mathrm{d}x\mathrm{d}y \geqslant \lim_{r\to\infty} \frac{r^2(K+o(1))}{r^2(1+o(1))} A(P) = KA(P).$$

On the other hand, $|f_x|^2 \leqslant KJ$, therefore

$$(KA(P))^2 \leqslant A(P) \int_P \left|\tilde{f}_x\right|^2 \mathrm{d}x\mathrm{d}y \leqslant KA(P)A(\tilde{f}(P)).$$

And since $\tilde{f}(P)$ is the fundamental domain of $S_{\tau'}$, *i.e.* $A(\tilde{f}(P)) = A(g(P)) = KA(P)$.
Hence $\left|\tilde{f}_x\right|^2 = KJ$ almost everywhere, *i.e.* $\tilde{f} = g$ is affine. □

Lemma 4.5.3. *Let $\theta : G_\tau \to G_{\tau'}$ be an isomorphism generated by K-quasiconformal map f, then*

$$d_{\mathbb{H}}\left(\tau, \frac{\tilde{f}(\tau)}{\tilde{f}(1)}\right) \leqslant \frac{1}{2}\log K$$

the equality holds iff $\tilde{f}$ is affine.

Proof. WLOG, we assume $\frac{f(\tau)}{f(1)} = \tau'$.
Let g homotopic to f and $\tilde{g} = \lambda(z + \mu\bar{z})$ is affine, then

$$\mu = \frac{\tau - \tau'}{\tau' - \bar{\tau}}.$$

So we obtain

$$\log K_g = \log\frac{1+|\mu|}{1-|\mu|} = \tanh\left(\frac{|\tau-\tau'|}{|\tau'-\bar{\tau}|}\right) = 2d_{\mathbb{H}}(\tau, \tau').$$

And by the above theorem, $K_g \geqslant K_f$.
Hence we complete the proof. □

Thm 4.5.2. *Teichmuller space of torus is isometric to $\mathbb{H}$ with hyperbolic metric.*

Proof. Consider $\psi : T(S_\tau) \to \mathbb{H}$ given by

$$\psi([f]) = \frac{\tilde{f}(\tau)}{\tilde{f}(1)}.$$

By three lemmas above, we can easily check that ψ is bijective.
Consider $[f_1], [f_2] \in T(S_\tau)$, let $f : f_1(S_\tau) \to f_2(S_\tau)$ be the extremal homotopic to $f_2 \circ f_1^{-1}$. Then by lemma 4.5.3,

$$d_T([f_1], [f_2]) = \frac{1}{2} \log K_f = d_{\mathbb{H}} \left(\frac{\tilde{f}_1(\tau)}{\tilde{f}_1(1)}, \frac{\tilde{f}_2(\tau)}{\tilde{f}_2(1)} \right) = d_{\mathbb{H}}(\psi([f_1], [f_2])).$$

□

4.6 Teichmuller theorem

Def 4.6.1. Let S be a Riemann surface and $\nu \in \mathrm{Belt}(S)$, we say ν annihilates $QD^1(S)$ if

$$\int_S \gamma\phi = 0$$

for all $\phi \in QD^1(S)$.

Def 4.6.2. Consider the linear functional

$$\Lambda_\mu(\phi) = \int_M \mu(z)\phi(z)\mathrm{d}x\mathrm{d}y.$$

We say μ and ν are infinitesimally equivalent, or $\mu \sim_* \nu$ if $\Lambda_\mu(\phi) = \Lambda_\nu(\phi)$ for $\phi \in \mathrm{QD}^1(S)$.
The infinitesimally equivalent class of μ is denoted by $[\mu]_*$.
An element μ is said to be infinitesimally extremal if for all $\nu \in [\mu]_*$, $\|\mu\|_\infty \leqslant \|\nu\|_\infty$.

Thm 4.6.1 (Hamilton–Krushkal)**.** *Suppose $\mu \in \mathrm{Belt}(S)$ is extremal in $[\mu] \in T(S)$, then μ is infinitesimally extremal.*

Proof. Let $\|\mu\|_\infty = k$, $\alpha \in [\mu]_*$ and $\|\alpha\|_\infty = k_1$.
Suppose $k_1 < k$, let $\nu = \mu - \alpha$
Then ν annihilates $\mathrm{QD}^1(S)$.
By theorem 4.1.2, take $[\delta(t)] = [t\nu] \in T(S)$ such that $\|\delta(t)\|_\infty \leqslant 12t^2$, let $\nu_t = t\nu - \delta(t)$.
So $[\nu_t] = [0]$ in $T(S)$.
Moreover, take $f^\lambda = f^\mu \circ (f^{\nu_t})^{-1}$, then

$$\begin{aligned}
|\lambda \circ f^{\nu_t}| &= \left| \frac{\mu - \nu_t}{1 - \mu\overline{\nu_t}} \right| \\
&= \sqrt{\frac{|\mu|^2 - 2\,\mathrm{Re}\,(\mu\overline{\nu_t}) + |\nu_t|^2}{1 - 2\,\mathrm{Re}\,(\mu\overline{\nu_t}) + |\nu_t\mu|^2}} \\
&= \sqrt{|\mu|^2 - 2\left(1 - |\mu|^2\right)\mathrm{Re}(\mu\bar{\nu}) + O(t^2)} \\
&= |\mu| - \frac{1 - |\mu|^2}{|\mu|}\,\mathrm{Re}\,(\mu\bar{\nu})\,t + O(t^2)
\end{aligned}$$

Consider $E_1 = \left\{ z \in S \middle| |\mu(z)| \leqslant \frac{k+k_1}{2} \right\}$, $E_2 = S \backslash E_1$.
On E_1, $|\lambda\left(f^{\nu_t}(z)\right)| \leqslant \frac{k+k_1}{2} + O(t) < k - C_1 t$ for small $t > 0$.

On E_2, since

$$\operatorname{Re}(\mu\bar{\nu}) = t\operatorname{Re}(|\mu|^2 - \mu\bar{\alpha}) \geqslant t|\mu|\operatorname{Re}(|\mu| - |\alpha|).$$

So we deduce

$$\frac{1-|\mu|^2}{|\mu|}\operatorname{Re}(\mu\bar{\nu}) \geqslant (1-|\mu|^2)(|\mu|-|\alpha|) \geqslant \frac{1}{2}(1-k^2)(k-k_1)$$

Therefore $|\lambda(z)| < k - C_2 t$ for small $t > 0$ and $z \in E_2$.
Hence $|\lambda(z)| < k - Ct$ for some constant C and $z \in S$, contradiction! □

Thm 4.6.2 (Reich-Strebel inequality)**.** *Suppose $f^\mu : S \to S$ and $[\mu] = 0$, then for $\phi \in \mathrm{QD}^1(S)$,*

$$\|\phi\|_1 \leqslant \int_S |\phi| \frac{\left|1 + \mu\frac{\phi}{|\phi|}\right|^2}{1-|\mu|^2} \mathrm{d}x\mathrm{d}y$$

Remark 4.6.1. The proof uses the theory of trajectories of quadratic differentials, so we omit the proof.

Exam 4.6.1. *Consider $\lambda = k\frac{|\phi|}{\phi}$ and $\phi_0 = -\phi$, then*

$$\int_S |\phi| \frac{\left|1 - \mu\frac{\phi}{|\phi|}\right|^2}{1-|\mu|^2} \mathrm{d}x\mathrm{d}y = \int_S |\phi| \frac{1-k}{1+k} \mathrm{d}x\mathrm{d}y.$$

So the Reich-Strebel inequality cannot hold, i.e. $[\lambda] \neq 0$.

We now proof the special case of Reich-Strebel inequality for square torus.

Prop 4.6.1 (Grötzsch)**.** *Let S be a square torus, $f^\mu : S \to S$ and $[\mu] = 0$, then for $\phi \in \mathrm{QD}^1(S)$,*

$$\|\phi\|_1 \leqslant \int_S |\phi| \frac{\left|1 + \mu\frac{\phi}{|\phi|}\right|^2}{1-|\mu|^2} \mathrm{d}x\mathrm{d}y$$

Proof. Take $\gamma_y(t) = (t, y)$.
Then $l(f(\gamma_y)) \geqslant l(\gamma_y) = 1$, so

$$\begin{aligned}
\int_0^1 1\mathrm{d}y &\leqslant \int_0^1 \int_0^1 |f_z(x,y)||1+\mu(x,y)|\mathrm{d}x\mathrm{d}y \\
&= \int_S |f_z||1+\mu|\mathrm{d}x\mathrm{d}y \\
&= \int_S \frac{|f_z||1+\mu|}{\sqrt{|f_z|^2 - |f_{\bar{z}}|^2}} \sqrt{|f_z|^2 - |f_{\bar{z}}|^2}\mathrm{d}x\mathrm{d}y \\
&\leqslant \int_S \frac{|1+\mu|^2}{1-|\mu|^2}\mathrm{d}x\mathrm{d}y \cdot \int_S \left(|f_z|^2 - |f_{\bar{z}}|^2\right)\mathrm{d}x\mathrm{d}y \\
&= \int_S \frac{|1+\mu|^2}{1-|\mu|^2}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

And since $\mathrm{QD}^1(S) = \mathbb{C}\mathrm{d}z^2$.
So we conclude the proof. □

Coro 4.6.1 (main inequality)**.** *Suppose $f : S \to f(S)$ and $\tilde{f} : f(S) \to S$ have Beltrami differentials μ and μ_1 resp. and $\tilde{f} \circ f$ is homotopic to the identity, then for any $\phi \in \mathrm{QD}^1(S)$*

$$\|\phi\|_1 \leqslant \int_S \frac{\left|1+\mu\frac{\phi}{|\phi|}\right|^2}{1-|\mu|^2} \frac{\left|1+\theta\mu_1(f)\frac{\phi}{|\phi|}\right|^2}{1-|\mu_1(f)|^2} |\phi| \mathrm{d}x\mathrm{d}y.$$

Here

$$\theta = p\left(1+\frac{\overline{\mu\phi}}{|\phi|}\right)\left(1+\frac{\mu\phi}{|\phi|}\right)^{-1}, p = \frac{\overline{f_z}}{f_z}$$

Proof. The Beltrami differential λ of $\tilde{f} \circ f$ is

$$\lambda = \frac{\mu + \mu_1(f)p}{1+\bar{\mu}\mu_1(f)p}$$

So by Reich-Strebel inequality,

$$\begin{aligned}
\|\phi\|_1 &\leqslant \int_S |\phi| \frac{\left|1+\lambda\frac{\phi}{|\phi|}\right|^2}{1-|\lambda|^2} \mathrm{d}x\mathrm{d}y \\
&= \int_S |\phi| \frac{\left|1+\bar{\mu}\mu_1(f)p + (\mu+\mu_1(f)p)\frac{\phi}{|\phi|}\right|^2}{|1+\bar{\mu}\mu_1(f)p|^2 - |\mu+\mu_1(f)p|^2} \mathrm{d}x\mathrm{d}y \\
&= \int_S |\phi| \frac{\left|1+\mu\frac{\phi}{|\phi|}\right|^2 \left|1+\frac{\bar{\mu}+\frac{\phi}{|\phi|}}{1+\mu\frac{\phi}{|\phi|}}\mu_1(f)p\right|^2}{\left(1-|\mu|^2\right)\left(1-|\mu_1(f)p|^2\right)} \mathrm{d}x\mathrm{d}y \\
&= \int_S \frac{\left|1+\mu\frac{\phi}{|\phi|}\right|^2}{1-|\mu|^2} \frac{\left|1+\theta\mu_1(f)\frac{\phi}{|\phi|}\right|^2}{1-|\mu_1(f)|^2} |\phi| \mathrm{d}x\mathrm{d}y
\end{aligned}$$

□

Thm 4.6.3. *Suppose $f : S_0 \to S$ with Beltrami differential of Teichmuller type,* i.e. $\mu = k\frac{\bar{\phi}}{|\phi|}$ *for some $\phi \in \mathrm{QD}^1(S)$, and $g : S_0 \to S$ is quasiconformal such that $g^{-1} \circ f$ homotopic to the identity, then either there exists a set of positive measure on S_0 for which $|\mu_g(z)| > k$ or $\mu_g(f) = \mu$ almost everywhere.*

Proof. Let $\nu(z) = \mathrm{Belt}(g^{-1})(f(z))$ and WLOG, we assume $\|\phi\|_1 = 1$.

If $\|\mu_g\|_\infty = \|\nu\|_\infty \leqslant k$, then

$$\begin{aligned}
1 &\leqslant \int_S \frac{\left|1-\mu\frac{\phi}{|\phi|}\right|^2}{1-|\mu|^2} \frac{\left|1-\theta\nu\frac{\phi}{|\phi|}\right|^2}{1-|\nu|^2} |\phi| \mathrm{d}x\mathrm{d}y \\
&= \int_S \frac{1-k}{1+k} \frac{\left|1-\frac{\overline{f_z}}{f_z}\nu\frac{\phi}{|\phi|}\right|^2}{1-|\nu|^2} |\phi| \mathrm{d}x\mathrm{d}y \\
&\leqslant \frac{1}{K} \int_S \frac{1+|\nu|^2}{1-|\nu|^2} |\varphi| \mathrm{d}x\mathrm{d}y \\
&\leqslant \frac{1}{K} \frac{1+\|\nu\|_\infty}{1-\|\nu\|_\infty} \|\varphi\|_1 \leqslant 1
\end{aligned}$$

So each inequality must be equality, *i.e.*

$$\left|1+\mu_g(f)\frac{\phi}{|\phi|}\right|^2 = \left|1-\frac{\overline{f_z}}{f_z}\nu\frac{\phi}{|\phi|}\right|^2 = 1+|\nu|^2, |\nu| \equiv k$$

hold almost everywhere.

Hence $\mu_g(f) = \mu$ almost everywhere. □

Thm 4.6.4 (Teichmuller existence theorem). *Let X Riemann surface with $\dim \mathrm{QD}^1(X) < +\infty$, then every point in $T(X)$ has a uniquely extremal representative, which is of Teichmuller type.*

Proof. Take $[\mu] \in T(X)$ such that μ is extremal in $[\mu]$, consider the linear functional

$$\lambda(\phi)(\phi) = \int_X \mu\phi.$$

Then there exists $\phi_0 \in \mathrm{QD}^1(X)$ such that $\|\lambda\| = \lambda(\phi_0)$ and $\|\phi_0\| = 1$.

By Hamilton-Krushkal condition, μ is infinitesimal extremal, *i.e.*

$$\|\mu\|_\infty = \|\lambda\| = \int_X \mu\phi.$$

So $\mu = k\frac{\overline{\varphi_0}}{|\varphi_0|}$.

By theorem 4.6.3, μ is the unique extremal. □

Prop 4.6.2. *Let $\tau_1, \tau_2 \in T(X)$, then there exists a geodesic between τ_1, τ_2 and moreover, if $\dim \mathrm{QD}^1(X) < +\infty$, then this geodesic is unique.*

Proof. By proposition 4.3.3, the geodesic exists.

And if $\dim \mathrm{QD}^1(X) < +\infty$, by the Teichmuller existence theorem, the geodesic is unique. □

Remark 4.6.2. For $\mu = k\frac{\bar{\phi}}{|\phi|}$, the unique geodesic from 0 to μ is given by

$$t \mapsto \left[\tanh(t)\frac{\bar{\phi}}{|\phi|}\right].$$

So the tangent bundle of $T(X)$ is given by $\mathrm{Belt}(X)/\sim_*$, which is the dual space of $\mathrm{QD}^1(X)$. Moreover, $T(X)$ is has a Finsler norm

$$\|[\mu]_*\|_{\mathrm{Fin}} = \inf_{\nu\in[\mu]_*} \|\nu\|_\infty.$$

Def 4.6.3. Consider $\mu \in \mathrm{Belt}(\mathbb{H}/\Gamma)$, we can extend μ on $\mathbb{C}$ with $\mu(z) = \overline{\mu(\bar{z})}$, then $f^\mu(z) = \overline{f^\mu(\bar{z})}$, *i.e.* we can induce $f^\mu : X \to X^]mu = \mathbb{H}/\Gamma^\mu$.

On the other hand, if we extend μ on $\mathbb{C}$ such that $\mu(z) = 0$ for $\mathrm{Im}\, z < 0$, we can define $f_\mu : \mathbb{C} \to \mathbb{C}$.

4.7 Schwartzian derivative

Def 4.7.1. If f is a conformal map, or more generally, for holomorphic map with $f'(z) \neq 0$, we define

$$(Sf)(z) = 6 \lim_{w\to z} \frac{\partial^2}{\partial z \partial w} \log \frac{f(z) - f(w)}{z - w}.$$

Remark 4.7.1. Notice that

$$\int_{z_1}^{z_2}\int_{w_1}^{w_2} \frac{\partial^2}{\partial z\partial w} \log \frac{f(z)-f(w)}{z-w} \mathrm{d}w\mathrm{d}z = \log \frac{\mathrm{Cr}(f(z_1), f(w_2), f(z_2), f(w_1))}{\mathrm{Cr}(z_1, w_2, z_2, w_1)}$$

So In some sence, $S(f)$ measures the amounts by which f distorts the cross ratios.

Prop 4.7.1.

$$(Sf)(z) = \frac{f'''}{f'}(z) - \frac{3}{2}\left(\frac{f''}{f'}\right)^2(z).$$

Proof.

$$\begin{aligned}
(Sf)(z) &= 6\lim_{t\to 0}\frac{f'(z)f'(z+t)}{(f(z)-f(z+t))^2} - \frac{1}{t^2} \\
&= 6\lim_{t\to 0}\frac{f'(z)\left(f'(z)+f''(z)t+\frac{1}{2}f'''(z)t^2+o(t^2)\right)}{t^2\left(f'(z)+\frac{1}{2}f''(z)t+\frac{1}{6}f'''(z)t^2+o(t^2)\right)^2} - \frac{1}{t^2} \\
&= 6\lim_{t\to 0}\frac{f'(z)}{t^2}\cdot\frac{f'(z)+f''(z)t+\frac{1}{2}f'''(z)t^2+o(t^2)}{f'(z)^2+f'(z)f''(z)t+\left(\frac{1}{4}f''(z)^2+\frac{1}{3}f'(z)f'''(z)\right)t^2+o(t^2)} - \frac{1}{t^2} \\
&= 6\lim_{t\to 0}\frac{1}{t^2}\left(1+\frac{f''}{f'}(z)t+\frac{1}{2}\frac{f'''}{f'}(z)t^2+o(t^2)\right) \\
&\qquad\cdot\left(1-\frac{f''}{f'}(z)t+\left(\left(\frac{f''}{f'}\right)^2(z)-\frac{1}{4}\left(\frac{f''}{f'}\right)^2(z)-\frac{1}{3}\frac{f'''}{f'}(z)\right)+o(t)\right) - \frac{1}{t^2} \\
&= 6\lim_{t\to 0}\frac{1}{t^2}\left(1+\left(\frac{1}{6}\frac{f'''}{f'}(z)-\frac{1}{4}\left(\frac{f''}{f'}\right)^2(z)\right)t^2+o(t)\right) - \frac{1}{t^2} \\
&= \frac{f'''}{f'}(z) - \frac{3}{2}\left(\frac{f''}{f'}\right)^2(z)
\end{aligned}$$

□

Prop 4.7.2.

$$S(f\circ g) = ((Sf)\circ g)\cdot(g')^2 + Sg$$

and $Sf \equiv 0$ iff f is a Mobius transformation. In particular, for Mobius transformation A,

$$S_{f\circ A} = (S_f\circ A)\cdot(A')^2.$$

Proof.

$$\begin{aligned}
S(f\circ g)(z) &= 6\lim_{w\to z}\frac{(f\circ g)'(z)(f\circ g)'(w)}{(f(g(z))-f(g(w)))^2} - \frac{1}{(z-w)^2} \\
&= 6\lim_{w\to z}\frac{f'(g(z))f'(g(w))}{(f(g(z))-f(g(w)))^2}g'(z)g'(w) - \frac{1}{(z-w)^2} \\
&= (Sf)(g(z))\cdot g'(z)^2 + 6\lim_{w\to z}\frac{g'(z)g'(w)}{(g(z)-g(w))^2} - \frac{1}{z-w} \\
&= (Sf)(g(z))\cdot g'(z)^2 + (Sg)(z)
\end{aligned}$$

Moreover, $Sf \equiv 0$ iff f preserved cross-ratio, *i.e.* f is a Mobius transformation. □

Def 4.7.2. Define

$$\begin{aligned}
\tilde{B}\colon \mathrm{Belt}(\mathbb{H}/\Gamma) &\longrightarrow \mathrm{QD}(\mathbb{H}^-/\Gamma) \\
\mu &\longmapsto S\left(f_\mu\big|_{\mathbb{H}^-}\right)
\end{aligned}$$

Then the Bers embedding is given by

$$\begin{aligned}
B\colon \mathrm{Belt}(\mathbb{H}/\Gamma)/\sim &\longrightarrow \mathrm{QD}(\mathbb{H}^-/\Gamma) \\
[\mu] &\longmapsto \tilde{B}(\mu)
\end{aligned}$$

Proof. Take $\mu \in \mathrm{Belt}(\mathbb{H}/\Gamma)$.
Then $f_\mu \circ \gamma \circ f_\mu^{-1}$ is a Mobius transformation for any $\gamma \in \Gamma$, so

$$S\left(f_\mu\big|_{\mathbb{H}^-}\right) = S\left(f_\mu \circ \gamma \circ f_\mu^{-1} \circ f_\mu\big|_{\mathbb{H}^-}\right) = S\left(f_\mu \circ g\big|_{\mathbb{H}^-}\right) = \left(S\left(f_\mu\big|_{\mathbb{H}^-}\right) \circ \gamma\right) \cdot (\gamma')^2$$

Hence $S\left(f_\mu\big|_{\mathbb{H}^-}\right) \in \mathrm{QD}(\mathbb{H}^-/\Gamma)$ is well-defined.
Moreover, by proposition 4.3.1, if μ homotopic to ν, then $f_\mu\big|_{\mathbb{H}^-} = f_\nu\big|_{\mathbb{H}^-}$. □

Prop 4.7.3 (Nehari)**.** *Let ρ be the hyperbolic metric on $X = \mathbb{H}/\Gamma$, for every $\mu \in \mathrm{Belt}(\mathbb{H}/\Gamma)$,*

$$\left\|\rho^{-2} S(f_\mu)\right\|_\infty \leqslant \frac{3}{2}.$$

Proof. $\rho^{-2}S(f_\mu)$ is invariant under Mobius transformations.
WLOG, we can only estimate $\rho(0)^{-2}S(f_\mu)(0)$ and assume $f'(0) = 1$.
Consider the function

$$\frac{1}{f\left(z^{-1}\right)} = z + \sum_{n=0}^{\infty} b_n z^{-n}.$$

By the Gronwall area theorem, we obtain that

$$|b_1| = \left|a_2^2 - a_3\right| \leqslant 1.$$

So $|S_f(0)| = 6\left|a_3 - a_2^2\right| \leqslant 6$.
Hence we complete the proof. □

Exam 4.7.1. *Take $f(z) = z^2$, then $Sf = -\frac{3}{2z^2}$, so the bound can be attained.*

Chapter 5

Additional topics

This chapter are some additional topics about teichmuller theory, most of the proposition has no proof. You can try to find them if you are interested.

5.1 Bicanonical embedding

Def 5.1.1. Suppose S is a Riemann surface and $a_1, \ldots, a_n \in S$, a divisor is

$$D = \sum_{i=1}^{n} m_i a_i,$$

and its degree is given by

$$\deg(D) = \sum_{i=1}^{n} m_i.$$

Def 5.1.2. A holomorphic map $f : S \to \mathbb{CP}^1$ is called meromorphism and we define

$$\operatorname{ord}_a(f) = \begin{cases} k & f \text{ has a zero of order } k \text{ at } a \\ -k & f \text{ has a pole of order } k \text{ at } a \end{cases}$$

The principal divisor of f is given by

$$(f) = \sum_{a \in S} \operatorname{ord}_a(f) a$$

Prop 5.1.1. $\deg((f)) = 0$.

Def 5.1.3. Let $L(D)$ be the set of meromorphic functions on S such that $(f) + D \geqslant 0$ and we denote $l(D) = \dim L(D)$.

Def 5.1.4. If ω is abelian differential, *i.e.* meromorphic 1-form, we say (ω) is canonical.
If $\phi \in \mathrm{QD}(s)$, we say (ϕ) is bicanonical.

Prop 5.1.2. $\deg(\omega) = -\chi(S)$.

Def 5.1.5. $I(D)$ is the space of Abelian differentials with $(\omega) \geqslant D$ and $i(D) = \dim I(D)$.

Thm 5.1.1 (Riemann-Roch)**.**

$$l(D) - i(D) = \deg(D) + 1 - g.$$

Thm 5.1.2. *S is a closed Riemann surface with genus $g \geqslant 2$, then* $\dim \mathrm{QD}(S) = 3g - 3$.

Proof. Let ω be an Abelian differential and $D = 2(\omega)$, then

$$\mathrm{QD}(S) \cong \{\text{meromorphic } f : S \to \mathbb{CP}^1 | (f) + D \geqslant 0\} = L(D)$$

So by Riemann-Roch,

$$\dim \mathrm{QD}(S) = l(D) = i(D) + \deg(D) + 1 - g = i(D) + 3g - 3.$$

And since for any Abelian differential α, $\deg(\alpha) = \deg(\omega)$, *i.e.* we cannot have $(\alpha) \geqslant 2(\omega)$. Hence $i(D) = 0$, *i.e.* $\dim \mathrm{QD}(S) = 3g - 3$. □

Prop 5.1.3 (canonical embedding)**.** *Let $\omega_1, \cdots, \omega_g$ be the basis of $\Omega(S)$,* i.e. *the set of all abelian differentials, if S is non-hyperelliptic, then*

$$f : S \to \mathbb{CP}^{g-1}, f = [\omega_1 : \cdots : \omega_g]$$

is an embedding.

Prop 5.1.4 (bicanonical embedding)**.** *Let $\phi_1, \ldots, \phi_{3g-3}$ be the basis of quadratic differential forms on S, if $g > 2$, then*

$$f : S \to \mathbb{CP}^{3g-4}, f = [\phi_1 : \cdots : \phi_{3g-3}]$$

is an embedding.

5.2 Automorphism of T_g

In this section, we assume $g > 2$.

For a biholomorphism (isometry) $I : T_g \to T_g$ with $\tau \in T_g$, there is an induced isometric

$$I^* : T^*_{I(\tau)} T_g \to T^*_\tau T_g$$
$$L : \mathrm{QD}(Y) \to \mathrm{QD}(X)$$

where $X \in \tau, Y \in I(\tau)$.

Thm 5.2.1. *Suppose $(X, \mu), (Y, \nu)$ are two finite measure spaces, $f_1, \ldots, f_k \in L^p(\mu)$ and $g_1, \ldots, g_k \in L^p(\nu)$ such that*

$$\int_X \left|1 + \sum \lambda_j f_j\right|^p \mathrm{d}\mu = \int_Y \left|1 + \sum \lambda_j g_j\right|^p \mathrm{d}\nu$$

for any $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$, then

$$\langle 1, f_1, \ldots, f_k\rangle \leftrightarrow \langle 1, g_1, \ldots, g_k\rangle$$

is isometric w.r.t. L^p-norm.

Moreover, if $0 < p < \infty$ and p is not even integer, then $\mu(F^{-1}(E)) = \nu(G^{-1}(E))$ for every Borel set $E \subset \mathbb{C}^k$ with $F = (f_1, \ldots, f_k), G = (g_1, \ldots, g_k)$.

Prop 5.2.1. *If $L : \mathrm{QD}(X) \to \mathrm{QD}(Y)$ is a complex linear isomorphism w.r.t. L^1-norm, then there exists biholomorphic $f : Y \to X$ such that $L(\phi) = f^*\phi$.*

Proof. Let $\phi_0, \ldots, \phi_k$ be the basis of $\mathrm{QD}^1(X)$ and $\psi_0, \ldots, \psi_k$ be the basis of $\mathrm{QD}^1(Y)$.

Then $L(\phi_i) = \psi_i$.

Consider $f_i = \frac{\phi_i}{\phi_0}, g_i = \frac{\psi_i}{\psi_0}$, so

$$\int_X \left|1 + \sum \lambda_i f_i\right| \mathrm{d}\mu = \int_Y \left|1 + \sum \lambda_i g_i\right| \mathrm{d}\nu$$

where $\mathrm{d}\mu = |\phi_0| \mathrm{d}x\mathrm{d}y, \mathrm{d}\nu = |\psi_0| \mathrm{d}x\mathrm{d}y$.

By the above theorem, this is possible only when there exists biholomorphic $f : Y \to X$ such that $f^*\mu = \nu$. □

5.3 Quasi-Fuchsian group

Consider Fuchsian group Γ and $\mathbb{H}^+/\Gamma = X, \mathbb{H}^-/\Gamma = \bar{X}$ in $\mathbb{S}^2$.
Take $\mu^+ \in \mathrm{Belt}(X), \mu^- \in \mathrm{Belt}(\bar{X})$ and

$$\mu = \begin{cases} \mu^+(z) & z \in \mathbb{H}^+ \\ \mu^-(z) & z \in \mathbb{H}^- \end{cases}$$

Then we obtain a homeomorphism $f^\mu : \mathbb{S}^2 \to \mathbb{S}^2$.

Prop 5.3.1. *μ is equivariant under Γ, i.e. $(\mu \circ A)\frac{\overline{A'}}{A'} = \mu$ on $\mathbb{S}^2$.*

So there exists a group of Mobius transformation Γ^μ such that f^μ conjugates Γ to Γ^μ.

Def 5.3.1. $\mathbb{H}^3/\Gamma$ is called hyperbolic Fuchsian 3-manifold.
Γ^μ is called quasi-Fuchsian group and $\mathbb{H}^3/\Gamma^\mu$ is called hyperbolic quasi-Fuchsian 3-manifold.

Remark 5.3.1. We say Γ^μ is "quasi-Fuchsian" because $f^\mu(\mathbb{R})$ is a quasicircle and Γ^μ preserve this curve since $\Gamma^\mu = f^\mu \Gamma (f^\mu)^{-1}$.

Notice that if $\mu^+ \sim \nu^+, \mu^- \sim \nu^-$, then $\mu \sim \nu$ and $\Gamma^\mu \cong \Gamma^\nu$. So we have a map

$$\begin{aligned} \mathcal{B} \colon T(X) \times T\left(\overline{X}\right) &\longrightarrow \{\text{Quasi-Fuchsian 3-manifolds}\} \\ ([\mu^+], [\mu^-]) &\longmapsto \mathbb{H}^3/\Gamma^\mu \end{aligned}$$

Prop 5.3.2. *Fuchsian and quasi-Fuchsian 3-manifolds are homeomorphic to $\Sigma_g \times (0,1)$ for some g.*

Proof. Consider Fuchsian group Γ and denote $H = \mathbb{D}^3 \cap \mathbb{R}^2$.
Then on every equi-distance surface of H, the quotient is homeomorphic to $\mathbb{H}^2/\Gamma$.
Hence $\mathbb{H}^3/\Gamma \cong (\mathbb{H}^2/\Gamma) \times \mathbb{R} = \Sigma_g \times (0,1)$. □

Def 5.3.2. Suppose M is a hyperbolic 3−manifold, the convex core $C(M)$ is the smallest closed convex subset of M containing every closed geodesic on M.

Prop 5.3.3. *Suppose $M = \mathbb{H}^3/\Gamma$, then $C(M)$ is the quotient of the convex hull of the limit set $\Lambda(\Gamma) \subset \mathbb{S}^2$ by Γ.*

Proof. Since every closed geodesic on M can be lifted to an axis γ of Γ.
So the endpoints of γ lie in $\Lambda(\Gamma)$, *i.e.* $\gamma \subset \mathrm{Ch}(\Lambda(\Gamma))$.
Conversely, for $\xi \in \Lambda(\Gamma)$, there exists a sequence of $g_n \in \Gamma$ whose fixed point approach ξ.
So the axis γ_n of g_n converges to a geodesic γ start at ξ.
Therefore the quotient of γ_n converges to the quotient γ, *i.e.* it is contained in $C(M)$.
Hence $C(M) = \mathrm{Ch}(\Lambda(\Gamma))/\Gamma$. □

Thm 5.3.1. *If M is a hyperbolic 3-manifold which is homeomorphic to $\Sigma_g \times (0,1)$ and $C(M)$ is compact in M, then M is quasi-Fuchsian.*

Prop 5.3.4. *$p \in \mathbb{S}^2 \backslash \Lambda(\Gamma)$ iff there exists a neighborhood U of p such that*

$$\left\{ A\big|_U : A \in \Gamma \right\}$$

is a normal family.

5.4 Complex dynamics

Def 5.4.1. Rational maps are of the form $f : \mathbb{CP}^1 \to \mathbb{CP}^1, z \mapsto \frac{P(z)}{Q(z)}$ where P, Q are relatively prime polynomial, its degree is given by $\deg f = \max\{\deg P, \deg Q\}$.

Let Rat_d be the space of degree d rational maps.

Now take $f \in \mathrm{Rat}_d$, the iterations of f is a sequence $\{f^n\}_{n\in\mathbb{Z}}$.

Def 5.4.2. The orbit of iteration is defined as

$$O^+(z) = \{f^n(z)|n \in \mathbb{N}\},$$

$$O^-(z) = \{f^n(z)|n \in \mathbb{Z}^-\},$$

$$O_{grand}(z) = \bigcup_{n\geqslant 0} O^-(f^n(z)).$$

Def 5.4.3. Two maps are called Mobius equivalent if $g = A \circ f \circ A^{-1}, A \in \mathrm{PSL}(2, \mathbb{C})$.

Def 5.4.4. A point is called periodic if $f^p(z) = z$ for some $p \geqslant 1$ and its multiplier $\lambda = (f^p)'(z)$, then there are three cases:

(1) when $0 < |\lambda| < 1$, we say z is attracting

(2) when $|\lambda| = 1$, we say z is parabolic

(3) when $|\lambda| > 1$, we say z is repelling.

A point is called critical point if $f'(z) = 0$ and a point is called exceptional if $\mathcal{O}^-(z)$ is finite.

Def 5.4.5. Fatou set is $F_f = \Big\{z \in \hat{\mathbb{C}}|\{f^n\}$ is a normal family in some neighborhood of $z\Big\}$ and Julia set is $J_f = \hat{\mathbb{C}}\backslash F_f$.

Thm 5.4.1. *$J(f)$ is non-empty closed subset of $\hat{\mathbb{C}}$, both $J(f)$ and $F(f)$ are completely invariant under f.*

Proof. Suppose $J(f) = \varnothing$, *i.e.* $F(f) = \hat{\mathbb{C}}$.
Then a subsequence $\{f^{n_k}\}$ uniformly converges to some $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.
But $\deg f^{n_k} \to \infty$, contradition! □

Def 5.4.6. For an attracting point z_0, we associate it with the basin of attraction

$$B(z_0) = \{z \in \hat{\mathbb{C}}|f^{pn}(z) \to z_0 \text{ as } n \to \infty\}.$$

Lemma 5.4.1. *$B(z_0) \subset F(f)$.*

Lemma 5.4.2. *Repelling periodic points must be contained in $J(f)$.*

Proof. Since $f^p(z_0) = z_0$ and $(f^p)'(z_0) > 1$.
So $(f^{pn})'(z_0) \to \infty$. □

Lemma 5.4.3. *$J(f)$ is accumelation set of periodic points.*

Proof. Let $w \in J(f)$, we may assume there exists v such that $f(v) = w$ and $f'(v) \neq 0$.
In a neighborhood U of w, consider

$$h_k(z) = \frac{f^k(z) - z}{f^{-1}(z) - z}.$$

Then h_k is not a normal family in U since $w \in J(f)$.
So by Montel theorem, for some k, $h_k(z) = 0$ or $h_k(z) = 1$ for $z \in U$.
Hence $f^k(z) = z$ or $f^k(z) = f^{-1}(z)$, *i.e.* $f^{k+1}(z) = z$. □

Exam 5.4.1. *Consider* $f_c = z^2 + c$.

(1) If $c = 0$, *then* $J(f_c) = \mathbb{S}^1$.

(2) If $0 < |c| < \frac{1}{4}$, *then* $J(f_c)$ *is a quasicircle.*

Def 5.4.7. A connected component U of $F(f)$ is called a Fatou component. It is periodic if $f^p(U) = U$ and it is pre-periodic if $f^n(U)$ is periodic, otherwise it is called wandering.

Thm 5.4.2. *A rational map of degree greater than one has no wandering domain.*

Def 5.4.8. Suppose μ is a Beltrami differential on $\hat{\mathbb{C}}$ and $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map, we say μ is equivariant if $(\mu \circ f)\frac{\bar{f}'}{f'} = \mu$ almost everywhere on $\hat{\mathbb{C}}$.

Prop 5.4.1. *Suppose* $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *such that* $\mathrm{Belt}(\phi) = \mu$ *is equivariant,* $g = \phi \circ f \circ \phi^{-1}$ *is rational.*

Proof. $\mathrm{Belt}(g \circ \phi) = \mathrm{Belt}(\phi \circ f) = (\mu \circ f)\frac{\bar{f}'}{f'} = \mu$.
So $\mathrm{Belt}(g) = 0$, *i.e.* g is conformal. □

Sketch of proof of theorem 5.4.2. Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has a wandering domain U, take μ on U.
Then can be defined on $O_{grand}(U)$ to be invariant under f and set $\mu = 0$ outside $O_{grand}(U)$.
So we get μ on $\hat{\mathbb{C}}$ which is invariant under f.
Let $f_t = \phi_t \circ f \circ \phi_t^{-1}$ where $\mathrm{Belt}(\phi_t) = t\mu$, $V(z) = \frac{\partial}{\partial t}\phi_t(z)$ and $\dot{f} = \frac{\partial}{\partial t} f_t$.
By dimension count, there exists μ such that $\dot{f} \equiv 0$ and $V \not\equiv 0$ on $\partial U \subset J(f)$.
So $[\mu] \neq 0$ in the tangent space of $\mathrm{Teich}(U)$.
And since $\dot{f}(z) = (V \circ f)(z) - f'(z)V(z)$.
Therefore $V \circ f = f' \cdot V$.
Notice that $J(f)$ is the closure of repelling periodic points.
So there exists $z \in \partial U \subset J(f)$ such that $f^n(z) = z, V(z) \neq 0$ and $(f^n)'(z) = \lambda$ with $|\lambda| > 1$.
Thus we obtain

$$\prod_{j=1}^{n} V(f^j(z)) = f'(f^{n-1}(z)) \cdots f'(f(z))f'(z) \prod_{j=0}^{n-1} V(f^j(z)) = (f^n)'(z) \prod_{j=1}^{n} V(f^j(z))$$

$$(\lambda - 1) \prod_{j=1}^{n} V(f^j(z)) = 0$$

But for every j,

$$V(z) = V(f^n(z)) = f'(f^{n-1}(z)) \cdots f'(f^j(z))V(f^j(z)) \neq 0,$$

contradiction! □

Appendix A

Quasi-isometry and Mostow rigidity

A.1 Boundary extension of quasi-isometry

Lemma A.1.1. *$f : \mathbb{H}^n \to \mathbb{H}^n$ is an (L, A)-quasi-isometry, γ is a geodesic ray and γ' is the geodesic ray within distance R from the quasigeodesic $f(\gamma)$. Let $\pi : \mathbb{H}^n \to \gamma, \pi' : \mathbb{H}^n \to \gamma'$ be nearest-point restriction maps, then we have*

$$d(f \circ \pi(x), \pi' \circ f(x)) \leqslant C,$$

for every $x \in \mathbb{H}^n$ and some constant C.

This lemma is called the quasi-commute property, once we have done this proof, we can complete the proof of theorem 2.1.2.

Thm A.1.1. *$f : \mathbb{H}^n \to \mathbb{H}^n$ is a quasi-isometry then f extend continuous to a homeomorphism $\partial f : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$.*

Proof. By Morse lemma, let γ be the geodesic ray from 0 to $a \in \partial\mathbb{H}^n$, $f \circ \gamma$ be within a bounded distance R from δ and define $\partial f(\gamma(+\infty)) = \delta(+\infty)$.

Consider a sequence $\{x_i\}$ converging to a.

Then the boundary sphere Σ_i of $H_i = \pi^{-1}(x_i)$ bound balls $B_i \subset \mathbb{S}^{n-1}$ containing a.

So $\{B_i\}$ forms a neighborhood basis of a.

And by the definition of quasi-isometry map, $y_i = f(x_i)$ converges to $\partial f(a)$.

Let $H_i' = (\pi')^{-1}(z_i)$ where $d\,(z_i, \pi'(y_i)) = C$ and $z_i, y_i, \partial f(a)$ are in order.

Then by lemma A.1.1, $f(B_i) \subset B_i'$.

So $\partial f(B_i)$ forms a neighborhood basis of $\partial f(a)$, *i.e.* f is continuous.

Similarly, we can take balls $B_i'' \subset \mathbb{S}^{n-1}$ such that $B_i'' \subset \partial f(B_i)$.

So f is open.

Moreover, suppose $\partial f(a) = \partial f(b)$ and let γ_1, γ_2 be the geodesic ray from 0 to a, b resp.

Then by Morse lemma $f \circ \gamma_i$ are both within a bounded distance R from $[f(0), \partial f(a)]$.

So for any $x \in \gamma_1$, there exists $y \in \gamma_2$ such that $d(f(x), f(y)) \leqslant 2R$.

Therefore $d(x, y) \leqslant Ld(f(x), f(y)) + LA \leqslant 2LR + LA$ is bounded, *i.e.* γ_1, γ_2 are asymptotic.

Thus $a = b$, *i.e.* f is injective.

Hence f is homeomorphic. □

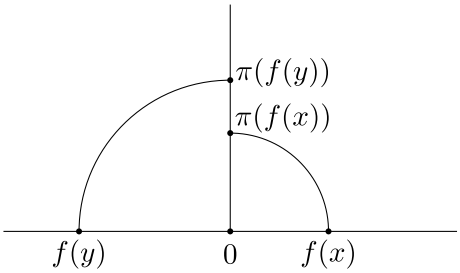
Proof of theorem 2.1.2. WLOG, assume $0, \infty \in \partial\mathbb{H}^n$ are the fixed points of ∂f.

Take x, y such that $|x| = |y| = r$ and $p \in \gamma$ where γ is the geodesic line connecting 0 and ∞.

Then by lemma A.1.1,

$$d(\pi(\partial f(x)), \pi(\partial f(y))) \leqslant d(f(\pi(x)), f(\pi(y))) + 2C \leqslant 2C$$

Notice that as the figure below, $d(\pi(\partial f(x)), \pi(\partial f(y))) = |\ln|f(x)| - \ln|f(y)||$.



So $\frac{|f(x)-0|}{|f(y)-0|} \leqslant e^{2C}$, *i.e.* ∂f is e^{2C}-quasiconformal. □

Now we only need to prove lemma A.1.1, we provide two different proofs. The second proof is more straightforward than the first one but I spent a week to complete the first proof before I suddenly conceived the second proof, whose most crucial step was inspired by the first proof. (Maybe this is because I am not so familiar with the hyperbolic geometry) So I decide to retain both.

For the first prove, we need some technique from Gromov. To simplify, we denote $[x, y]$ as the geodesic segment from x to y below.

Def A.1.1. For three points $u, v, w \in \mathbb{H}^n$, the Gromov product of u, v at w is defined by

$$(u, v)_w = \frac{1}{2}(d(u, w) + d(v, w) - d(u, v)).$$

Lemma A.1.2. *Let γ be a geodesic in $\mathbb{H}^n$ and $y \in \gamma$, for a point $x \in \mathbb{H}^n$, take $z \in \gamma$ such that $d(x, z) = d(x, \gamma)$, then prove that $[x, z] + [z, y]$ is $(\sqrt{2}, 0)$-quasi-isometry.*

Proof. Let $\delta = [x, z] + [z, y]$ and π be the nearest point restriction map to $[x, y]$.

Reparametrize δ by the length of image under π.

Since we have

$$\begin{aligned}\cosh^2 d(z, \pi(z)) &= \frac{\cosh d(x, z) \cosh d(y, z)}{\cosh d(x, \pi(z)) \cosh d(y, \pi(z))} \\ &= \frac{\cosh d(x, y)}{\cosh d(x, \pi(z)) \cosh d(y, \pi(z))} \\ &= 1 + \tanh d(x, \pi(z)) \tanh d(y, \pi(z)) \leqslant 2\end{aligned}$$

So for any $p \in \delta$, $d(p, \pi(p)) \leqslant \cosh^{-1} \sqrt{2}$.

Hence

$$|s - t| \leqslant d(\delta(s), \delta(t)) \leqslant \sqrt{2} d(\pi(\delta(s)), \pi(\delta(t))) = \sqrt{2}|s - t|$$

□

Lemma A.1.3. *Let $\gamma : \mathbb{R} \to \mathbb{H}^n$ be an (L, A)-quasigeodesic and p, q, r are 3 points in order on γ, prove that there exists a constant K such that $(p, r)_q \leqslant K$.*

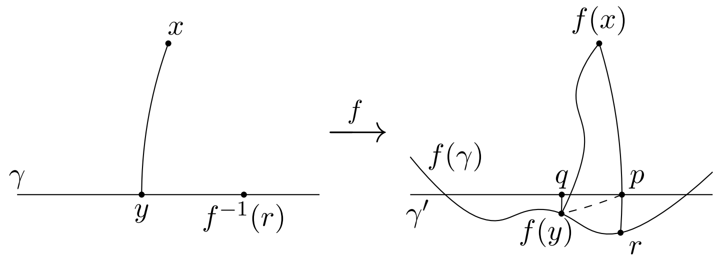
Proof. Let π be the nearest point restriction map to $[p, r]$.

By Morse lemma, $d(q, \pi(q)) \leqslant R$ for some R dependent only on L, A, so we have

$$\begin{aligned}(p, r)_q &= \frac{1}{2}(d(p, q) + d(r, q) - d(p, r)) \\ &= \frac{1}{2}(d(p, q) + d(r, q) - d(p, \pi(q)) - d(r, \pi(q))) \\ &\leqslant d(q, s) \leqslant R\end{aligned}$$

□

proof of lemma A.1.1.



Let $y = \pi(x), p = \pi'(f(x)), q = \pi'(f(y))$ and $r \in f \circ \gamma$ such that $d(r,p) \leqslant R$.
By lemma A.1.2, $[f(x),p] + [p,q]$ and $[x,y] + [y, f^{-1}(r)]$ are quasi-isometry.
So by lemma A.1.3, $(f(x),q)_p \leqslant K_1, (f(x),r)_{f(y)} \leqslant K_2$.
Hence by Morse lemma,

$$\begin{aligned} d(f(y),p) &\leqslant d(f(y),q) + d(p,q) = d(f(y),q) + (q, f(x))_p + (p, f(x))_q \\ &\leqslant R + K_1 + \frac{1}{2}(d(p,q) + d(f(x),q) - d(p,f(x))) \\ &\leqslant R + K_1 + \frac{1}{2}(d(r,f(y)) + d(p,r) + d(f(y),q) \\ &\qquad + d(f(x),f(y)) + d(f(y),q) - d(r,f(x)) + d(p,r)) \\ &= R + K_1 + d(f(y),q) + d(p,r) + (f(x),r)_f(y) \\ &\leqslant K_1 + K_2 + 3R \end{aligned}$$

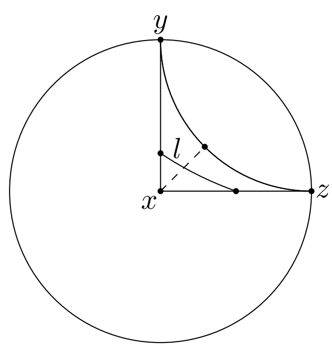
□

Notice that in lemma A.1.2, we use a counter-intuitive property of $\mathbb{H}^n$:

Prop A.1.1. *Consider a geodesic right triangle $\triangle$ in $\mathbb{D}^n$, let x be the right angle vertex and l be the hypotenuse, then $d(x,l) \leqslant \operatorname{arccosh}\sqrt{2}$.*

Proof. We have given a proof by some hyperbolic trigonometry in the proof of lemma A.1.2, we now give a proof with some kind of geometric intuition.

WLOG, we assume x is at the origin.
Then legs of $\triangle$ are on some radius of $\mathbb{D}^n$, denote them by l_1, l_2.
Let y, z be the boundary points of l_1, l_2 resp.
So $d(x,l) \leqslant d(x,[y,z]) = \operatorname{arccosh}\sqrt{2}$.



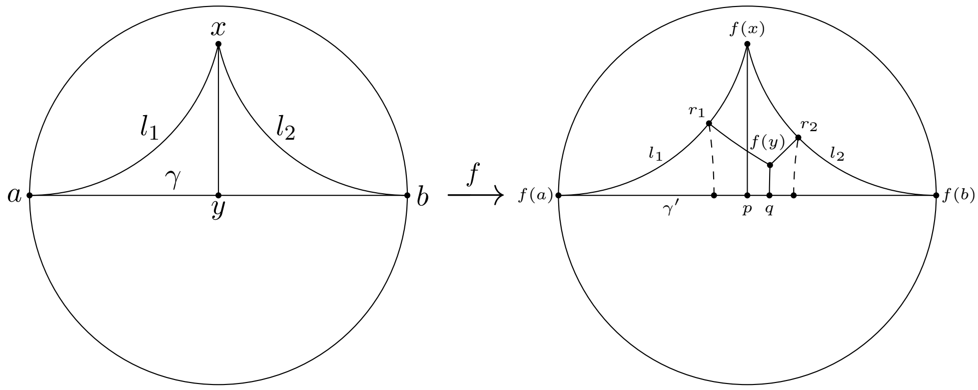
□

Alternative proof of lemma A.1.1. Let a, b be the endpoints of γ and $l_1 = [a,x], l_2 = [b,x]$.

Then by Morse lemma, $l_1' = [f(a), f(x)], l_2' = [f(b), f(y)]$ are within distance R from the quasigeodesic $f(l_1), f(l_2)$ resp.

Let $y = \pi(x), p = \pi'(f(x)), q = \pi'(f(y))$ and $r_i \in l_i'$ such that $d(f(y), l_i) = d(f(y), r_i)$, so

$$\begin{aligned} d(f(y), p) &\leqslant d(f(y), q) + d(p, q) \\ &\leqslant R + d(q, \pi(r_1)) + d(q, \pi(r_2)) \\ &\leqslant R + d(f(y), r_1) + d(f(y), r_2) \\ &\leqslant R + d(f(y), f(l_1)) + R + d(f(y), f(l_2)) + R \\ &\leqslant 3R + Ld(y, l_1) + A + Ld(y, l_2) + A \\ &\leqslant 3R + 2L\text{arccosh}\sqrt{2} + 2A \end{aligned}$$



□

Prop A.1.2. *Suppose $f : \mathbb{H}^n \to \mathbb{H}^n$ is a (L, A)-quasi-isometry, then f is quasi-dense: there exists a constant C such that for $y \in \mathbb{H}^n$, there is some $x \in \mathbb{H}^n$ satisfying that*

$$d(f(x), y) < C.$$

Proof. Take a geodesic line γ containing y which ends at a, b.

By theorem 2.1.2, $p = \partial f^{-1}(a)$ and $q = \partial f^{-1}(b)$ are well-defined.

Take the geodesic line γ' ending at p, q.

Then $f \circ \gamma'$ is within a bounded distance R from γ where R only depends on L, A.

Let π be the nearest point restriction map π for γ.

So $\pi^{-1}(y) \cap f \circ \gamma'$ is nonempty, take p in it.

Hence $p = f(x)$ for some x and

$$d(f(x), y) = d(p, y) = d(p, \pi(p)) \leqslant R.$$

□

Remark A.1.1. For general definition of quasi-isometry, we require the quasi-dense property. But since we can prove this property for $f : \mathbb{H}^n \to \mathbb{H}^n$, we did not required it when defining quasi-isometry $f : \mathbb{H}^n \to \mathbb{H}^n$.

Prop A.1.3. *(L, A)-quasi-isometry map $f : \mathbb{H}^n \to \mathbb{H}^n$ has a quasi-inverse: a (L, A_1)-quasi-isometry map $g : \mathbb{H}^n \to \mathbb{H}^n$ such that*

$$d(f \circ g(y), y) \leqslant B, d(g \circ f(x), x) \leqslant B.$$

Proof. Take $g(y) \in \mathbb{H}^n$ such that $d(f \circ g(y), y) \leqslant A$, then

$$\begin{aligned} d(g(y_1), g(y_2)) &\leqslant Ld(f \circ g(y_1), f \circ g(y_2)) + LA \\ &L(d(f \circ g(y_1), y_1) + d(f \circ g(y_2), y_2) + d(y_1, y_2)) + LA \\ &\leqslant Ld(y_1, y_2) + 3LA \end{aligned}$$

Similarly, $d(y_1, y_2) \leqslant Ld(g(y_1), g(y_2)) + 3A$.
And since

$$d(g(f(x)), x) \leqslant Ld(f(g(f(x))), f(x)) + LA \leqslant 2LA.$$

So g is (L, A_1)-quasi-isometry for some A_1 and it is quasi-inverse of f. □

Prop A.1.4. *Suppose $f, g : \mathbb{H}^n \to \mathbb{H}^n$ are two (L, A)-quasi-isometry such that $\partial f = \partial g$, then there exists $C > 0$ such that $d(f(x), g(x)) \leqslant C$ for every $x \in \mathbb{H}^n$.*

Proof. Consider $x \in \mathbb{H}^n$ and a geodesic line γ ending at $a, b \in \partial\mathbb{H}^n$.
Since $\partial f(a) = \partial g(a), \partial f(b) = \partial g(b)$.
So both $f \circ \gamma, g \circ \gamma$ are within a bounded distance R from the geodesic line γ' ending at $\partial f(a), \partial f(b)$, where R only depends on L, A.
Therefore $d(f(x), \gamma'), d(g(x), \gamma') \leqslant R$.
Notice that ∂f is surjective.
In the other words, for every point $p \in \partial\mathbb{H}^n$, there exists a geodesic line γ_p ending at p such that γ_p intersects with both $B_R(f(x))$ and $B_R(g(x)))$.
Suppose $U = B_R(f(x))$ and $V = B_R(g(x))$ are disjoint.
let δ be the geodesic segment from $f(x)$ to $g(x)$ and take a point $q \in \delta\backslash(U \cup V)$.
Consider the nearest point restriction map π for δ and the hypersurface $H = \pi^{-1}(q)$.
Then H is totally geodesic and divides $\overline{\mathbb{H}^n}$ into two components C_1, C_2 where $y_i \in C_i$.
Moreover, $U \subset C_1$ and $V \subset C_2$ since δ is perpendicular to H, $d(q, y_i) > R$.
Take $p \in H \cap \partial\mathbb{H}^n$.
Since both $\overline{C_1}, \overline{C_2}$ are convex.
So any geodesic line γ_p ending at p disjoints with one of U and V, contradiction!
Hence $d(f(x), g(x)) \leqslant \mathbb{C}$. □

A.2 Mostow rigidity theorem

Thm A.2.1 (Mostow rigidity). *M, N are closed hyperbolic n-manifold with $n \geqslant 3$, then every homotopy equivalent $f : M \to N$ must be homeomorphism to an isometry*

Proof. Let $g : N \to M$ be the homotopy inverse of f.
By differential topology, f, g can be homotopy to some smooth maps, *i.e.* WLOG, we assume f, g are smooth.
And we can lift f, g to $\tilde{f}, \tilde{g} : \mathbb{H}^n \to \mathbb{H}^n$.
Then $\tilde{f}, \tilde{g}$ are Lipschitz.
Moreover, there exists a smooth homotopy $H : M \times I \to M$ such that $g \circ f \overset{H}{\simeq} \mathrm{Id}_M$.
With some suitably $\tilde{g}$, we can lift H to $\tilde{H}$ such that $\tilde{g} \circ \tilde{f} \overset{\tilde{H}}{\simeq} \mathrm{Id}_{\mathbb{H}^n}$.
So $\left|\mathrm{d}\tilde{H}\right|$ is bounded, *i.e.* there exists constant C such that

$$d\left(\tilde{g} \circ \tilde{f}(x), x\right) \leqslant C.$$

Thus $\tilde{f}$ is quasi-isometry with inverse $\tilde{g}$.
By theorem 2.1.2, $\tilde{f}$ can be extended to quasiconformal $\partial\tilde{f} : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$.
We claim that $\partial\tilde{f}$ is actually conformal.
Once we have proved this, we can extend $\partial\tilde{f}$ to a conformal map $h : \mathbb{H}^n \to \mathbb{H}^n$, *i.e.* it is an isometry under the hyperbolic metric.
Since for $A \in \Gamma_X$, there exists $B \in \Gamma_Y$ such that $\partial\tilde{f} \circ A = B \circ \partial\tilde{f}$.
So $h \circ A = B \circ h$ on $\mathbb{H}^n$, take

$$H_0 : \mathbb{H}^n \times I \to \mathbb{H}^n, (x, t) \mapsto (1 - t)\tilde{f}(x) + th(x),$$

then $\tilde{f} \overset{H_0}{\simeq} h$ and $H_0 \circ A = B \circ \tilde{H}_0$.

Hence $H_0 : M \times I \to N$ is well-defined and $f \overset{H_0}{\simeq} h$. □

We now prove the claim in the proof of Mostow rigidity, it is a direct consequence of the theorem below.

Thm A.2.2. *Suppose X, Y are closed hyperbolic n-manifolds with $n \geqslant 3$ and $f : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a homeomorphism which is differentiable at x_0 with nonzero derivative. If $f \circ \gamma \circ f^{-1} \in \Gamma_Y$ for any $\gamma \in \Gamma_Y$, then f is a Mobius transformation.*

Proof. WLOG, assume $x_0 = 0$ and $f(0) = 0$.

Similar to proposition 2.2.2, we let $A_n(x) = \frac{x}{n}$, take $y \in \mathbb{H}^n$ and a fundamental domain F of X, then we have $\gamma_n \circ A_n(y) \in F, \gamma_{n_i} \circ A_{n_i} \to \sigma_1, \delta_n \circ f = f \circ \gamma_n$ and $\delta_{n_i} \circ A_{n_i} \to \sigma_2$.

Let $f_n = A_n^{-1} \circ f \circ A_n$ converges to $h : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ and $J = f'(0)$.

Then $\sigma_2^{-1} \circ \delta \circ \sigma_2 \circ h = h \circ \sigma_1^{-1} \circ \gamma \circ \sigma_1$ and $h(x) = Jx$ for $x \in \partial\mathbb{H}^n$.

Notice that

$$\begin{aligned} \sigma_2 \circ h \circ \sigma_1^{-1} &= \lim_{i\to\infty} \sigma_{n_i} \circ A_{n_i} \circ A_{n_i}^{-1} \circ f \circ A_{n_i} \circ A_{n_i}^{-1} \circ \gamma_{n_i}^{-1} \\ &= \lim_{i\to\infty} \sigma_{n_i} \circ f \circ \gamma_{n_i}^{-1} = f \end{aligned}$$

So we only need to prove that h is a Mobius transformation.

By Liouville theorem, this is equivalent to prove that h is an Euclidean similarity, which follows from the lemma below since Γ_X is cocompact. □

Lemma A.2.1. *Suppose $\gamma \in \mathrm{SO}^+(n,1)$ such that $\gamma(\infty) \neq \{0, \infty\}$ and $A \in \mathrm{GL}(n-1)$ which conjugates γ to $A\gamma A^{-1} \in \mathrm{SO}^+(n,1)$, then A is an Euclidean similarity.*

Proof. Suppose A is not an Euclidean similarity.

Since $A\gamma(\infty) \neq 0$, let P be a hyperplane in $\mathbb{R}^{n-1}$ with $0 \in P, A\gamma(\infty) \notin P$.

So $\gamma A^{-1}(P)$ must be a sphere $S \subset \mathbb{R}^{n-1}$ as γ is conformal.

But $A(S)$ is an ellipsoid which is not a sphere.

Hence $A\gamma A^{-1}$ is not conformal, contradiction! □